Double Dimers and the Super Ptolemy Relation

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(joint with Gregg Musiker and Sylvester Zhang)
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University of Minnesota

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Quantum Curves, Integrability, and Cluster Algebras

Ptolemy's Theorem

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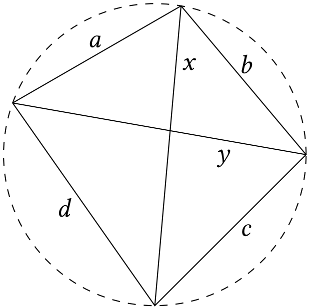
Take a quadrilateral inscribed in a circle, with lengths labelled as in the picture.



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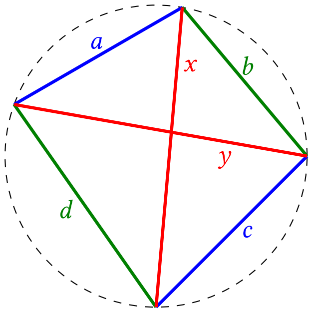

Then $xy = ac + bd$.

Triangulated Polygons

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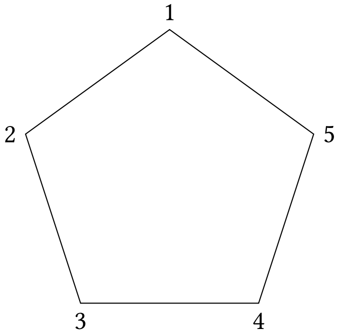
$$x_{ij} = \text{ length of diagonal } (i, j)$$



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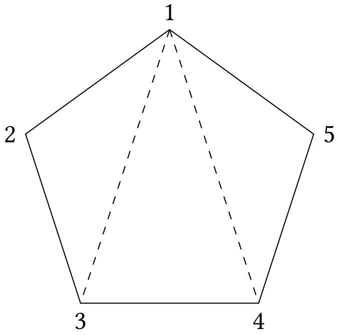

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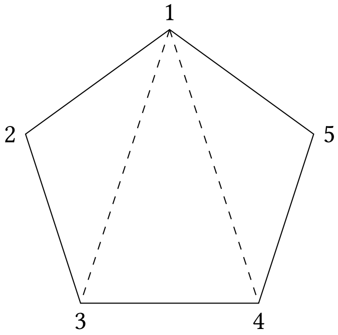

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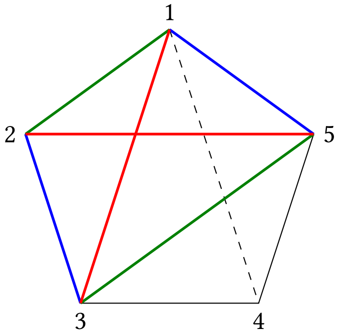

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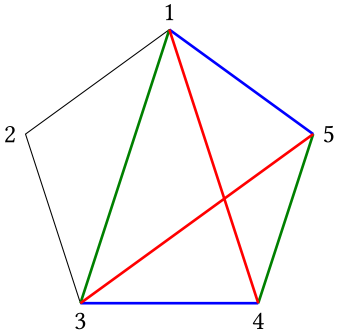

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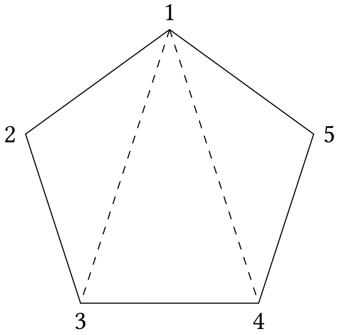

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Answer: They are generating functions of "*dimer covers*" of certain graphs.

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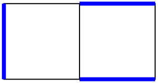

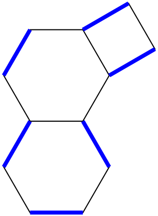

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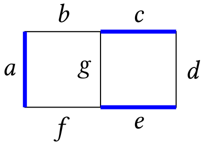
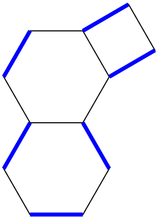


weight $= ace$



If the edges have weights, then the "*weight*" of a dimer cover is the product of the edge weights.

