Hurwitz theory, with (a) spin

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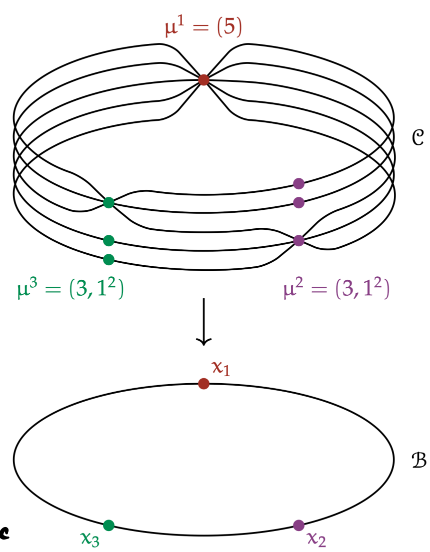
Overview:
1) (Spin) Hurwitz theory
2) Fermion formalism & topological recursion
3) Intersection theory on $\overline{\mathcal{M}}_{g,n}$
4) Applications to Gromov-Witten theory

1) (SPIN) HURWITZ THEORY

"(Spin) Hurwitz theory is the (signed) enumeration of branched covers of Riemann surfaces."



Defn. Fix a base $\mathcal{B} \ni x_1, \dots, x_k$. Fix $\mu^1, \dots, \mu^k \vdash d$.

$$\boxed{H_d(\mathcal{B}; \mu^1, \dots, \mu^k) := \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|}}$$

Hurwitz numbers

iso classes of $f : \mathcal{C} \xrightarrow{d:1} \mathcal{B}$, $\mathcal{C}$ cmpct, cnntd, Riemann srfc

- ramified over x_i, ramification profile μ^i
- unramified everywhere else

Questions:
- Properties?
- How to compute?
- What are they used for?

Answers:
- Relation to rep. theory of S_d
- Integrable hierarchies
- Topological recursion
- Intersection theory on $\overline{\mathcal{M}}_{g,n}$

$\Big\}\ \mathcal{B} = \mathbb{P}^1$
- Gromov-Witten theory of curves
- Volumes of moduli spaces of holomorphic differentials $\Big\}\ \mathcal{B} = \mathbb{T}$

$\vdots$

Motivated by the geometry of the moduli space of holom. diffnts, Eskin-Okounkov-Pandharipande introduced spin Hurwitz numbers.

Defn. A spin structure on B is a line bundle $\vartheta \to B$ s.t. $\vartheta^{\otimes 2} \cong \omega_B$. Define the parity $p(\vartheta) \equiv h^0(B,\vartheta) \pmod 2$

positive/negative

Expl:

$\mu = (2m_1+1, \dots, 2m_\ell+1)$

- $\mathcal{O}(-1)$ is the only spin strctr on $\mathbb{P}^1$
- If $\vartheta \to B$ is a spin structure and $C \xrightarrow{f} B$ a ramified cover with ODD ramifications,
$$\vartheta_{f,B} = f^*\vartheta \otimes \mathcal{O}\left(\tfrac{1}{2}\mathrm{Ram}\right)$$
is a spin structure on C.

Defn. Fix a base $B \ni x_1,\dots,x_k$ and $\vartheta \to B$ a spin strctr. Fix $\mu^1,\dots,\mu^k \vdash d$ ODD partitions.

$$H_d(B,\vartheta;\mu^1,\dots,\mu^k) := \sum_{[f]} \frac{(-1)^{p(\vartheta_{f,B})}}{|\mathrm{Aut}(f)|}$$

spin Hurwitz numbers

We have a monodromy map to the spin symmetric group: $0 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{S}_d \to S_d \to 0$

$$\left\{ \left[f: C \xrightarrow{d:1} B\right] \ \text{positive or negative} \right\} \xleftrightarrow{\ \sim\ } \left\{ \text{repres of } S_d \text{ w/ or w/o a lift to } \tilde{S}_d \right\}$$

$$f \longmapsto \rho: \pi_1(B - \{\text{brnch pnts}\}, 0) \to S_d$$

$f^{-1}(0) = \{p_1,\dots,p_d\}$

$$\gamma \longmapsto [p_i \mapsto \tilde{\gamma}_i(1)]$$

unique lift of γ starting at p_i

$\lambda_1 > \lambda_2 > \dots$

FACTS: 1) $\{\text{irreps of } \tilde{S}_d\} \xleftrightarrow{\sim} \{\lambda \vdash d \text{ strict partition}\}$

