Counting meromorphic differentials on the Riemann sphere and the KP hierarchy

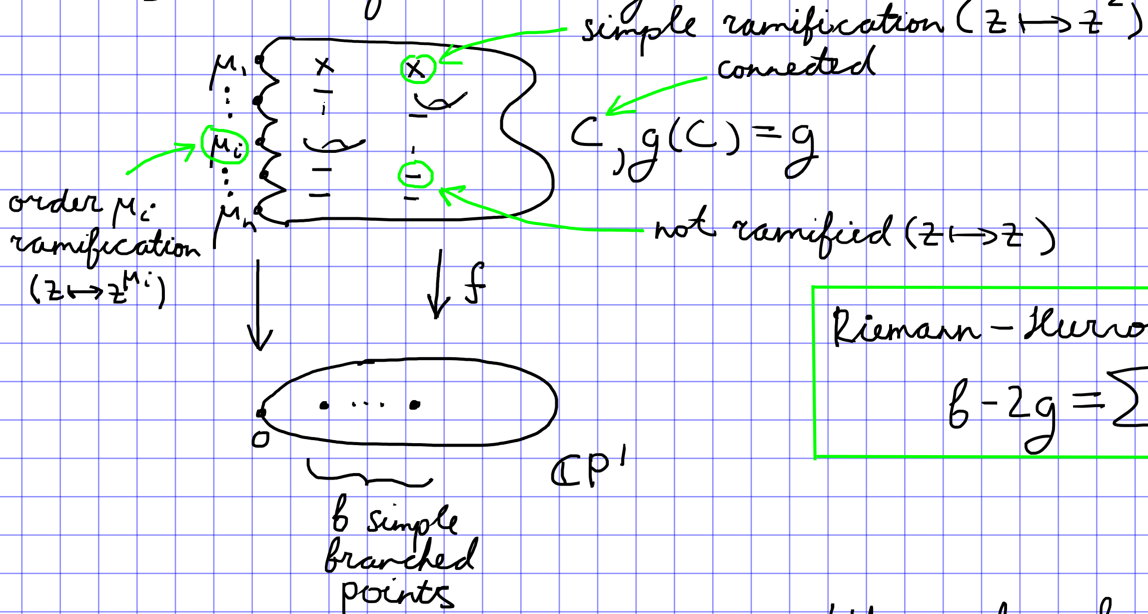
Curve counting invariants (Hurwitz numbers, Gromov–Witten invariants, ...) $\longleftrightarrow$ Integrable systems

Typically: the generating series of invariants gives a solution of an integrable system (often, KP and its analogs)

Example

$\mu_1, \dots, \mu_n \geq 1$, $\overline{\mu} = (\mu_1, \dots, \mu_n)$

Consider ramified coverings



Riemann–Hurwitz formula

$$b - 2g = \sum \mu_i + n - 2$$

Simple Hurwitz number $h_{b,\overline{\mu}} := \left(\begin{array}{l} \text{the number of isomorphism} \\ \text{classes } [f : C \to \mathbb{C}P^1] \\ \text{weighted by } \dfrac{1}{|\mathrm{Aut}(f)|} \end{array} \right)$

Theorem (Okounkov, 2000)

$p_1, p_2, \dots, \beta$ - formal variables

$$\exp\left(\sum_{n \geq 1} \sum_{\overline{\mu} = (\mu_1, \dots, \mu_n)} \sum_{b \geq 0} \frac{\beta^b}{b!} h_{b,\overline{\mu}} \frac{p_{b_1} p_{b_2} \cdots p_{b_n}}{n!} \right)$$

is a tau function of the KP hierarchy

Much less studied phenomenon:

The coefficients in the equations defining IS contain an information about curve counting invariants

Goal of the talk (j/w Paolo Rossi and Dimitri Zvonkine)
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① Genus 0.

