Quantisation of spectral curves of arbitrary rank and genus via topological recursion

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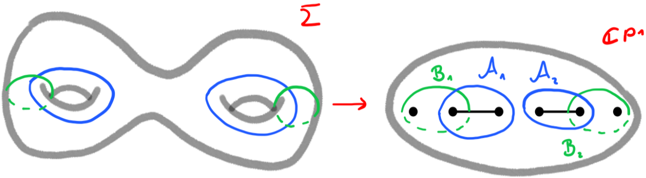
(based on joint work with B. Eynard, O. Marchal and N. Orantin)



Quantum Curves, Integrability and Cluster Algebras (13-18 December), MATRIX

December 16, 2021

Outline

1. Quantum curves and topological recursion
2. Spectral curves
3. Topological recursion and loop equations
4. Perturbative wave function and KZ equations
5. Non-perturbative wave functions KZ equations and Lax system
6. Link with isomonodromic systems
7. Questions and future work

Outline

Topological recursion (TR, Chekhov–Eynard–Orantin '04-'07)

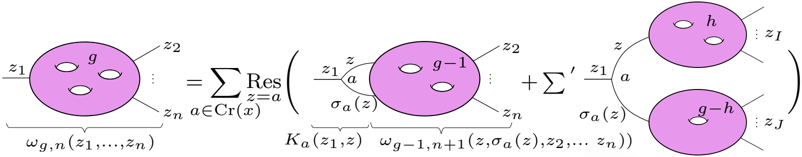
Goal: "Count surfaces $S_{g,n}$ of genus g with n boundaries (topology (g,n))."

Spectral curve

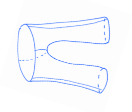
TR :
$$\begin{cases} \Sigma \text{ Riemann surface} \\ x\colon \Sigma \to \mathbb{CP}^1 \\ \omega_{0,1} = y\,\mathrm{d}x \text{ 1-form} \quad \text{(discs)} \\ \omega_{0,2} \quad (1,1)\text{-form} \quad \text{(cylinders)} \end{cases}$$

$\rightsquigarrow$ **recursion on** $|\chi(S_{g,n})| = 2g-2+n$

Differential forms $\omega_{g,n}(z_1,\dots,z_n)$, $z_i \in \Sigma$, $\forall g,n \geq 0$.



- Terms in correspondence with the ways of cutting a **pair of pants** $(0,3)$ from $S_{g,n}$.



- Related to many interesting geometric problems: Hurwitz covers, graphs embedded on surfaces, Gromov–Witten invariants, volumes of moduli spaces, statistical physics models, knot invariants...

Presentation of the problem

$P \in \mathbb{C}[x,y]$ and $\Sigma = \{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\}$ plane curve of genus $\hat{g}$.

A quantization of Σ is a differential operator $\widehat{P}$ of the form

$$\widehat{P}(\widehat{x},\widehat{y};\hbar) = P_0(\widehat{x},\widehat{y}) + O(\hbar),$$

where $\widehat{x} = x\cdot$, $\widehat{y} = \hbar\frac{\mathrm{d}}{\mathrm{d}x}$, such that $P_0(x,y) = P(x,y)Q(x,y)$, for some $Q \in \mathbb{C}[x,y]$ (often 1).

- The operators $\hat{x}$ and $\hat{y}$ satisfy $[\hat{y},\hat{x}] = \hbar$.
- $\hat{P}(\hat{x},\hat{y})\psi(x,\hbar) = 0$. Schrödinger equation: $\left(\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \hat{R}(x,\hbar)\right)\psi(z,\hbar) = 0$.

WKB asymptotic expansion $\rightsquigarrow \log\psi(x,\hbar) = \sum_{k\geq -1} \hbar^k S_k(x) \in \hbar^{-1}\mathbb{C}[[\hbar]]$.

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Conjecture

Both $\hat{P}$ and ψ can be constructed from Σ using topological recursion.

History and literature

- Proved for many particular cases $\rightsquigarrow$ genus $\hat{g} = 0$ spectral curves.
- Bouchard–Eynard '17 $\rightsquigarrow$ spectral curves whose Newton polygon has $N_I := \#\{\text{interior points}\} = 0$ (Fact: $\hat{g} \leq N_I$).

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- Iwaki–Marchal–Saenz '18, Marchal–Orantin '19 (reversed approach) $\rightsquigarrow$ Lax pairs associated with $\hbar$-dependent Painlevé equations and any $\hbar \partial_x \Psi(x,\hbar) = \mathcal{L}(x,\hbar)\Psi(x,\hbar)$, with $\mathcal{L}(x,\hbar) \in \mathfrak{sl}_2(\mathbb{C})$, satisfy the topological type property from Bergère–Borot–Eynard '15 ($\hat{g} = 0$).
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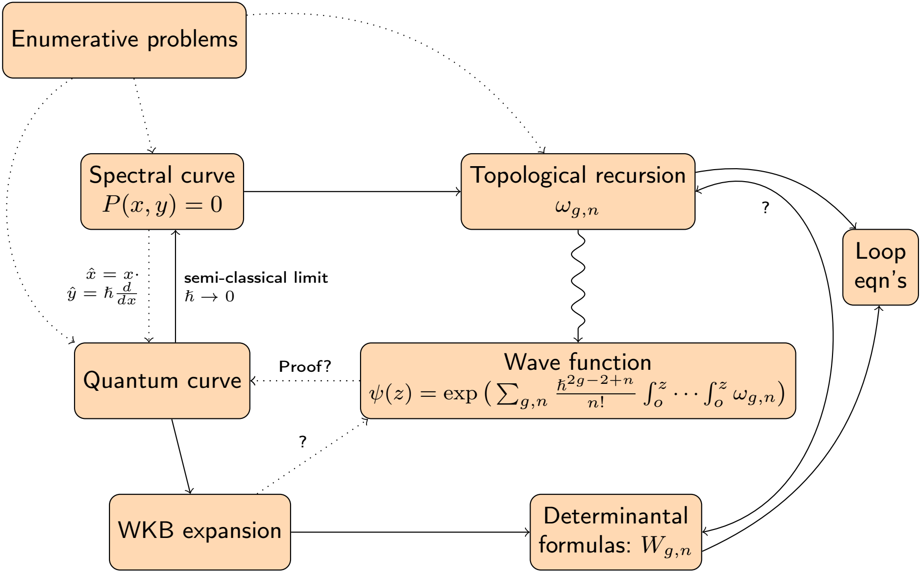
- Eynard–GF–Marchal–Orantin '21 $\rightsquigarrow$ any algebraic curve with simple ramifications.

Context



Enumerative problems

Spectral curve
$P(x,y)=0$

Topological recursion
$\omega_{g,n}$

Loop eqn's

$\hat{x} = x\cdot$
$\hat{y} = \hbar\frac{d}{dx}$

semi-classical limit
$\hbar \to 0$

?

Quantum curve

Proof?

Wave function
$\psi(z) = \exp\left(\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_o^z \cdots \int_o^z \omega_{g,n}\right)$

?