2) basis of the spin class algebra $\tilde{Z}_d$ $\xleftrightarrow{\sim}$ $\{\mu \vdash d \text{ odd partition}\}$ $\rightsquigarrow$ $\tilde{f}_\mu(\lambda)$ = spin central character $\sim \tilde{\chi}_\lambda(\mu)$

Thm (Character formula, Eskin-Okounkov-Pandharipande '08, Gunningham '16). Disconnected spin Hurwitz numbers of $(B,\vartheta) = (\mathbb{P}^1, \mathcal{O}(-1))$ are given by

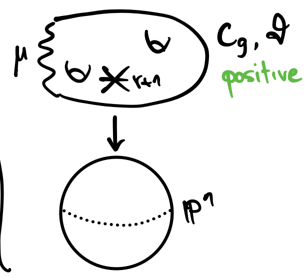
$$\boxed{H^\bullet_d(\mathbb{P}^1,\mathcal{O}(-1);\mu^1,\dots,\mu^k) = 2^{\frac{\sum_i (\ell(\mu^i)-d)}{2} - d} \sum_{\substack{\lambda \vdash d \\ \text{strict part.}}} \left(\frac{\dim(\lambda)}{2^{p(\lambda)/2}\, d!}\right)^2 \prod_{i=1}^{k} \tilde{f}_{\mu^i}(\lambda)}$$

RESTRICTION:
- $(B,\vartheta) = (\mathbb{P}^1, \mathcal{O}(-1))$
- $\mu^1 = \mu$ generic ODD partition of d
- $\mu^2 = \mu^3 = \cdots = (r+1, 1, \ldots, 1)$ + lower order terms $=: \tilde{C}_r$ $\quad$ r EVEN

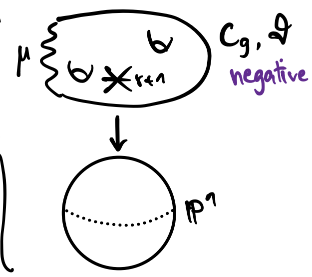
$$h^r_{g;\mu} := \frac{|\mathrm{Aut}(\mu)|}{b!} H_d(\mathbb{P}^1, \mathcal{O}(-1); \mu, \underbrace{\tilde{C}_r, \ldots, \tilde{C}_r}_{b \text{ times}}) \qquad b = \frac{2g-2+\ell(\mu)+d}{r}$$

genus of the cover

$$= \# \left\{ \right. $$



$$\left. \right\} - \# \left\{ \right.$$



$$\left. \right\}$$

Character formula:
$$h^{0,r}_{g;\mu} = \frac{|\mathrm{Aut}(\mu)|}{b!} 2^{1-g-2d} \sum_{\substack{\lambda \vdash d \\ \text{strict part.}}} \left(\frac{\dim(\lambda)}{2^{p(\lambda)/2} d!} \right) \tilde{f}_\mu(\lambda) \left(\frac{p_{r+1}(\lambda)}{r+1} \right)^b$$

power sum

UPSHOT: $\boxed{\text{signed count of Hurwitz covers}}$ $\overset{\text{monodromy}}{\rightsquigarrow}$ $\boxed{\text{repr. theory of } \tilde{S}_d}$

2) FERMION FORMALISM & TOPOLOGICAL RECURSION

FACTS: $\exists$
- $\left(\mathcal{F}^B_0 = \mathrm{span}\{ |\lambda\rangle \mid \lambda \text{ strict part.} \}, \langle \cdot | \cdot \rangle, |0\rangle \right)$ = Fock space of type B

  vect. space $\quad$ pairing $\quad$ distinguished element (vacuum)

- $\hat{D}_\infty \curvearrowright \mathcal{F}^B_0$ with explicit elements $J^B_{-\mu}$, $\mu \vdash d$ odd, F^B_{r+1}, r even, s.t.