The numbers of residueless meromorphic differentials on $\mathbb{C}P^1$	$\longleftrightarrow$	The coefficients in the equations of the dispersionless KP hierarchy

② Arbitrary genus

Intersection numbers with the classes of meromorphic differentials in $\overline{\mathcal{M}}_{g,n}$	$\longleftrightarrow$	The coefficients in the equations of the full KP hierarchy

Genus 0

$n \geq 3$

$A = (a_1, \dots, a_n)$, $a_i \in \mathbb{Z}$, $\sum a_i = -2$

$Z = (z_1, \dots, z_n)$, $z_i \in \mathbb{C}P^1$

There is a unique, up to multiplicative constant, meromorphic differential $\omega_{Z,A}$ with $(\omega_{Z,A}) = \sum a_i z_i$:

$$\omega_{Z,A} = \prod (z - z_i)^{a_i} dz. \quad \text{(formula in the case } z_i \in \mathbb{C}\text{)}$$

$$\boxed{\mathcal{H}_0^{res}(A) := \left\{ Z = (z_1, \dots, z_n) \in (\mathbb{C}P^1)^n \,\middle|\, \begin{array}{l} z_i \neq z_j \\ \operatorname{res}_{z_i} \omega_{Z,A} = 0 \end{array} \right\} \Big/ PGL(2,\mathbb{C})}$$

Well known:

If $\mathcal{H}_0^{res}(A) \neq \emptyset$, then $\dim \mathcal{H}_0^{res}(A) = \{1 \leq i \leq n \mid a_i \geq 0\} - 2$

Interested in counting problem $\rightsquigarrow$

Question:

How to compute the numbers $|\mathcal{H}_0^{res}(a,b,-c_1,\dots,-c_n)|$?

$a,b\geq 1,\ c_1,\dots,c_n\geq 2$

$a+b-\sum c_i=-2$

Example

$|\mathcal{H}_0^{res}(2,2,-3,-3)|=?$

$$(z_1,z_2,z_3,z_4)\xmapsto{\text{unique } \varphi\in PGL(2,\mathbb{C})}(1,t,0,\infty)$$

$z_i\neq z_j$

For $\omega=\dfrac{(z-1)^2(z-t)^2}{z^3}$ we have

$(\omega)=2[1]+2[t]-3[0]-3[\infty]$

$\mathrm{res}_0\,\omega=1+4t+t^2$, two roots $\Rightarrow |\mathcal{H}_0^{res}(2,2,-3,-3)|=2$

Dispersionless KP hierarchy (dKP hierarchy)

p - formal variable

$f_i=f_i(x)$ - smooth functions of x, $i\geq 1$

$$H:=\Big\{\overbrace{\sum_{i\leq m}Q_i(x)p^i}^{Q}\Big\}\qquad Q_+:=\sum_{i=0}^{m}Q_i(x)p^i$$

$Q_i(x)$ ← smooth functions

Poisson bracket on H: $\{p^a,p^b\}=\{f(x),g(x)\}:=0$

$\{p^a,g(x)\}=-\{g(x),p^a\}:=ag'p^{a-1}$

$\hat{L}:=p+f_1(x)p^{-1}+f_2(x)p^{-2}+\dots$

dKP hierarchy

$$\boxed{\frac{\partial\hat{L}}{\partial T_n}=\{(\hat{L}^n)_+,\hat{L}\},\quad n\geq 1}$$

Example

- $n=1$

$\hat{L}_+=p$, $\{p,p+\sum_{i\geq1}f_ip^{-i}\}=\sum_{i\geq1}f_i'p^{-i}$

Hence, $\dfrac{\partial f_i}{\partial T_1}=f_i'=\partial_x f_i$

- $n=2$, $(\hat{L}^2)_+=p^2+2f_1$

$$\frac{\partial\hat{L}}{\partial T_2}=\{p^2+2f_1,p+\sum_{i\geq1}f_ip^{-i}\}=\underbrace{2f_2'}_{\overset{\shortparallel}{\frac{\partial f_1}{\partial T_2}}}p^{-1}+\underbrace{(2f_3'+2f_1f_1')}_{\overset{\shortparallel}{\frac{\partial f_2}{\partial T_2}}}p^{-2}+\dots$$

Standard change of variables: $w_i := \mathrm{res}_{p=0} \hat{L}^i$

$$\boxed{\begin{array}{l}\underline{\text{Example}}\\ w_1 = f_1 \qquad w_2 = 2f_2 \qquad w_3 = 3f_3 + 3f_1^2 \qquad w_4 = 4f_4 + 12 f_1 f_2\end{array}}$$