WKB expansion

Determinantal formulas: $W_{g,n}$

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Input of topological recursion (TR) (Chekhov–Eynard–Orantin, '04-'07)

Input: *Spectral curve* $\mathcal{S} = (\Sigma, x, y\mathrm{d}x, B)$:

- Σ Riemann surface of genus $\hat{g}$.
- Two meromorphic functions $x, y : \Sigma \to \mathbb{C} \Rightarrow P(x,y) = 0,\ P \in \mathbb{C}[x,y]$.
- Symplectic basis of non-contractible cycles $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{\hat{g}}$ on Σ.
- A symmetric bidifferential $B = \omega_{0,2}$ on $\Sigma \times \Sigma$ such that $\omega_{0,2}(z_1,z_2) \underset{z_2 \to z_1}{\sim} \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1-z_2)^2}$ + holomorphic with vanishing $\mathcal{A}$-periods.

Output: $\omega_{\mathbf{g},\mathbf{n}}(\mathbf{z_1},\ldots,\mathbf{z_n}) \in \mathrm{H}^0(\Sigma^n, (K_\Sigma(*\mathrm{Cr}(x)))^{\boxtimes n})^{\mathfrak{S}_n}$, for all $g, n \geq 0$.

Important properties: For $2g-2+n>0$, the $\omega_{g,n}$ are symmetric meromorphic differentials with poles at ramifications points.

Regularity condition: $x : \Sigma \to \mathbb{C}$ meromorphic function with finitely many and simple ramification points (denoted $\mathcal{R}(x)$), and $y : \Sigma \to \mathbb{C}$ holomorphic on a neighborhood of every $a \in \mathcal{R}(x)$ and $\mathrm{d}y(a) \neq 0 \Rightarrow$ Existence of a local involution σ around every ramification point: $x(z) = x(\sigma(z))$.

Spectral curves

Fix N distinct points $\Lambda_1, \ldots, \Lambda_N \in \mathbb{P}^1 \setminus \{\infty\}$. Let $\mathcal{H}_d(\Lambda_1, \ldots, \Lambda_N, \infty)$ be the Hurwitz space of degree d ramified coverings $x \colon \Sigma \to \mathbb{P}^1$, where Σ is the Riemann surface of genus $\hat{g}$:

$$\Sigma := \overline{\left\{(\lambda, y) \mid P(\lambda, y) = 0\right\}},$$

where $x(\lambda, y) := \lambda$ and

$$P(\lambda, y) = \sum_{l=0}^{d} (-1)^l y^{d-l} P_l(\lambda),$$

with each coefficient P_l being a rational function with possible poles at $\lambda \in \mathcal{P} := \{\Lambda_i\}_{i=1}^{N} \bigcup \{\infty\}$ and $P_0 = 1$.

Classical spectral curve $\rightsquigarrow (\Sigma, x)$.

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- Local coordinates (in the base): $\{\xi_q(\lambda)\}_{q \in \mathcal{P}}$ around $q \in \mathcal{P}$ are defined by
$$\forall\, i \in [\![1, N]\!] \,:\, \xi_{\Lambda_i}(\lambda) := (\lambda - \Lambda_i) \qquad \text{and} \qquad \xi_\infty(\lambda) := \lambda^{-1}.$$
- Local coordinates (in the cover): near any $p \in x^{-1}(q)$, let $d_p := \mathrm{ord}_p(\xi_q)$
$$\zeta_p(z) = \xi_q(x(z))^{\frac{1}{d_p}}.$$

$\{d_p\}_{p \in x^{-1}(q)}$ is called the ramification profile of q. We have $\sum_{p \in x^{-1}(P)} d_p = d$.

Admissible spectral curves

Expansion of the 1-form $\omega_{0,1} = ydx$ around any pole $p \in x^{-1}(\mathcal{P})$:

$$ydx = \sum_{k=0}^{r_p-1} t_{p,k}\zeta_p^{-k-1}d\zeta_p + \text{analytic at } p.$$

The $t_{p,k}$'s are called the spectral times (or *KP times*).

Ramification points: $\mathcal{R}_0 \coloneqq \left\{p \in \Sigma \mid 1 + \mathrm{order}_p\, dx \neq \pm 1\right\}$,

$$\mathcal{R} \coloneqq \left\{p \in \Sigma \mid dx(p) = 0\ ,\ x(p) \notin \mathcal{P}\right\} = \mathcal{R}_0 \setminus x^{-1}(\mathcal{P}).$$

Critical values: $x(\mathcal{R})$.

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Definition (Admissible classical spectral curves)

A classical spectral curve (Σ, x) is *admissible* if:

- $P(\lambda, y) = 0$ is an irreducible algebraic curve;
- $a \in \mathcal{R}$ are simple, i.e. dx has only a simple zero at $a \in \mathcal{R}$;
- $\forall (a_i, a_j) \in \mathcal{R} \times \mathcal{R}$ with $a_i \neq a_j$, $x(a_i) \neq x(a_j)$;
- $\forall a \in \mathcal{R}$, $dy(a) \neq 0$;
- $\forall p \in x^{-1}(\mathcal{P})$ ramified, the 1-form ydx has a pole of degree $r_p \geq 3$ at p and $t_{p,r_p-2} \neq 0$.

Torelli marking and filling fractions

For any symplectic basis $(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^g$ of $H_1(\Sigma,\mathbb{Z})$, let

$$B^{(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^g} \in H^0(\Sigma^2, K_\Sigma^{\boxtimes 2}(2\Delta))^{\mathfrak{S}_2} \subset \mathcal{M}_2(\Sigma^2)$$

be the unique symmetric bidifferential on Σ^2 with a unique double pole on the diagonal Δ, without residue, bi-residue equal to 1 and normalized on the $\mathcal{A}$-cycles by

$$\forall\, i \in [\![1,g]\!]\,,\ \oint_{z_1\in\mathcal{A}_i} B^{(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^g}(z_1,z_2) = 0.$$

Remark

Choice of Torelli marking can be thought of as a choice of polarisation from a geometric quantisation point of view.

Let $((\Sigma,x),(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^g)$ be some admissible initial data. We define $(\epsilon_i)_{i=1}^g$ the tuple of *filling fractions* by

$$\forall\, i \in [\![1,g]\!]\,,\quad \epsilon_i \coloneqq \frac{1}{2\pi i}\oint_{\mathcal{A}_i} y dx.$$

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Properties of TR

- $\omega_{g,n}$ are invariant under permutations of their n arguments.
- $\omega_{0,1}(z_1)$ may only have poles at $x^{-1}(\mathcal{P})$. $\omega_{0,2}(z_1,z_2)$ may only have poles at $z_1 = z_2$. For $(h,n) \in \mathbb{N} \times \mathbb{N}^* \setminus \{(0,1),(0,2)\}$, $\omega_{h,n}(z_1,\dots,z_n)$ may only have poles at $z_i \in \mathcal{R}$, for $i \in [\![1,n]\!]$.
- For all $i \in [\![1,\hat{g}]\!]$,

$$\frac{\partial}{\partial \epsilon_i}\omega_{h,n}(z_1,\dots,z_n) = \oint_{z\in\mathcal{B}_i} \omega_{h,n+1}(z,z_1,\dots,z_n).$$

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Ramification points at poles:

- In the definition of TR, residues at $a \in \mathcal{R} = \mathcal{R}_0 \setminus x^{-1}(\mathcal{P})$.
- But the points of $\mathcal{P}$ could also be ramified (many interesting examples, like the Airy curve $y^2 = x$).
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Lemma (Ramified poles don't contribute for admissible curves)

Let $\omega'_{h,n}$ be the topological recursion differential forms defined by taking residues at all $a \in \mathcal{R}_0$ (including $a \in x^{-1}(\mathcal{P})$). If $\forall p \in x^{-1}(\mathcal{P})$, we have $r_p \geq 3$ and $t_{p,r_p-2} \neq 0$, then $\omega'_{h,n} = \omega_{h,n}$, and $\omega_{h,n}$ with $(h,n) \neq (0,1),(0,2)$ have poles only at $\mathcal{R} = \mathcal{R}_0 \setminus x^{-1}(\mathcal{P})$.