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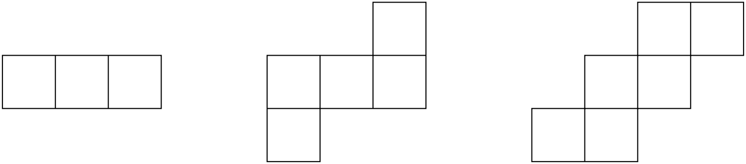

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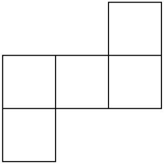
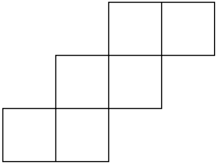
Examples:

$W(G) = RR$



$W(G) = URRU$



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To each snake graph G, we can associate a word $W(G)$ in the alphabet $\{R, U\}$ (for "*right*" and "*up*").

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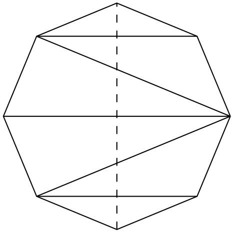

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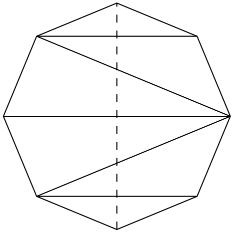

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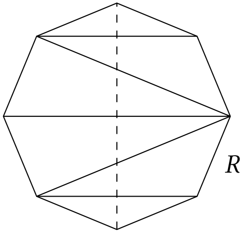

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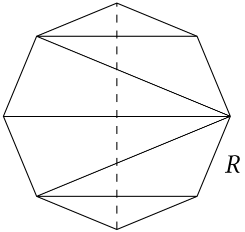

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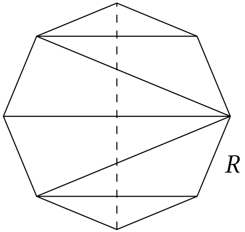

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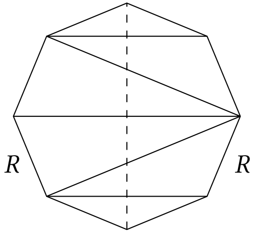

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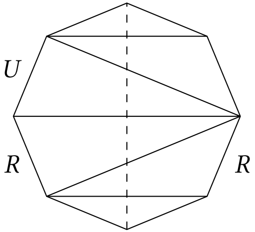

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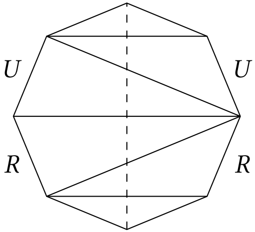

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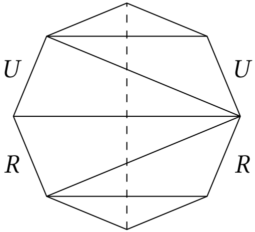

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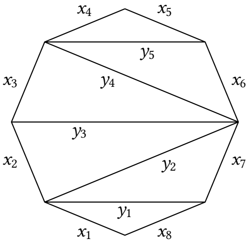

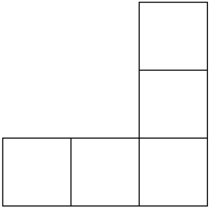

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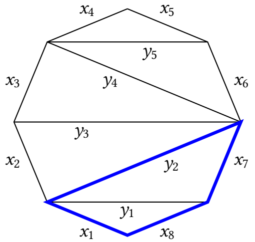

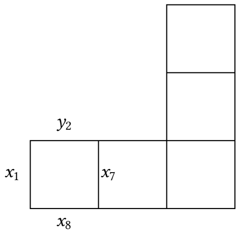

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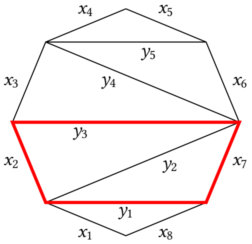

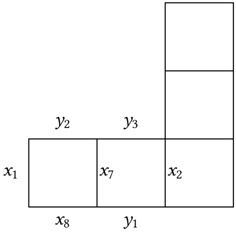

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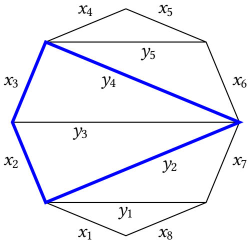

