Yau Mathematical Sciences Center

Welcome to super hyperbolic surface theory!

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Outline

1. Hyperbolic surfaces
2. McShane identity
3. Super hyperbolic geometry

The hyperbolic plane

There are many models for the hyperbolic plane, e.g.:

- **Upper half-plane model**:
  $\{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, with metric $\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$.
- **Poincaré disk model**:
  $\{z \in \mathbb{C} \mid |z| < 1\}$, with metric $\frac{4|\mathrm{d}z|^2}{(1-|z|)^2}$.
- The **paraboloid model**:
  $\{(x_1,x_2,y) \in \mathbb{R}^{2,1} \mid x_1x_2 = y^2 \text{ and } x_1+x_2 > 0\}$, with metric $\mathrm{d}x_1\mathrm{d}x_2 - \mathrm{d}y^2$.

Isometry group

- The isometry group of the hyperbolic plane is given by the Möbius group, and can be realised as $\mathrm{PSL}_2(\mathbb{R})$ (for the upper half-plane model).
- It's realised by $\mathrm{SO}^+(1,2)$ for the paraboloid model.
- It's more convenient to determine hyperbolic trigonometric identities via $\mathrm{SO}^+(1,2)$ since there are equivalences up to sign.
- Good reference: Buser's *Geometry and spectra of compact Riemann surfaces*.

Hyperbolic surfaces

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- All ideal triangles are isometric, so it's just a matter of how we glue them together.
- One way to specify the gluing is using *shearing lengths* $\Rightarrow$ one parameter for each edge.
- Another way, when the surface is cusped, is to add decorating horocycles at each cusp and measure the length of each edge after truncation by the decorating horocycles.

Hyperbolic surfaces can be built from ideal triangles!

Fact: changes in triangulation correspond to highly structured changes in parameters used for gluing.

Curves on surfaces

Negatively curved surfaces$\Rightarrow$ closed curves have unique geodesic representatives.

$\Rightarrow$hyperbolic surfaces (of finite type) admit countably many closed geodesics.

Hyperbolic trigonometry $\Rightarrow$ the geometry of a hyperbolic surface is determined by a finite collection of curves.

Geodesics and geodesics laminations

What do simple geodesics on hyperbolic surfaces look like?

- Simple closed geodesics: countably many.
- Simple “spiralling” geodesics: countably many.
- Geodesic laminations: uncountably many.

Question

What does the set of simple geodesics on a surface (as a subset of the surface) look like?

Birman–Series geodesic sparsity

The set of points on a hyperbolic surface which lie on simple geodesics is a (closed) set of measure 0 (in fact, Hausdorff dimension 1).

Idea of proof:

- Cover all simple geodesics with "fattened" geodesic arcs of length $\leq L$, s.t.
- the number of such fattened arcs is polynomial in L, with width $o(e^{-cL})$.
- Area covered by thin neighborhoods of order $\text{poly}(L) \cdot o(e^{-cL})$, which tends to 0 as $L \to \infty$.

Observation

The proof of Birman–Series hints that the growth rate of the number of geodesics of length $\leq L$ is bounded by a polynomial in L.

McShane and Rivin examined the 1-cusped torus case, and showed (using Zagier's work) that

Polynomial growth rate

Given a 1-cusped torus $S_{1,1}$, the number

$$N_{S_{1,1}}(L) := \{ \text{ s.c.g. } \gamma \text{ on } S_{1,1} \ \mid \ \ell_\gamma(S_{1,1}) \leq L \}$$

asymptotically behaves as $N_{S_{1,1}}(L) = \eta(S_{1,1}) L^2 + o(L^2)$.

- Rivin: for a hyperbolic surface $S_{g,n}$ of genus g with n cusps or boundaries

$$N_{S_{g,n}}(L) := \left\{ \text{ s.c.g. } \gamma \text{ on } S_{g,n} \;\middle|\; \ell_\gamma(S_{g,n}) \leq L \right\}$$

is $O(L^{6g-6+2n})$ (weaker result).

- Mirzakhani:

$$N_{S_{g,n}}(L) = \eta(S_{g,n}) L^{6g-6+2n} + o(L^{6g-6+2n}),$$

where η depends continuously on $S_{g,n}$.

McShane identity for $S_{1,1}$

Let $\mathcal{C}(S_{1,1})$ denote the set of simple closed geodesics on $S_{1,1}$, then

$$1 = \sum_{\gamma \in \mathcal{C}(S_{1,1})} \frac{2}{e^{\ell_\gamma} + 1}.$$

- **Idea of proof:** the summand indexed by γ is the probability that a geodesic shot out from the cusp self-intersects without hitting γ.
- **N.B.:** the Birman–Series ensures that almost all geodesics self-intersect.

Generalizations:

The original identity is due to Greg McShane, and generalizations incl.:

- McShane: cusped surfaces $S_{g,n}$ of general type.
- Mirzakhani: bordered hyperbolic surfaces
  Tan-Wong-Zhang: the above (indep.) and surfaces with small cone-angles.
- Norbury: non-orientable surfaces.
- Bowditch, Akiyoshi-Miyachi-Sakuma, H., H.-Norbury: Quasifuchsian representations (hyp. 3-manifolds homeo. to $S_{g,n} \times I$); pseudo-Anosov mapping tori.
- Labourie-McShane, Huang-Sun: cusped finite area convex real projective surfaces, higher rank positive representations $\rho : \pi_1(S_{g,n}) \to \mathrm{PGL}_d(\mathbb{R})$.