Rewrite the dKP hierarchy in the variables w_i:

$$\frac{\partial w_i}{\partial T_j} = \partial_x R_{ij}$$

polynomials in $w_1, w_2, \dots$

Properties:
- $R_{ij} = R_{ji}$
- $R_{1,i} = R_{i,1} = w_i$
- $R_{ij} = -\mathrm{res}_{p=0}\left((\hat{L}^i)_+ \partial_p(\hat{L}^j)\right)$
- $\deg R_{ij} = i+j$ where $\deg w_i := i+1$ $\Rightarrow$

$\Rightarrow$ $\boxed{\text{the coefficient of } w_{c_1-1}\cdots w_{c_n-1} \text{ in } R_{a+1,b+1} \text{ is } 0 \text{ unless } a+b+2 = \sum c_i}$

$$\boxed{\begin{array}{l}\underline{\text{Example}}\\ R_{2,2} = \frac{4}{3} w_3 - 2w_1^2 \qquad R_{2,3} = \frac{3}{2} w_4 - 3 w_1 w_2 \qquad R_{3,3} = \frac{9}{5} w_5 - 3w_1 w_3 - \frac{9}{4} w_2^2 + 3w_1^3\end{array}}$$

Theorem (B.–Rossi–Zvonkine, 2021)

$a, b \geq 1$, $c_1, \dots, c_n \geq 2$, $a+b-\sum c_i = -2$.

Then

$$|\mathcal{H}_0^{\mathrm{res}}(a, b, -c_1, \dots, -c_n)| = (-1)^{n+1} \frac{(c_1-1)\cdots(c_n-1)}{(a+1)(b+1)} \left.\frac{\partial^n R_{a+1,b+1}}{\partial w_{c_1-1}\cdots \partial w_{c_n-1}}\right|_{w_*=0}$$

Idea of the proof

$\mathbb{Z}^* := \mathbb{Z} \setminus \{-1\}$

t^α, $\alpha \in \mathbb{Z}^*$ are formal variables

$$F(t^*) := \sum_{n \geq 3} \sum_{\substack{\alpha_1, \dots, \alpha_n \in \mathbb{Z}^* \\ \sum \alpha_i = -2 \\ |\{1 \leq i \leq n \mid \alpha_i \geq 0\}| = 2}} |\mathcal{H}_0^{\mathrm{res}}(\alpha_1, \dots, \alpha_n)| \frac{t^{\alpha_1}\cdots t^{\alpha_n}}{n!}.$$

Then F satisfies the WDVV equations

$$\sum_{\mu+\nu=-2} \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu} \frac{\partial^3 F}{\partial t^\nu \partial t^\gamma \partial t^\delta} = \sum_{\mu+\nu=-2} \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\mu} \frac{\partial^3 F}{\partial t^\nu \partial t^\gamma \partial t^\alpha}, \quad \alpha,\beta,\gamma,\delta \in \mathbb{Z}^*.$$

Fully determines F starting from a small set of numbers $|\mathcal{H}_0^{res}(\alpha_1,\dots,\alpha_n)|$, which we computed explicitly

Remark

F gives a Dubrovin–Frobenius manifold underlying the KP hierarchy! The metric is $\eta_{\alpha\beta} = \delta_{\alpha+\beta,-2}$, the unit is $\frac{\partial}{\partial t^0}$.

Arbitrary genus

General framework:

cohomological field theory (CohFT) $\rightarrow$ Dubrovin–Zhang hierarchy (geometric interpretation of a special solution)

cohomological field theory (CohFT) $\rightarrow$ DR hierarchy (geometric interpretation of equations)

Dubrovin–Zhang hierarchy $\leftrightarrow$ DR hierarchy: conjecturally equivalent