Loop equations

For $(h,n,l) \in \mathbb{N}^3$, $\lambda \in \mathbb{P}^1$ and $\mathbf{z} := (z_1,\ldots,z_n) \in \Sigma^n$,

$$Q^{(l)}_{h,n+1}(\lambda;\mathbf{z}) := \sum_{\beta \underset{l}{\subseteq} x^{-1}(\lambda)} \sum_{\mu \in \mathcal{S}(\beta)} \sum_{\bigsqcup_{i=1}^{l(\mu)} J_i = \mathbf{z}} \sum_{\sum_{i=1}^{l(\mu)} g_i = h + l(\mu) - l} \left[\prod_{i=1}^{l(\mu)} \omega_{g_i,|\mu_i|+|J_i|}(\mu_i, J_i) \right],$$

differential with possible poles at $\lambda \in \mathcal{P} \cup x(\mathcal{R})$, $z_i \in \mathcal{R}$ and $z_i \in x^{-1}(\lambda)$.

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Particular cases:

- $Q^{(l)}_{0,1}(\lambda) = \sum_{\beta \underset{l}{\subseteq} x^{-1}(\lambda)} \prod_{z\in\beta} \omega_{0,1}(z) = P_l(\lambda)\,(d\lambda)^l$.
- $Q^{(l)}_{0,2}(\lambda; z_1) = \sum_{\beta \underset{l}{\subseteq} x^{-1}(\lambda)} \sum_{z\in\beta} \omega_{0,2}(z,z_1) \prod_{\substack{\tilde{z}\in\beta \\ \tilde{z}\neq z}} \omega_{0,1}(\tilde{z})$.

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For $(h, n, l) \in \mathbb{N}^3$, $\lambda \in \mathbb{P}^1$ and $\mathbf{z} := (z_1, \dots, z_n) \in \Sigma^n$,

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differential with possible poles at $\lambda \in \mathcal{P} \cup x(\mathcal{R})$, $z_i \in \mathcal{R}$ and $z_i \in x^{-1}(\lambda)$.

$$Q^{(l)}_{h,n+1}(\lambda; \mathbf{z}) = 0, \text{ for } l \geq d + 1.$$

Particular cases:

- $Q^{(l)}_{0,1}(\lambda) = \sum_{\beta \underset{l}{\subseteq} x^{-1}(\lambda)} \prod_{z \in \beta} \omega_{0,1}(z) = P_l(\lambda) \, (d\lambda)^l$.
- $Q^{(l)}_{0,2}(\lambda; z_1) = \sum_{\beta \underset{l}{\subseteq} x^{-1}(\lambda)} \sum_{z \in \beta} \omega_{0,2}(z, z_1) \prod_{\substack{\tilde{z} \in \beta \\ \tilde{z} \neq z}} \omega_{0,1}(\tilde{z})$.

Theorem (Loop equations)

The function $\lambda \mapsto \frac{Q^{(l)}_{h,n+1}(\lambda; \mathbf{z})}{(d\lambda)^l}$ *has no poles at* $\lambda \in x(\mathcal{R})$, $\forall \mathbf{z} \in (\Sigma \setminus \mathcal{R})^n$.

- $Q^{(1)}_{h,n+1}(\lambda; \mathbf{z}) = \sum_{z \in x^{-1}(\lambda)} \omega_{h,n+1}(z, \mathbf{z}) = \delta_{n,0}\delta_{h,0} P_1(\lambda) d\lambda + \delta_{n,1}\delta_{h,0} \frac{d\lambda \, dx(z_1)}{(\lambda - x(z_1))^2}$.

Loop equations

$$\hat{Q}^{(l)}_{h,n+1}(z;\mathbf{z}) := \sum_{\beta \underset{l}{\subseteq} (x^{-1}(x(z))\setminus\{z\})} \ \sum_{\mu\in\mathcal{S}(\beta)} \ \sum_{\bigsqcup_{i=1}^{l(\mu)} J_i=\mathbf{z}} \ \sum_{\sum_{i=1}^{l(\mu)} g_i=h+l(\mu)-l} \ \prod_{i=1}^{l(\mu)} \omega_{g_i,|\mu_i|+|J_i|}(\mu_i, J_i)$$

Possible poles $\rightsquigarrow$ z with $x(z)\in x(\mathcal{R})$, $z\in x^{-1}(\mathcal{P})$, and $z_i \in \mathcal{R}\cup \left(x^{-1}\left(x(z)\right)\setminus\{z\}\right)$.

Lemma

For $\mathbf{z} := (z_1,\dots,z_n)\in\Sigma^n$ such that $x(z_i)\neq x(z_j)$ for any $i\neq j$, the functions

$$\widetilde{Q}^{(l)}_{h,n+1}(\lambda;\mathbf{z}) := \frac{Q^{(l)}_{h,n+1}(\lambda;\mathbf{z})}{(d\lambda)^l} - \sum_{j=1}^{n} d_{z_j}\left(\frac{1}{\lambda - x(z_j)}\,\frac{\hat{Q}^{(l-1)}_{h,n}(z_j;\mathbf{z}\setminus\{z_j\})}{(dx(z_j))^{l-1}}\right)$$

are rational functions of λ with no poles at $\lambda\in x(\mathcal{R})$ and at $\lambda\in\bigcup_{i=1}^{n}\{x(z_i)\}$.

For $z\in\Sigma\setminus\left(\mathcal{R}\bigcup x^{-1}(\mathcal{P})\right)$ and $\mathbf{z}\in\left[\Sigma\setminus\left(\mathcal{R}\bigcup x^{-1}(x(z))\right)\right]^n$, we have

$$Q^{(l)}_{h;n+1}(x(z);\mathbf{z}) = \hat{Q}^{(l)}_{h;n+1}(z;\mathbf{z}) + \hat{Q}^{(l-1)}_{h-1;n+2}(z;z,\mathbf{z}) + \sum_{A\sqcup B=\mathbf{z}} \ \sum_{h_1+h_2=h} \hat{Q}^{(l-1)}_{h_1,|A|+1}(z;A)\,\omega_{h_2,|B|+1}(z,B).$$

Outline

Perturbative wave function over a divisor

$D = \sum_{i=1}^{s} \alpha_i [p_i]$ a generic divisor (of degree$= \sum_i \alpha_i = 0$) on $\widetilde{\Sigma_{\mathcal{P}}}$, $\Sigma_{\mathcal{P}} := \Sigma \setminus x^{-1}(\mathcal{P})$.