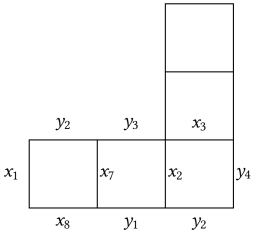

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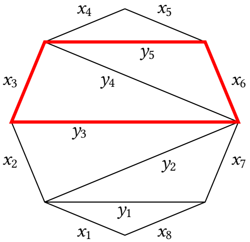

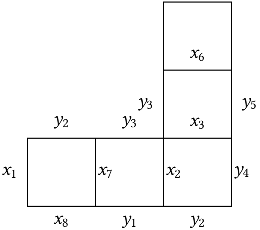

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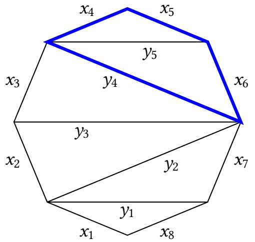

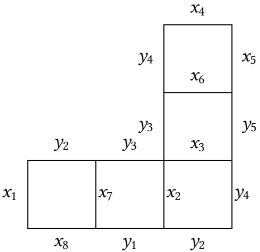

The Laurent Formula

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Theorem [Musiker, Schiffler][1]

$$x_\gamma = \frac{1}{\text{cross}(\gamma)} \sum_{M \in D(G_\gamma)} \text{wt}(M)$$

where $\text{cross}(\gamma)$ is the product of all edges of the triangulation which γ crosses.

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Corollary

Each x_γ is a Laurent polynomial in the lengths of the diagonals from any fixed triangulation.

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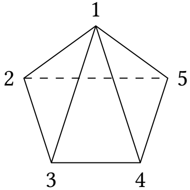
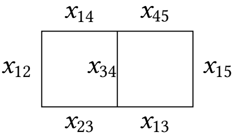
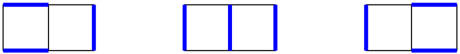
Example







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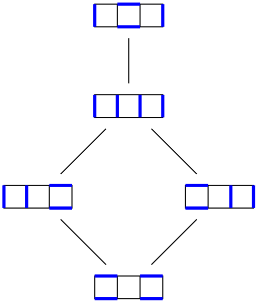
Distributive Lattice Structure



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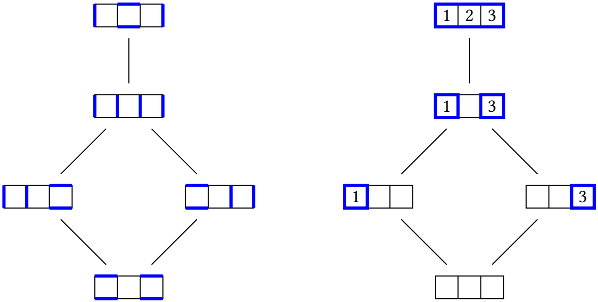

Superimpose the minimal dimer cover (but don't draw doubled edges) to see this is isomorphic to a lattice of subsets ordered under inclusion.

Super Algebras

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In this example, A_0 is spanned by monomials with an even number of θ's, and A_1 is spanned by monomials containing an odd number of θ's.

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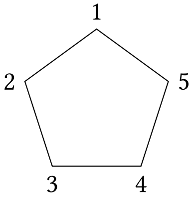

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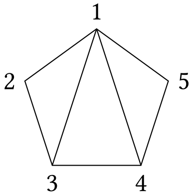

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- an orientation of each edge in T
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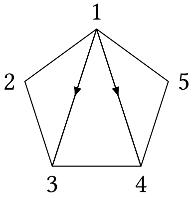

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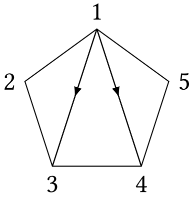

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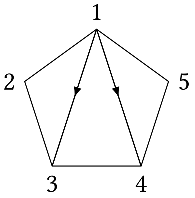

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The example above would have 7 even generators and 3 odd generators.