$\alpha_1,\dots,\alpha_n \in \mathbb{Z}^*$, $\sum \alpha_i = 2g-2$

$$\boxed{\overline{\mathcal{H}}_g^{res}(\alpha_1,\dots,\alpha_n) := \overline{\left\{ [C, x_1,\dots,x_n] \in \mathcal{M}_{g,n} \;\middle|\; \text{there is a residueless meromorphic differential } \omega \text{ with } (\omega) = \textstyle\sum \alpha_i x_i \right\}} \subset \overline{\mathcal{M}}_{g,n}}$$

$$\operatorname{codim} \overline{\mathcal{H}}_g^{res}(\alpha_1,\dots,\alpha_n) = g-1+\underbrace{|\{1\le i\le n \mid \alpha_i<0\}|}_{N_\alpha}$$

$$\rightsquigarrow \left[\overline{\mathcal{H}}_g^{res}(\alpha_1,\dots,\alpha_n)\right] \in H^{2(g-1+N_\alpha)}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$$

$u_1(x), u_2(x), \dots$ functions of x

$u_i^{(k)} := \partial_x^k u_i$

$\alpha, \beta \geq 1$

$$P_{\alpha\beta} := \sum_{g,n} \frac{\varepsilon^{2g}}{n!} \sum_{\substack{k_1,\dots,k_n \geq 0 \\ \alpha_1,\dots,\alpha_n \geq 1}} \prod_{i=1}^n u_{\alpha_i}^{(k_i)} \times$$

$$\times \operatorname{Coef}_{a_1^{k_1}\cdots a_n^{k_n}} \int_{\overline{\mathcal{M}}_{g,n+2}} \lambda_g \left[\mathcal{H}_g^{res}(\alpha-1, \beta-1, -\alpha_1-1, \dots, -\alpha_n-1)\right] DR_g\left(-\sum a_i, 0, a_1, \dots, a_n\right)$$

$\in H^{2g}(\overline{\mathcal{M}}_{g,n+2}, \mathbb{Q})$ $\qquad$ $DR_g(\dots) \in H^{2g}(\overline{\mathcal{M}}_{g,n+2}, \mathbb{Q})$

polynomials in ε and $u_1, u_2, \dots$

$\int_{\overline{\mathcal{M}}_{g,n+1}} DR_g(-\sum a_i, a_1, \dots, a_n) \lambda_g \Theta$ is a polynomial in $a_1, \dots, a_n$ of degree $2g$, $\Theta \in H^*(\overline{\mathcal{M}}_{g,n+1}, \mathbb{Q})$

Example

$P_{1,2} = u_2, \quad P_{1,3} = u_3 + \frac{\varepsilon^2}{8} u_1^{(2)}, \quad P_{2,2} = u_3 + \frac{u_1^2}{2} + \frac{\varepsilon^2}{24} u_1^{(2)}.$

Full KP hierarchy

$$\tilde{H} := \Big\{ \underbrace{\sum_{i \leq m} Q_i(x) \partial_x^i}_{Q} \Big\} \qquad Q_+ := \sum_{i=0}^m Q_i(x) \partial_x^i \qquad \operatorname{res} Q := Q_{-1}$$

Q ← pseudo-differential operator

$Q_i(x)$ ← smooth functions

Multiplication rule: $\partial_x^k \circ f(x) := \sum_{i \geq 0} \binom{k}{i} f^{(i)} \partial_x^{k-i}, \; k \in \mathbb{Z} \implies$

$\implies \tilde{H}$ is an associative noncommutative algebra

$L := \partial_x + f_1(x)\partial_x^{-1} + f_2(x)\partial_x^{-2} + \dots$

KP hierarchy

$$\frac{\partial L}{\partial T_n} = \left[(L^n)_+, L\right], \; n \geq 1$$

Theorem (B.–Rossi–Zvonkine, 2021)

① The PDEs

$$(*) \quad \frac{\partial u_\alpha}{\partial t^\beta} = \partial_x P_{\alpha\beta}, \quad \alpha, \beta \geq 1,$$

are pairwise compatible (the flows $\frac{\partial}{\partial t^\beta}$ commute)

② The change of variables

$$P_{1,\alpha} = -\frac{1}{\alpha} \operatorname{res} L^\alpha, \qquad t^\beta = \beta T_\beta,$$

relates the system $(*)$ to the full KP hierarchy.

Remark

$P_{1,\alpha} = u_\alpha + O(\varepsilon^2) \implies u_\alpha \mapsto P_{1,\alpha}$ can be considered as a change of variables