Perturbative wave function $\psi(D,\hbar) = \psi_{0,i}(D,\hbar)$ associated to D:

$$\exp\left(\sum_{h\geq 0}\sum_{n\geq 0}\frac{\hbar^{2h-2+n}}{n!}\int_D\cdots\int_D\left(\omega_{h,n}(z_1,\ldots,z_n) - \delta_{h,0}\delta_{n,2}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\right)\right).$$

$$e^{-\hbar^{-2}\omega_{0,0}}e^{-\hbar^{-1}\int_D \omega_{0,1}}\psi(D,\hbar) \in \mathbb{C}[[\hbar]].$$

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$$e^{-\hbar^{-2}\omega_{0,0}} e^{-\hbar^{-1}\int_D \omega_{0,1}} \psi(D,\hbar) \in \mathbb{C}[[\hbar]].$$

$$\forall\, i \in [\![1,s]\!]\,,\ l \geq 1\ :\ \psi_{l,i}(D,\hbar) := \left[\sum_{h \geq 0} \sum_{n \geq 0} \frac{\hbar^{2h+n}}{n!} \overbrace{\int_D \cdots \int_D}^{n} \frac{\hat{Q}^{(l)}_{h,n+1}(p_i;\cdot)}{(dx(p_i))^l} \right] \psi(D,\hbar).$$

Perturbative partition function $Z(\hbar) = \psi(D = \emptyset, \hbar)$:

$$Z(\hbar) := \exp\left(\sum_{h \geq 0} \hbar^{2h-2} \omega_{h,0} \right), \text{ with } e^{-\hbar^{-2}\omega_{0,0}} Z(\hbar) \in \mathbb{C}[[\hbar]].$$

Remark

Wave functions are meant to be solutions to a differential equation; the partition function is expected to play the role of an associated tau function from the point of view of isomonodromic or integrable systems.

KZ equations

Theorem (General KZ equations)

For $i \in [\![1,s]\!]$ *and* $l \in [\![0,d-1]\!]$,

$$\begin{aligned}\frac{\hbar}{\alpha_i}\frac{d\psi_{l,i}(D,\hbar)}{dx(p_i)} &= -\psi_{l+1,i}(D,\hbar) - \hbar \sum_{j\in[\![1,s]\!]\setminus\{i\}} \alpha_j \frac{\psi_{l,i}(D,\hbar)-\psi_{l,j}(D,\hbar)}{x(p_i)-x(p_j)} \\ &\quad + \sum_{h\geq 0}\sum_{n\geq 0}\frac{\hbar^{2h+n}}{n!}\int_{z_1\in D}\cdots\int_{z_n\in D}\widetilde{Q}^{(l+1)}_{h,n+1}(x(p_i);\mathbf{z})\,\psi(D,\hbar) \\ &\quad + \left(\frac{1}{\alpha_i}-\alpha_i\right)\left[\sum_{(h,n)\in\mathbb{N}^2}\frac{\hbar^{2h+n+1}}{n!}\overbrace{\int_D\cdots\int_D}^{n}\frac{d}{dx(p_i)}\left(\frac{\hat{Q}^{(l)}_{h,n+1}(p_i;\cdot)}{(dx(p_i))^l}\right)\right]\psi(D,\hbar).\end{aligned}$$

If $\alpha_i = \pm 1$,

$$\begin{aligned}\frac{\hbar}{\alpha_i}\frac{d\psi_{l,i}(D,\hbar)}{dx(p_i)} &= -\psi_{l+1,i}(D,\hbar) - \hbar \sum_{j\in[\![1,s]\!]\setminus\{i\}} \alpha_j \frac{\psi_{l,i}(D,\hbar)-\psi_{l,j}(D,\hbar)}{x(p_i)-x(p_j)} \\ &\quad + \sum_{h\geq 0}\sum_{n\geq 0}\frac{\hbar^{2h+n}}{n!}\int_{z_1\in D}\cdots\int_{z_n\in D}\widetilde{Q}^{(l+1)}_{h,n+1}(x(p_i);\mathbf{z})\,\psi(D,\hbar).\end{aligned}$$

Regularised KZ equations

Let $z \in \widetilde{\Sigma_{\mathcal{P}}}$ be a generic point and $x^{-1}(\infty) = \{\infty^{(\alpha)}\}_{\alpha \in [\![1,\ell_\infty]\!]}$.
When $D = [z] - [p_2]$, $\psi(D,\hbar)$ has an essential singularity as $p_2 \to \infty^{(\alpha)}$.
Need to regularise the wave functions: $\psi_l^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar)$.

Theorem (KZ equations for regularized wave functions)

For $\alpha \in [\![1,\ell_\infty]\!]$, $l \in [\![0, d-1]\!]$, the regularised wave functions satisfy

$$\hbar \frac{d}{dx(z)} \psi_l^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar) + \psi_{l+1}^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar)$$

$$= \Bigg[\sum_{h \geq 0} \sum_{n \geq 0} \frac{\hbar^{2h+n}}{n!} \sum_{P \in \mathcal{P}} \sum_{k \in S_P^{(l+1)}} \xi_P(x(z))^{-k} \operatorname*{Res}_{\lambda \to P} \xi_P(\lambda)^{k-1} d\xi_P(\lambda)$$

$$\int_{z_1 = \infty^{(\alpha)}}^{z_1 = z} \cdots \int_{z_n = \infty^{(\alpha)}}^{z_n = z} \frac{Q_{h,n+1}^{(l+1)}(\lambda; \mathbf{z})}{(d\lambda)^{l+1}} \Bigg] \psi^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar).$$

Generalised cycles and algebra of symbols

Generalized cycles: $\mathcal{E} := \{\mathcal{C}_{p,k}\}_{p\in\Sigma, k\in\mathbb{Z}} \cup \{\mathcal{C}_o^p\}_{p\in\Sigma} \cup \{\mathcal{A}_i, \mathcal{B}_i\}_{i=1}^g$, where the integration of a meromorphic form ω along such cycles is defined as:

- $\forall\ p \in \Sigma$, and $\forall\ k \in \mathbb{Z}$,

$$\int_{\mathcal{C}_{p,k}} \quad : \quad \omega \mapsto \operatorname*{Res}_{p} \zeta_p^{-k}\, \omega\,.$$

- Let γ be a Jordan arc from a point $o \in \Sigma$ to a point $p \in \Sigma$.

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Commutative algebra freely generated by a set of symbols consisting of a pair (h,n) and a symbol $\int_{C_1}\cdots\int_{C_n}$, labeled by generalised cycles $C_i\in\mathcal{E}$:

$$\check{\mathcal{W}} = \mathbb{C}\left[\left\{\int_{C_1}\cdots\int_{C_n}\omega_{h,n}\right\}_{h,n\geq 0}\right] \quad / \text{ (cycle linearity relations).}$$

Evaluation map:

$$\begin{array}{rccc} \text{ev}: & \check{\mathcal{W}} & \to & \mathbb{C} \\ & \int_{C_1}\cdots\int_{C_n}\omega_{h,n} & \mapsto & \int_{z_1\in C_1}\cdots\int_{z_n\in C_n}\omega_{h,n}(z_1,\dots,z_n). \end{array}$$

$\mathcal{W} \rightsquigarrow$ extension to formal Laurent power series in $\hbar$, exponentials and inverses.