The Super Ptolemy Relation

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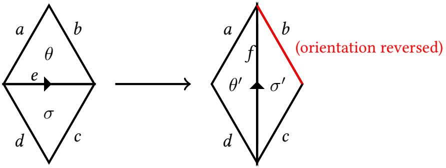
Given two adjacent triangles, we can "*flip*" the diagonal:



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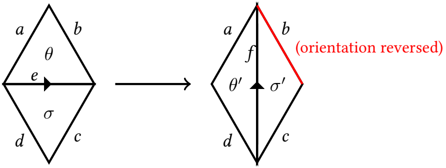

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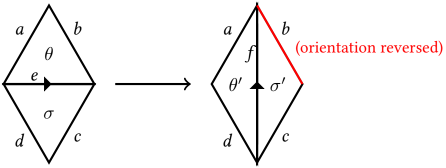

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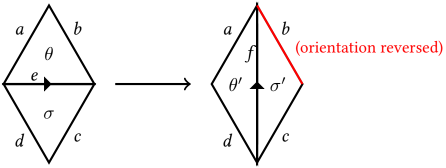

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$$\theta' = \frac{\sqrt{bd}\,\theta + \sqrt{ac}\,\sigma}{\sqrt{ac+bd}}$$

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Question: Can we explicitly describe these algebraic expressions?

Answer: Yes! They are generating functions for "*double dimer covers*" of the snake graph.

Double Dimer Covers

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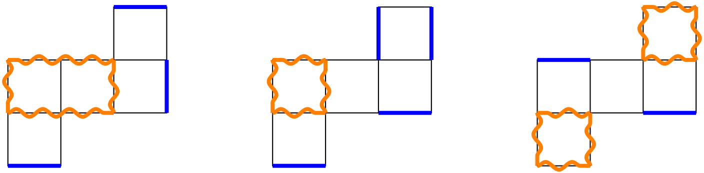
Dimers will be drawn as wavy orange lines, and double dimers will be drawn as straight blue lines.

Examples:



Double Dimer Covers on Snake Graphs

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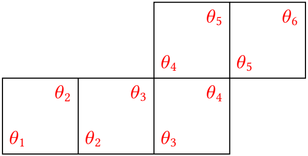
Every square tile in a snake graph represents two triangles in the triangulation. We will label the tiles with the odd variables of those triangles.



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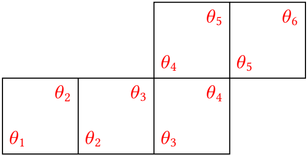

The *weight* of a double dimer cover is the product of the square roots of the edge weights, times the odd variables at the beginning and end of cycles.

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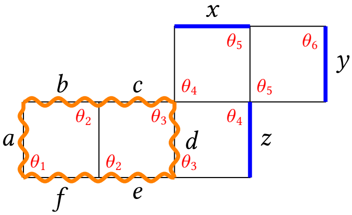

$$\text{weight} = xyz\sqrt{abcdef}\,\theta_1\theta_3$$

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Laurent Formula

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Theorem[Musiker, O., Zhang][1]

Given a fixed triangulation (with the default orientation),

$$x_\gamma = \frac{1}{\mathrm{cross}(\gamma)} \sum_{M \in DD(G_\gamma)} \mathrm{wt}(M)$$

Moreover, there is an ordering of the θ's which makes all terms positive.

[1] Ovenhouse Musiker and Zhang. "An Expansion Formula for Decorated Super-Teichmüller Spaces". In: *arXiv preprint arXiv:2102.09143* (2021)

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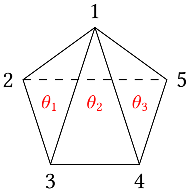
Corollary ("Laurent Phenomenon")

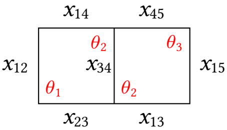
Each term of x_γ is a Laurent monomial in the square roots of the x's, times a monomial in the θ's.

[1] Ovenhouse Musiker and Zhang. "An Expansion Formula for Decorated Super-Teichmüller Spaces". In: *arXiv preprint arXiv:2102.09143* (2021)

Example



1
2
5
θ_1
θ_2
θ_3
3
4



x_{14}
x_{45}
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x_{15}
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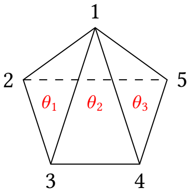
x_{13}

Example



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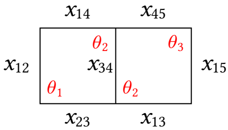