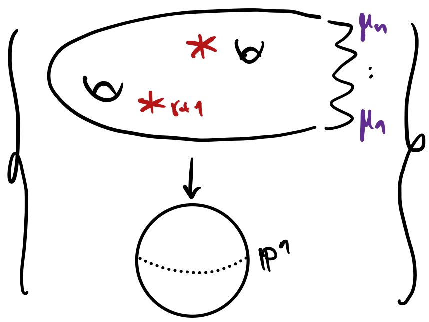
  Lie algebra

$$J^B_{-\mu}|0\rangle = \sum_{\substack{\lambda \vdash d \\ \text{strict part.}}} \frac{\tilde{\chi}_\lambda(\mu)}{2^{p(\lambda)/2 + \ell(\mu)}} |\lambda\rangle \qquad F^B_{r+1}|\lambda\rangle = p_{r+1}(\lambda)|\lambda\rangle$$

Prop (Spin H#s as vacuum expectation values).

$$\boxed{h^{\bullet,r}_{g;\mu} = \frac{2^{1-g}}{b!} \left\langle 0 \right| e^{J_1^B} \left(\frac{\mathcal{F}^B_{r+1}}{r+1} \right)^b \prod_{i=1}^{n} \frac{J^B_{-\mu_i}}{\mu_i} \left| 0 \right\rangle}$$

signed #



UPSHOT: signed count of Hurwitz covers $\overset{\text{monodromy}}{\rightsquigarrow}$ repr. theory of $\tilde{S}_d$ $\rightsquigarrow$ vacuum exp. val. on $\mathcal{F}_0^B$

ADVANTAGE:
- We can use commutation relations of operators in $\hat{\mathfrak{a}}_\infty$ to get closed formulae for low $(g, \ell(\mu))$
- The generating series of spin H#s is a BKP τ-fnctn

Prop. For $(g, \ell(\mu)) = (0,1)$ and $(0,2)$, we get explicit formulae:

$$h^r_{0;\mu} = \frac{\mu^{\frac{\mu-1}{r}-2}}{\left(\frac{\mu-1}{r}\right)!}, \qquad h^r_{0;\mu_1,\mu_2} = \frac{r}{\mu_1+\mu_2} \frac{\mu_1^{\lfloor \frac{\mu_1}{r} \rfloor} \mu_2^{\lfloor \frac{\mu_2}{r} \rfloor}}{\lfloor \frac{\mu_1}{r} \rfloor ! \lfloor \frac{\mu_2}{r} \rfloor !}$$

$\mu \in \mathbb{Z}_+^{odd}$, $r \mid \mu$ $\qquad$ $\mu_1, \mu_2 \in \mathbb{Z}_+^{odd}$, $r \mid \mu_1+\mu_2$

RECALL: given some initial data $\mathcal{S}$ (a spectral curve $\mathcal{S} = (\Sigma, x, y : \Sigma \to \mathbb{C}, B)$), topological recursion is a recursive procedure that constructs a sequence of differentials $(\omega_{g,n})_{g,n}$ on the curve:

$$\mathcal{S} \xrightarrow{\ TR\ } (\omega_{g,n})_{g,n}$$

input: $(0,1)$ + $(0,2)$

output: all (g,n) with $2g-2+n>0$

Thm (Topological recursion for spin HH's, conjectured in [GKL], proved by Alexandrov–Shadrin).
Spin HH's are computed by topological recursion on the spectral curve on $\mathbb{P}^1$

$$x(z) = \log(z) - z^r, \qquad y(z) = z, \qquad B(z_1,z_2) = \frac{1}{2}\left(\frac{1}{(z_1-z_2)^2} + \frac{1}{(z_1+z_2)^2}\right) dz_1\, dz_2$$

by expanding the correlators near $e^{x(z_i)} = 0$:

$$\boxed{\omega_{g,n}(z_1,\dots,z_n) = d_1 \cdots d_n \sum_{\substack{\mu \\ \text{odd part.}}} h^r_{g;\mu}\, e^{\mu_1 x(z_1)} \cdots e^{\mu_n x(z_n)}}$$

UPSHOT: $\boxed{\text{signed count of Hurwitz covers}} \rightsquigarrow \boxed{\text{vacuum exp. val. on } \mathcal{F}^B_0} \rightsquigarrow \boxed{\text{topological recursion}}$