KZ equations with linear operators

Operators $(\mathcal{I}_C)_{C\in\mathcal{E}}$ acting on $\mathcal{W}$:

$$\forall\,(h,n)\in\mathbb{N}^2\;:\;\mathcal{I}_C\left[\int_{C_1}\cdots\int_{C_n}\omega_{h,n}\right]:=\int_{C_1}\cdots\int_{C_n}\int_C\omega_{h,n+1}.$$

Re-writing the RHS of the KZ equations with a multi-linear operator $\widetilde{\mathcal{L}}_l(x(z))$ that uses $\mathcal{I}_{\mathcal{C}_{p,k}}$ $\rightsquigarrow$ new system of KZ equations, for $\alpha\in[\![1,\ell_\infty]\!]$, $l\in[\![0,d-1]\!]$:

$$\begin{aligned}&\hbar\frac{d}{dx(z)}\psi_l^{\text{reg}}([z]-[\infty^{(\alpha)}])+\psi_{l+1}^{\text{reg}}([z]-[\infty^{(\alpha)}])\\&=\text{ev. }\widetilde{\mathcal{L}}_l(x(z))\left[\psi^{\text{reg symbol}}([z]-[\infty^{(\alpha)}])\right].\end{aligned}$$

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$$=\text{ev.}\ \widetilde{\mathcal{L}}_l(x(z))\left[\psi^{\text{reg symbol}}([z]-[\infty^{(\alpha)}])\right].$$

Degree 2 case (hyperelliptic):

$$\boxed{P(x,y)=R(x)-y^2=0,\text{ with }R(x)\in\mathbb{C}(x)}$$

$x:\Sigma\to\mathbb{C}\mathrm{P}^1$ is a double cover and we have a global involution

$$(x,y)\mapsto(x,-y).$$

Remark

In degree 2, the operators $\mathcal{I}_{\mathcal{C}_{p,k}}$ can be interpreted as derivatives with respect to the moduli of the classical spectral curve $\partial_{t_{p,k}}$.

KZ equations for $d = 2 \rightsquigarrow$ system of PDEs

Theorem (Eynard–GF,'19)

For $k = 1, 2,$

$$\hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_k^2} + \sum_{i \neq k} \frac{\frac{\mathrm{d}}{\mathrm{d}x_k} - \frac{\mathrm{d}}{\mathrm{d}x_i}}{x_k - x_i} \right) \psi = \left(R(x_k) + \mathcal{L}(x_k) \right) \psi.$$

Let $\zeta_\infty \in x^{-1}(\infty)$ and $\zeta_l \in x^{-1}(\Lambda_l)$ be poles of $\omega_{0,1}$ of orders m_∞ and m_l, $l = 1, \ldots, N$, respectively. Let $d_\infty := \mathrm{ord}_{\zeta_\infty}(x)$. The operator $\mathcal{L}(x) = \mathcal{L}_\infty(x) + \mathcal{L}_\Lambda(x)$ reads

$$\mathcal{L}_\infty(x) = \sum_{j=1-2d_\infty}^{m_\infty} t_{\zeta_\infty, j} \sum_{k=0}^{\frac{1-j}{d_\infty} - 2} x^k \left(- \frac{j}{d_\infty} - k - 2 \right) \frac{\partial}{\partial t_{\zeta_\infty, j + d_\infty (k+2)}},$$

$$\mathcal{L}_\Lambda(x) = \sum_{l=1}^{N} \left(\frac{1}{x - \lambda_l} \frac{\partial}{\partial \lambda_l} + \sum_{j=1}^{m_l - 1} t_{\zeta_l, j} \sum_{k=1}^{j} (x - \lambda_l)^{-(k+1)} (j + 1 - k) \frac{\partial}{\partial t_{\zeta_l, j+1-k}} \right).$$

Example

In the Airy case, $y^2 = x$, we have only one pole, at $\zeta_i = \infty$, of degree $m_i = 3$, with $d_i = -2$. The sum is empty and $\mathcal{L}(x) = 0$.

Airy and elliptic cases for two-point divisors

Divisor $D = [z_1] - [z_2]$:

- PDEs for Airy curve: $y^2 = x$. We had $\mathcal{L}(x) = 0$.

$$\begin{cases} \hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} + \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} \right) \psi & = x_1 \psi, \\ \hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_2^2} + \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} \right) \psi & = x_2 \psi. \end{cases}$$

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More generally, the admissible curves considered in Bouchard–Eynard, '17 (empty Newton polygon), are those for which $\mathcal{L}(x) = 0$.

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More generally, the admissible curves considered in Bouchard–Eynard, '17 (empty Newton polygon), are those for which $\mathcal{L}(x) = 0$.

- PDEs for elliptic curve: $R(x(z)) = y(z)^2 = x^3 + tx + V$, with

$$-V = \int_{\mathcal{B}_{\infty,1}} \omega_{0,1} = \frac{\partial}{\partial t_{\infty,1}} \omega_{0,0} = -\frac{\partial}{\partial t} \omega_{0,0}$$

$\Rightarrow R(x(z)) = x^3 + tx + \frac{\partial}{\partial t} \omega_{0,0}$.
We have $\mathcal{L}(x) = \frac{\partial}{\partial t}$.

$$\left(\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x_k^2} + \hbar^2 \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} \right) \psi = (x_k^3 + t x_k + V + \frac{\partial}{\partial t}) \psi,$$

for $k = 1, 2$.

Monodromies of the perturbative wave function ⇝ bad monodromies

Problem for genus $\hat{g} > 0$: $\int_o^z \cdots \int_o^z \omega_{g,n}$ are not invariant after z goes around a cycle. Very bad monodromies when z goes around a $\mathcal{B}_i$ (first type cycle).

Lemma

$$\forall p \in x^{-1}(\mathcal{P}) \,:\, \psi_l([z+\mathcal{C}_p]-[\infty^{(\alpha)}],\hbar) = (-1)^{\delta_{p,\infty^{(\alpha)}}} e^{\frac{2\pi i t_{p,0}}{\hbar}} \psi_l([z]-[\infty^{(\alpha)}],\hbar),$$

$$\forall j \in [\![1,\hat{g}]\!] \,:\, \psi_l([z+\mathcal{A}_j]-[\infty^{(\alpha)}],\hbar) = e^{\frac{2\pi i \epsilon_j}{\hbar}} \psi_l([z]-[\infty^{(\alpha)}],\hbar),$$

where $\mathcal{C}_p$ $(=\mathcal{C}_{p,0})$ *is a small circle around* p, *and*

$$\psi(D+\mathcal{B}_j,\hbar) = \exp\left(\sum_{(h,n,m)\in\mathbb{N}^3} \frac{\hbar^{2h-2+n+m}}{n!m!} \overbrace{\int_D \cdots \int_D}^{n} \overbrace{\int_{\mathcal{B}_j} \cdots \int_{\mathcal{B}_j}}^{m} \omega_{h,n+m}\right).$$