Applications

McShane identities are beautiful in and of themselves, and have the following applications:

- Discreteness of the simple length spectrum.
- (slightly weaker) collar lemma!
- Can be used to demonstrate length non-domination.
- Mirzakhani used them to compute the Weil-Petersson volumes of moduli spaces
- $\Rightarrow$ A proof of Witten's conjecture — relationships between intersection numbers for families of moduli spaces.
- She also uses this to prove the polynomial growth rates of simple closed geodesic length spectra.

Grassmann algebra

(Finite dimensional) Grassmann algebra $\mathbb{R}_{S[N]}$:

- real algebra with generators $1, \beta_{[1]}, \ldots, \beta_{[N]}$
- 1 commutes with everything
- write $\beta_{[1,3]}$ for $\beta_{[1]}\beta_{[3]}$
- $\beta_{[i]}$ multiply like wedge product for 1-forms:
$$(\tfrac{1}{5} + 3\beta_{[1]})(5 + \beta_{[1,3]}) = \left(1 + \tfrac{1}{5}\beta_{[1,3]}\right) + \left(15\beta_{[1]}\right).$$
- everything splits into even + odd terms
- can have division as long as the constant term is non-zero:
$$(\tfrac{1}{5} + 3\beta_{[1]})(5 - 15\beta_{[1]}) = 1$$

Super manifolds

Classical manifold theory is based on $\mathbb{R}$ (or $\mathbb{C}$):

Algebraic geometry goes further: use all types of different fields $\mathbb{F}$ instead of $\mathbb{R}$ (or $\mathbb{C}$).

Idea: super geometry uses Grassmann algebra $\mathbb{R}_{S[N]}$ instead.

- doesn't always have division;
- isn't commutative — has *even* and *odd* parts.
- locally modelled on m copies of the even part of $\mathbb{R}_{S[N]}$, n copies of the odd part of $\mathbb{R}_{S[N]}$.

Regarding super manifolds as real vector bundles over the "body" manifold (i.e.: constant part) ignores rich algebraic structure.

Classical Riemannian geometric objects still make sense:

- tangent vectors + tangent bundles + vector fields
- cotangent vectors + cotangent bundles + differential forms
- super Riemannian metrics — outputs bosons (i.e.: even elements).
- flat connections + holonomy representations
- super geodesics — variational principle approach: solutions to Euler-Lagrange equations.

Super-hyperbolic plane

Consider the super Minkowski space $\mathbb{R}^{2,1|2}$ consisting of vectors $(x_1, x_2, y|\phi, \theta)$,

- where x_1, x_2, y are even and
- ϕ, θ are odd

with pairing $u = (x_1, x_2, y|\phi, \theta)$ and $u' = (x_1', x_2', y'|\phi', \theta')$ is given by

$$\langle u, u' \rangle \;=\; \tfrac{1}{2}(x_1 x_2' + x_1' x_2) - yy' + \phi\theta' + \phi'\theta.$$

The super hyperbolic plane is given by

$$\{u = (x_1, x_2, y|\phi, \theta) \mid \langle u, u \rangle = 1 \text{ and } x_1 + x_2 > 0\}.$$

Orthosymplectic group

The *Orthosymplectic group* $\mathrm{OSp}(1|2)$ consists of matrices

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} \quad \text{satisfying} \quad g^{\mathrm{st}} J g = J,$$

Latin letters=even, Greek=odd, and

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \textit{supertranspose} \quad g^{\mathrm{st}} = \begin{pmatrix} a & c & \gamma \\ b & d & \delta \\ -\alpha & -\beta & f \end{pmatrix},$$

and the *superdeterminant*

$$\mathrm{sdet}\, g = f^{-1} \det \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + f^{-1} \begin{pmatrix} \alpha\gamma & \alpha\delta \\ \beta\gamma & \beta\delta \end{pmatrix} \right] \text{ also needs to be } 1.$$

Super Fuchsian representations: discrete faithful representations

$$\rho : \pi_1(S_{1,1}) \to \mathrm{OSp}(1|2).$$

= holonomy representations of (marked) super hyperbolic surfaces.

$\Rightarrow$ Super Teichmüller space

$$\mathcal{T}(S_{1,1}) = \mathrm{Hom}_{df}(\pi_1(S_{1,1}), \mathrm{OSp}(1|2))/\mathrm{OSp}(1|2).$$

Geometric invariants

- Penner–Zeitlin and give a characterisation for supergeodesics in the super hyperbolic plane in *Super McShane identity*.
- **Geodesic length** of a closed curve a versus supertrace of $\rho(a)$:

$$|\mathrm{str}(\rho(a)) + 1| = 2\cosh(\tfrac{1}{2}\ell_a).$$

- Super ideal triangles are the same are parametrised by an odd invariant! Each super ideal triangle is parametrised by three points in $\mathbb{R}^{1|1}$, then mod out by $\mathrm{Osp}(1|2)$, which is $3|2$-dimensional.
- This leads to the *W-invariant* for super (once-punctured) tori.

Super McShane identity

Given any super torus $S_{1,1}$, let $C(S_{1,1})$ denote the set of simple closed geodesics on $S_{1,1}$. Then,

$$\sum_{a \in \mathcal{C}(S_{1,1})} \left(\frac{1}{e^{\ell_a} + 1} + \frac{W_a}{4} \frac{\sinh \frac{\ell_a}{2}}{\cosh^2 \frac{\ell_a}{2}} \right) = \frac{1}{2}.$$

Also established the asymptotic growth rate of the simple length spectra of super tori.