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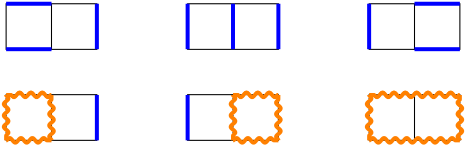

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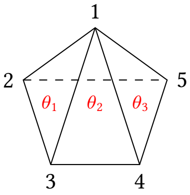
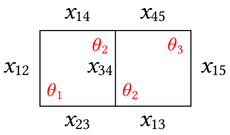
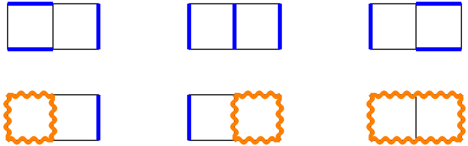
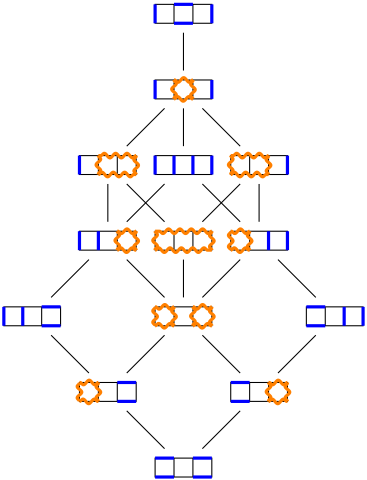






$$x_{25} = \frac{1}{x_{13}x_{14}}\Big(x_{14}x_{23}x_{15} + x_{12}x_{34}x_{15} + x_{12}x_{13}x_{45}$$
$$+ x_{15}\sqrt{x_{12}x_{14}x_{23}x_{34}}\,\theta_1\theta_2 + x_{12}\sqrt{x_{13}x_{15}x_{34}x_{45}}\,\theta_2\theta_3$$
$$+ \sqrt{x_{12}x_{13}x_{14}x_{15}x_{23}x_{45}}\,\theta_1\theta_3\Big)$$

Distributive Lattice Structure



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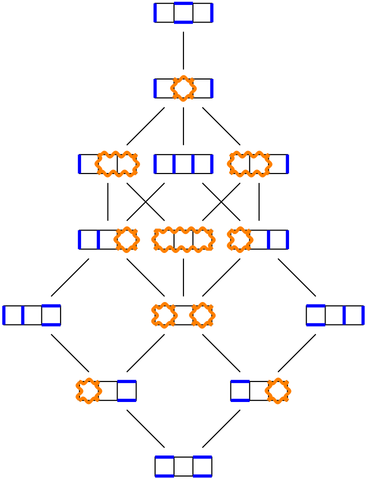

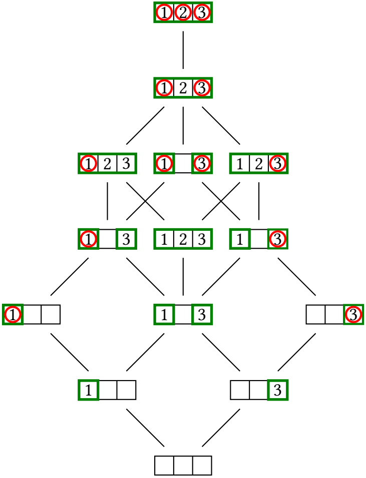

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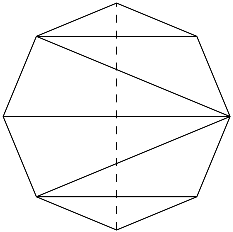

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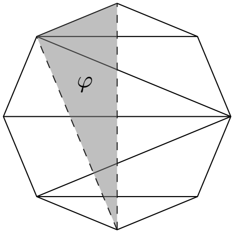
Let φ be a triangle with γ as a side, and also a boundary side.



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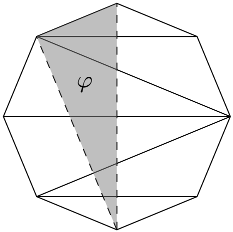
Can we express θ_φ in terms of the initial triangulation?



The Toggle Involution

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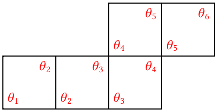
Recall that snake graphs are labelled with odd variables.