Using that the $\mathcal{B}_j$ period of $\omega_{h,n+1}$ is equal to the variation of $\omega_{h,n}$ wrt ϵ_j,

$$\psi(D+\mathcal{B}_j,\hbar) = \exp\left(\sum_{(h,n)\in\mathbb{N}^2} \frac{\hbar^{2h-2+n}}{n!} \overbrace{\int_D \cdots \int_D}^{n} \sum_{m\geq 0} \frac{1}{m!}\left(\hbar \frac{\partial}{\partial \epsilon_j}\right)^m \omega_{h,n}\right) \Rightarrow$$

$$\psi_l([z+\mathcal{B}_j]-[\infty^{(\alpha)}],\hbar) = e^{\hbar\frac{\partial}{\partial\epsilon_j}} \psi_l([z]-[\infty^{(\alpha)}],\hbar) = \psi_l([z]-[\infty^{(\alpha)}],\hbar,\epsilon_j \to \epsilon_j+\hbar).$$

Outline

Summing over the lattice

Remark

Our KZ equations do not depend on $z \in \widetilde{\Sigma}$ *but only on its image* $x(z) \Rightarrow$
For any finite family of c_γ*, the following sum satisfies the same KZ equations*

$$\psi_l([z] - [\infty^{(\alpha)}], \hbar, \{c_\gamma\}) \coloneqq \sum_{\gamma \in \pi_1(\Sigma \setminus x^{-1}(\mathcal{P}))} c_\gamma \; \psi_l([z] + \gamma - [\infty^{(\alpha)}], \hbar).$$

Goal: Build solutions to the same KZ equations but with better monodromies along the $\mathcal{B}_i$-cycles.

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Goal: Build solutions to the same KZ equations but with better monodromies along the $\mathcal{B}_i$-cycles.

Strategy: Sum over $\gamma = \sum_{i=1}^{g} n_i \mathcal{B}_i$, i.e. $\epsilon_i \to \epsilon_i + \hbar$. Formally $\rightsquigarrow$ discrete Fourier transform of the perturbative wave function:

$$\psi_l^{\infty^{(\alpha)}}(z, \hbar; \boldsymbol{\epsilon}, \boldsymbol{\rho}) := \sum_{\mathbf{n} \in \mathbb{Z}^g} e^{\frac{2\pi i}{\hbar} \sum_{j=1}^{g} \rho_j n_j} \; \psi_l([z] - [\infty^{(\alpha)}], \hbar, \epsilon + \hbar \mathbf{n}).$$

Trans-series with special ordering

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Remark (Limitations)

- *Filling fraction* $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_g) \rightsquigarrow$ *not a global coordinate on the space of classical spectral curves with fixed spectral times (only a local coordinate).*
- *Not a finite sum* $\rightsquigarrow$ *not necessarily defined in* $\mathcal{W}$.

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We need a special ordering of the trans-monomials:

$$\sum_{r \geq 0} \sum_{\mathbf{n} \in \mathbb{Z}^g} F_{\mathbf{n},r} \hbar^r e^{\frac{1}{\hbar} \sum_{j=1}^{g} n_j v_j}.$$

The partial sums $\sum_{\mathbf{n} \in \mathbb{Z}^g} F_{\mathbf{n},r} e^{\frac{1}{\hbar} \sum_{j=1}^{g} n_j v_j}$ will give rise to theta functions (through convergent series in the spirit of the trans-asymptotics of Costin–Costin, '10).

Non-perturbative wave functions

Riemann matrix of periods of Σ: $\tau_{i,j} = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{B}_i} \int_{\mathcal{B}_j} \omega_{0,2}$, $\forall (i,j) \in [\![1,\hat{g}]\!]^2$.

Riemann theta function (analytic function of $\mathbf{v} \in \mathbb{C}^{\hat{g}}$) and its derivatives:

$$\Theta^{(i_1,\dots,i_k)}(\mathbf{v},\tau) = \sum_{(n_1,\dots,n_g)\in\mathbb{Z}^{\hat{g}}} e^{2\pi \mathrm{i} \sum_{i=1}^{\hat{g}} n_i v_i} \, e^{\pi \mathrm{i} \sum_{(i,j)\in[\![1,\hat{g}]\!]^2} n_i \tau_{i,j} n_j} \prod_{j=1}^{k} n_{i_j}.$$

For $D = [z] - [\infty^{(\alpha)}]$, we define the non-perturbative wave function

$$\psi_{\mathrm{NP}}(D;\hbar,\boldsymbol{\rho}) := e^{\hbar^{-2}\omega_{0,0}+\omega_{1,0}} e^{\hbar^{-1}\int_D \omega_{0,1}} \frac{1}{E(D)} \sum_{r=0}^{\infty} \hbar^r G^{(r)}(D;\boldsymbol{\rho}),$$

where E is the prime form on Σ,

$$G^{(r)}(D;\boldsymbol{\rho}) := \sum_{k=0}^{3r} \sum_{i_1,\dots,i_k\in[\![1,\hat{g}]\!]^k} \Theta^{(i_1,\dots,i_k)}(\mathbf{v},\tau) G^{(r)}_{(i_1,\dots,i_k)}(D)$$

and where $v_j := \frac{\rho_j+\varphi_j}{\hbar} + \mu_j^{(\alpha)}(z)$, $\mathbf{v} = (v_1,\dots,v_{\hat{g}})$, with

$$\varphi_j := \frac{1}{2\pi i}\oint_{\mathcal{B}_j} \omega_{0,1} \qquad \text{and} \qquad \mu_j^{(\alpha)}(z) := \frac{1}{2\pi i}\int_D \oint_{\mathcal{B}_j} \omega_{0,2}.$$

Same KZ equations and good monodromies

- Non-perturbative wave functions satisfy the same KZ equations as their perturbative partners.

$$\hbar \frac{d\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho})}{dx(z)} + \psi_{l+1,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho}) =$$

$$\sum_{P\in\mathcal{P}} \sum_{k\in S_P^{(l+1)}} \xi_P^{-k}(x(z))\mathrm{ev.}\left[\widetilde{\mathcal{L}}_{P,k,l}\, \psi_{0,\mathrm{NP}}^{\infty^{(\alpha)},\mathrm{symbol}}(z,\hbar,\boldsymbol{\rho})\right].$$

- Non-perturbative wave functions ⇝ simple monodromy properties.

For $j \in [\![1,\hat{g}]\!]$, we have

$$\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z+\mathcal{A}_j,\hbar,\boldsymbol{\rho}) = e^{\frac{2\pi i \epsilon_j}{\hbar}}\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho}),$$

$$\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z+\mathcal{B}_j,\hbar,\boldsymbol{\rho}) = e^{-\frac{2\pi i \rho_j}{\hbar}}\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho})$$

and $\forall\, p \in x^{-1}(\mathcal{P})$

$$\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z+\mathcal{C}_p,\hbar,\boldsymbol{\rho}) = (-1)^{\delta_{p,\infty^{(\alpha)}}}\; e^{\frac{2\pi i t_{p,0}}{\hbar}}\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho}).$$

Lax systems

For $l \geq 0$, we define

$$\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho}) := \mathrm{ev.} \sum_{\beta \underset{l}{\subseteq} \left(x^{-1}(x(z)) \setminus \{z\}\right)} \frac{1}{l!} \left(\prod_{j=1}^{l} \mathcal{I}_{\mathcal{C}_{\beta_j},1} \right) \psi_{\mathrm{NP}}^{\mathrm{symbol}}(D;\hbar,\boldsymbol{\rho}).$$

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We use them to define a $d \times d$ matrix

$$\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar,\boldsymbol{\rho}) := \left[\psi_{l-1,\mathrm{NP}}^{\infty(\alpha)}(z^{(\beta)}(\lambda),\hbar,\boldsymbol{\rho})\right]_{1 \leq l,\beta \leq d},$$

where $z^{(\beta)}(\lambda)$ denotes the β^{th} preimage by x of λ.