3) INTERSECTION THEORY on $\overline{\mathcal{M}}_{g,n}$

RECALL: for g,n s.t. $2g-2+n>0$, the moduli space of stable curves

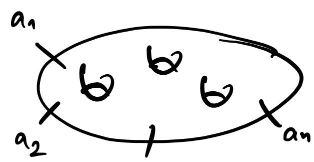
$$\overline{\mathcal{M}}_{g,n} := \left\{ (C, p_1,\dots,p_n) \;\middle|\; \begin{array}{l} C \text{ is a genus } g \text{ stable curve} \\ \text{with } n \text{ smooth marked pnts} \end{array} \right\} \Big/ \sim$$

is a smooth cmct cmplx orbifold of $\dim_{\mathbb{C}} = 3g-3+n$. Thus, we can consider intersection numbers:

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha \in \mathbb{Q}, \qquad \alpha \in H^{2(3g-3+n)}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

EXPECTATION: every curve-counting problem can be expressed as an intersection number on the moduli space of curve:

$$N_{g;a_1,\dots,a_n} = \#\left\{ \text{"structures" on} \right.$$



$$\left. \right\} \overset{?}{=} \int_{\overline{\mathcal{M}}_{g,n}} \alpha_{a_1,\dots,a_n}$$

Question. What is the RHS for spin H#'s ?

Thm (Spin ELSV formula). For $r=2$, spin H#s are given by double Hodge integrals:

$$\boxed{h^{r=2}_{g;\mu_1,\dots,\mu_n} = 2^{4g-4+2n}\left(\prod_{i=1}^{n} \frac{\mu_i^{\frac{\mu_i-1}{2}}}{\left(\frac{\mu_i-1}{2}\right)!}\right)\int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda(1)\,\Lambda(-1/2)}{\prod_{i=1}^{n}\left(1-\frac{\mu_i}{2}\psi_i\right)}}$$

The formula is equivalent to topological recursion for spin H#s.

More generally, for arbitrary r (even), $\Lambda(1)\Lambda(-1/2)$ is substituted by a product of Witten's 2-spin class and Chiodo's class.

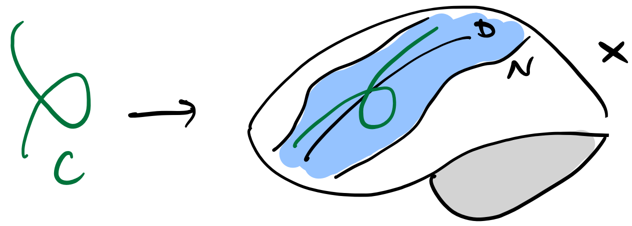
4) APPLICATIONS TO GROMOV-WITTEN THEORY (work in progress w/ Kramer, Lewański, Sauvaget)

Let X be a smooth variety, $\beta \in H_2(X,\mathbb{Z})$. GW invariants count curves in X of given genus and degree:

$$GW_g(X;\beta) \text{ "="} \ \#\left\{ C \xrightarrow{f} X \;\middle|\; \begin{array}{l} C \text{ is a curve of} \\ \text{genus } g \\ f_*[C]=\beta \end{array} \right\}$$

If X is a Kähler surface of general type with a smooth canonical divisor D.

then GW invariants of degree dD, $d \in \mathbb{Z}_+$, localise around D:



Here N is the normal bundle.

FACT: by the adjunction formula, N is a spin structure on D: $N^{\otimes 2} \cong \omega_D$.

Thm (Lee-Parker, Maulik-Pandharipande, Kiem-Li). $GW_g(X; dD)$ are determined by "local" invariants $GW_g^{spin}(D, N; d)$ that depend on the spin curve (D, N).

UPSHOT: $\boxed{\text{GW theory of Kähler surfaces}} \xrightarrow{\text{localisation}} \boxed{\text{GW theory of spin curves}}$

Question. How to solve GW theory of spin curves?

Conjecture (Spin GW/H correspondence):

$\boxed{\text{GW invariants of spin curves are equivalent to spin Hurwitz numbers}}$

The proof (in progress) makes use of the spin ELSV formula as a starting pnt:

$\boxed{\text{spin GW of } (B, \theta)} \xrightarrow{\text{degen.}} \boxed{\text{spin GW of } (\mathbb{P}^1, \mathcal{O}(-1))} \xrightarrow{\text{local.}} \boxed{\text{double Hodge integrals}} \xrightarrow{\text{spin ELSV}} \boxed{\text{spin Hurwitz numbers}}$