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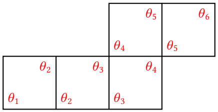

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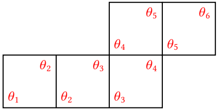

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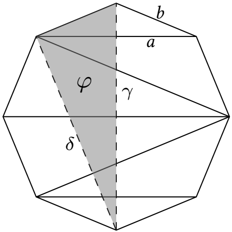
Examples:

$$(\theta_1\theta_2)^* = \theta_1\theta_2\theta_6, \qquad (\theta_4\theta_6)^* = \theta_4$$

Formula for Odd Variables



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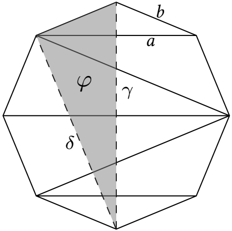


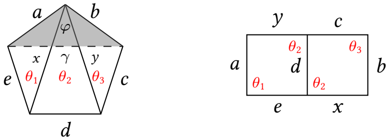
Theorem [Musiker, O, Zhang][1]

$$\sqrt{\gamma\delta}\,\theta_\varphi = \frac{1}{\mathrm{cross}(\gamma)}\frac{\sqrt{a}}{\sqrt{b}}\sum_{M\in D_t(G_\gamma)} \mathrm{wt}(M)^*$$

where D_t is the set of double dimer covers using the *top* edge of the last tile.

[1] Ovenhouse Musiker and Zhang. "Double Dimer Covers on Snake Graphs from Super Cluster Expansions". In: *arXiv preprint arXiv:2110.06497* (2021)

Example



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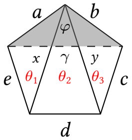
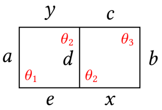
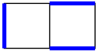
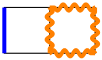
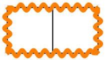




$D_t(G)$:







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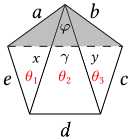

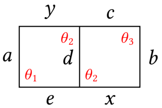

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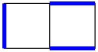

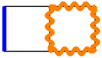

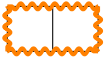

$$\sum_{M \in D_t(G)} \mathrm{wt}(M) = acx + a\sqrt{bcdx}\,\theta_2\theta_3 + \sqrt{abcexy}\,\theta_1\theta_3$$

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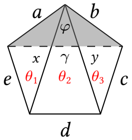

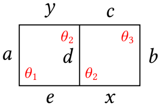

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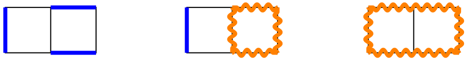

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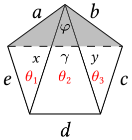

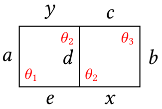

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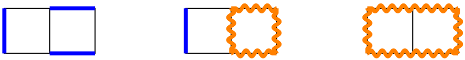

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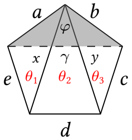

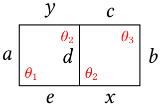

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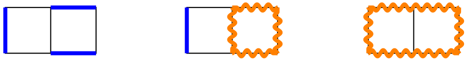

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$$\sqrt{a\gamma}\,\theta_\varphi = \frac{1}{xy}\left(ax\sqrt{cy}\,\theta_3 + a\sqrt{bdxy}\,\theta_2 + y\sqrt{abex}\,\theta_1\right)$$

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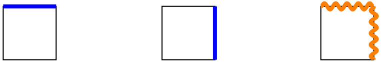
Looking at the top-right corner of the last tile of $G = G_\gamma$, there are 3 cases:



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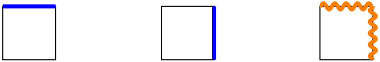

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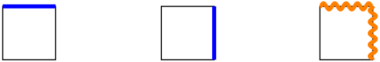

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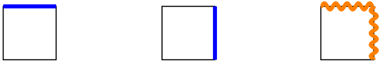

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The strategy of the proof is to show that

$$\frac{ac}{e} = \sum_{M \in D_T(G)} \mathrm{wt}(M)$$

$$\frac{bd}{e} = \sum_{M \in D_R(G)} \mathrm{wt}(M)$$

$$\frac{\sqrt{abcd}}{e}\,\sigma\theta = \sum_{M \in D_{tr}(G)} \mathrm{wt}(M)$$

Thank You!