Lax systems

$$\widetilde{\mathcal{L}}_l(x(z)) = \sum_{P\in\mathcal{P}} \sum_{k\in S_P^{(l+1)}} \xi_P(x(z))^{-k} \widetilde{\mathcal{L}}_{P,k,l}, \quad \mathcal{L}_{P,k,l} := \widetilde{\mathcal{L}}_{P,k,l} - P_{P,k}^{(l+1)}.$$

Theorem (ODE and Lax system)

Let $\hat{L}(\lambda,\hbar) := -\widehat{P}(\lambda) + \hbar \sum_{P\in\mathcal{P}} \sum_{k\in\mathbb{N}} \xi_P^{-k}(\lambda)\widehat{\Delta}_{P,k}(\lambda,\hbar)$*. Then,*

$$\hbar \frac{d\widehat{\Psi}_{\text{NP}}(\lambda,\hbar)}{d\lambda} = \hat{L}(\lambda,\hbar)\widehat{\Psi}_{\text{NP}}(\lambda,\hbar),$$

where

$$\widehat{P}(\lambda) := \begin{bmatrix} -P_1(\lambda) & 1 & 0 & \dots & 0 \\ -P_2(\lambda) & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -P_{d-1}(\lambda) & 0 & 0 & \dots & 1 \\ -P_d(\lambda) & 0 & 0 & \dots & 0 \end{bmatrix}$$

For any $P\in\mathcal{P}$, $k\in\mathbb{N}$, $l\in[\![0,d-1]\!]$*, one has the auxiliary systems*

$$\hbar^{-1}\text{ev}.\mathcal{L}_{P,k,l}\widehat{\Psi}_{\text{NP}}^{\text{symbol}}(\lambda,\hbar) = \widehat{A}_{P,k,l}(\lambda,\hbar)\widehat{\Psi}_{\text{NP}}(\lambda,\hbar),$$

where $\hat{L}(\lambda,\hbar)$ *and* $\widehat{A}_{P,k,l}(\lambda,\hbar)$ *are* $\hbar$*-trans-series functions that are rational functions of* λ*, with no poles at critical values* $\lambda\in x(\mathcal{R})$*.*

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- (1) $\rightsquigarrow$ linear differential system of size $d\times d$ whose formal fundamental solution can be computed by TR, with poles at the poles of the leading WKB term...
- $\hat{L}(\lambda,\hbar)$ has poles only at $\lambda\in\mathcal{P}$ and at zeros of the Wronskian $\det\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)$, *apparent singularities* of the system (can be computed thanks to the KZ eqns).

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$$\hbar \frac{d\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)}{d\lambda} = \hat{L}(\lambda,\hbar)\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar), \tag{2}$$

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- Most technical proof ⇝ by induction on the order of the transseries.
- Proof uses admissibility conditions (distinct critical values, smooth simple ramification points) ⇝ should adapt without them but involving more technical computations.

4 different interesting gauges

None of the gauge transformations modify the first line of the wave functions matrix (used to define the quantum curve).

- Gauge $\widehat{\Psi}$: Natural gauge coming from KZ equations and provides compatible auxiliary systems $(\mathcal{L}_{P,k,l})_{P\in\mathcal{P},l\in[\![0,d-1]\!],k\in S_P^{(l+1)}}$.
- Gauge $\widetilde{\Psi}$ ($\hbar^0$ gauge transformation from $\widehat{\Psi}$): Leading order in $\hbar$ of $\widetilde{L}$ is companion-like $\rightsquigarrow$ the classical spectral curve is directly recovered from its last line.
- Gauge Ψ: Corresponding Lax matrix L is companion-like at all orders in $\hbar$ $\rightsquigarrow$ both the quantum and classical curves are directly read from the last line of L and its $\hbar \to 0$ limit. Natural framework for Darboux coordinates and isomonodromic deformations.
- Gauge $\check{\Psi}$: Lax matrix $\check{L}$ has no apparent singularities. This allows to interpret $\check{L}(\lambda,\hbar)d\lambda$ as an $\hbar$-familly of Higgs fields giving rise to a flow in the corresponding Hitchin system.

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Spectral curves from integrable systems

Definition

Let $\hbar \frac{\partial}{\partial x}\Psi(x,\hbar) = \mathcal{L}(x,\hbar)\Psi(x,\hbar)$ be a (2×2) differential system (with $\det \Psi = 1$). We define the classical spectral curve associated to it by

$$P(x,y) \coloneqq \lim_{\hbar\to 0} \det(y\mathrm{Id} - \mathcal{L}(x,\hbar)) = 0,$$

which gives a polynomial equation. For a non-zero genus curve, this must be completed with a choice of symplectic basis of cycles and a bidifferential B.

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Different approach:

- $\hbar$-differential system.
- Define the classical spectral curve associated to it.
- Show that interesting quantities from the point of view of the differential system may be reconstructed from topological recursion applied to this classical spectral curve.
- Proof by showing that the differential system satisfies the topological type property (Bergère–Borot–Eynard '15).

Isomonodromic deformations

We consider isomonodromic deformations of the linear differential equation $\partial_x - \mathcal{L}(x)$, which depend on a number of continuous parameters t_k (times):

$$\begin{cases} \hbar \dfrac{\partial}{\partial x} \Psi(x, t_k; \hbar) = \mathcal{L}(x, t_k; \hbar) \Psi(x, t_k; \hbar), \\ \hbar \dfrac{\partial}{\partial t_k} \Psi(x, t_k; \hbar) = \mathcal{R}_k(x, t_k; \hbar) \Psi(x, t_k; \hbar) \end{cases}$$

We call such a (compatible integrable) system an isomonodromic system.

$$\frac{\partial^2}{\partial t_k \partial x} \Psi = \frac{\partial^2}{\partial x \partial t_k} \Psi \Leftrightarrow \hbar \frac{\partial \mathcal{L}}{\partial t_k} - \hbar \frac{\partial \mathcal{R}_k}{\partial x} + [\mathcal{L}, \mathcal{R}_k] = 0 \text{ (zero-curvature equation).}$$

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Consider the deformed spectral curve

$$P(x, y; \hbar) = \det(y \, \mathrm{Id} - \mathcal{L}(x, t_k; \hbar)) = P_0(x, y) + \sum_{m \geq 1} \hbar^m P_m(x, y).$$

Classical spectral curve $\rightsquigarrow P_0(x, y)$ (family of curves parametrized by t_k's).

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Remark

Painlevé equations $\rightsquigarrow$ *Isomonodromic deformations. Painlevé property* $\rightsquigarrow$ *Solutions have no movable singularities other than poles. Classification of all second order differential equations with the Painlevé property* $\rightsquigarrow$ 50 *solutions and only* 6 *which could not be integrated from already known functions.*

Painlevé I

In the family of elliptic curves $y^2 = x^3 + tx + V$, taking $t = -3u_0^2$ and $V = 2u_0^3$, amounts to pinching the $\mathcal{B}$-cycle (first kind). So in this case, we have genus $\hat{g} = 0$ and the **spectral curve** admits a rational parametrization:

$$\begin{cases} \Sigma = \mathbb{C}P^1, & x(z) = z^2 - 2u_0,\ y(z) = z^3 - 3u_0 z, \\ y\mathrm{d}x = (z^3 - 3u_0 z)2z\mathrm{d}z, & B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z^2)^2}. \end{cases}$$

TR: Witten–Kontsevich intersection numbers $\rightsquigarrow \omega_{g,n}(z_1, \ldots, z_n) =$

$$\sum_{d_1,\ldots,d_n} \frac{6^{2-2g-n} u_0^{5-5g-2n}}{(3g-3+n-\sum_i d_i)!} \left\langle \tau_2^{3g-3+n-\sum_i d_i} \tau_{d_1} \cdots \tau_{d_n} \right\rangle_g \prod_{i=1}^n \frac{u_0^{d_i} (2d_i+1)!!\mathrm{d}z_i}{z_i^{2d_i+1}}.$$

$$n = 0 \rightsquigarrow \mathcal{F}_g = \omega_{g,0} = u_0^{5-5g} \frac{6^{2-2g}}{(3g-3)!} \left\langle \tau_2^{3g-3} \right\rangle_g = (-t/3)^{\frac{5-5g}{2}} \frac{6^{2-2g}}{(3g-3)!} \left\langle \tau_2^{3g-3} \right\rangle_g.$$

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Then $U(t) = u_0 + \frac{\hbar^2}{48t^2} + \sum_{g \geq 2} \hbar^{2g} \frac{\partial^2 \mathcal{F}_g}{\partial t^2}$ satisfies the Painlevé I equation $\frac{\hbar^2}{2} \frac{\partial^2}{\partial t^2} U + 3U^2 = -t$, which is the compatibility equation of the Lax pair

$$\mathcal{L}(x, t; \hbar) := \begin{pmatrix} \frac{\hbar}{2} \dot{U} & x - U \\ (x - U)(x + 2U) + \frac{\hbar^2}{2} \ddot{U} & -\frac{\hbar}{2} \dot{U} \end{pmatrix} \text{ and } \mathcal{R}(x, t; \hbar) := \begin{pmatrix} 0 & 1 \\ x + 2U & 0 \end{pmatrix}.$$

From the PDE found we can get that $\psi_\pm(x) = e^{\sum_{g,n} \frac{(\pm 1)^n \hbar^{2g-2+n}}{n!} \int \cdots \int \omega_{g,n}}$:

$$\left(\hbar \frac{\partial}{\partial x} - \mathcal{L}(x, t; \hbar)\right) \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0\ , \quad \left(\hbar \frac{\partial}{\partial t} - \mathcal{R}(x, t; \hbar)\right) \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0.$$

Practical computations to quantise a classical spectral curve

1. Write down the KZ equations satisfied by the non-perturbative wave function.
2. Expand these KZ equations around each pole $\lambda \to P \in \mathcal{P} \rightsquigarrow$ expression of the coefficients of the asymptotic expansion of $\psi_{0,\mathrm{NP}}^{(\infty^{(\alpha)})}$ in terms of the action of the operators $\mathcal{I}_C$.
3. Use the latter expressions to compute the Wronskian of the system thanks to its expansion around its poles. This allows to compute the position of the apparent singularities $(q_i(\hbar))_{i=1}^{d}$.
4. Write down the linear system and the associated quantum curve, and use the compatibility of the system to recover its properties.

Example

- Reconstruction via TR of a 2-parameter family of formal transseries solutions to Painlevé 2 and quantisation. Classical spectral curve: $y^2 - P_1(\lambda)y + P_2(\lambda) = 0$, where $P_1(\lambda) = P^{(1)}_{\infty,2}\lambda^2 + P^{(1)}_{\infty,1}\lambda + P^{(1)}_{\infty,0}$ and $P_2(\lambda) = P^{(2)}_{\infty,4}\lambda^4 + P^{(2)}_{\infty,3}\lambda^3 + P^{(2)}_{\infty,2}\lambda^2 + P^{(2)}_{\infty,1}\lambda + P^{(2)}_{\infty,0}$.
- Quantisation of a degree 3, genus 1 classical spectral curve with a single singularity at infinity: $y^3 - (P^{(1)}_{\infty,1}\lambda + P^{(1)}_{\infty,0})y^2 + (P^{(2)}_{\infty,2}\lambda^2 + P^{(2)}_{\infty,1}\lambda + P^{(2)}_{\infty,0})y - P^{(3)}_{\infty,3}\lambda^3 - P^{(3)}_{\infty,2}\lambda^2 - P^{(3)}_{\infty,1}\lambda - P^{(3)}_{\infty,0} = 0$.

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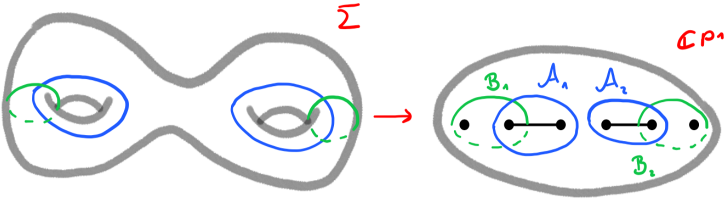
Future work

- Ongoing: More conceptual proof of the QC conjecture?
- Upgrade to trans-algebraic spectral curves (essential singularities) with the work of Bouchard–Kramer–Weller?
- Interesting enumerative geometry in higher genus TR problems?
- Get rid of admissibility conditions?
- Explore the connection with summability, exact WKB, Stokes phenomenon and resurgence. **Conjecture:** There exist values of ε and $\hbar$ making the transseries involved summable.
- **Conjecture:** The non-perturbative partition function is a tau function.
- How does the connection built as $d - \mathcal{L}(x, \hbar)\mathrm{d}x/\hbar$ depend on the choice of cycles $(\mathcal{A}_i, \mathcal{B}_i)$?
- Relation to the topological type property approach (can that be proved for higher genus spectral curves?).
- Extend the result to a ramified covering of surfaces other than $\mathbb{C}P^1$.
- Generalization to difference equations? (Subtleties including K_2 condition of Gukov–Sułkowski '12?). Non-algebraic curves, such as $P(e^x, e^y)$ (important for volume conjecture).
- General relation between Virasoro constraints (or even Kontsevich–Soibelman '17, ABCD of Andersen–Borot–Chekhov–Orantin '17) and quantum curves.

Thank you very much for your attention!



Articles:

- *From topological recursion to wave functions and PDEs quantizing hyperelliptic curves*, with B. Eynard, arXiv:1911.07795 (2019)
- *Quantizing generic algebraic spectral curves via topological recursion*, with B. Eynard, O. Marchal, N. Orantin, arXiv:2106.04339 (2021)