Bifundamental Baxter Operators

(joint w. A. Shapiro)

3D gauge theories with $G = GL_n \times GL_m$ and matter in bifundamental rep. $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$

quantum integrable systems

(open q-Toda chain)

cluster algebras

Quantum integrable systems

$$\{H_i\}_{i=1}^n \subset \mathcal{A} \circlearrowright \mathcal{H}$$

self-adjoint commuting Hamiltonians

$*$- algebra of observables

Hilbert space representation (States)

$\boxed{\text{Goal 1}}$ Construction of interesting families $\{H_i\}_{i=1}^n$

$\boxed{\text{Goal 2}}$ a complete set of orthonormal joint eigenfunctions $\{\Psi_\lambda\}$ for $\{H_i\}_{i=1}^n$

Often, Goal 1 achieved using quantum groups, and
Goal 2 using quantum inverse scattering method

We'll explain how to construct a particular quantum integrable system, the $\boxed{\text{open } q\text{-Toda chain}}$,

using a different point of view:

using cluster algebras and their symmetries

Open Toda chain: $\{H_i\}_{i=1}^n$ commuting

$[2,\infty)$

differential operators on $\mathbb{R}^n$

$x_1 + \cdots + x_n = 0$

$sl_2 \quad x_2 = -x_1$

$H_1 = T + T^{-1} + e^{2x}$

$$H_1 = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n-1} e^{x_i - x_{i+1}}$$

e^p

$2 \quad (1 + p + p^2/2)$

$+ (1 - p + p^2/2)$

n particles in 1D, nearest-neighbor interactions.

Difference operator version:

$U_q(gl_n)$

$$H_1 = \sum_{i=1}^{n} T_i + \sum_{i=1}^{n-1} e^{x_i - x_{i+1}}, \qquad T_i f = f(x_i + \delta)$$

i.e. $T_i \cdot e^{x_j} = q^{\delta_{ij}} e^{x_j}$, $\boxed{q = e^{\delta}}$.

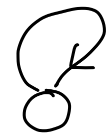
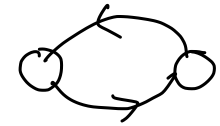
Basic input to define a cluster algebra:

1. a lattice Λ

2. a skew-symmetric bilinear form
$(\cdot\,,\,\cdot) : \Lambda \otimes \Lambda \longrightarrow \mathbb{Z}$

3. a basis $\{e_i\}$ for Λ

The data $\left(\Lambda, (\cdot,\cdot), \{e_i\}\right)$ defines a

$\boxed{\text{quiver:}}$ (directed graph without  or )

$$\text{vertices} \longleftrightarrow \{e_i\}$$

$$\text{signed \# arrows } e_i \longrightarrow e_j \;=\; (e_i, e_j)$$

e.g. $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, $(e_1, e_2) = 1$, basis $\{e_1, e_2\}$

$$\rightsquigarrow \quad \underset{e_1}{\circ} \longrightarrow \underset{e_2}{\circ}$$

<u>e.g.</u> (GL_n Toda chain)

$$P = GL_n \text{ weight lattice} \simeq \bigoplus_{i=1}^{n} \mathbb{Z}\{\varepsilon_i\}$$

with standard (<u>symmetric</u>!) Euclidean form $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$

n fundamental weights $\boxed{\omega_i = \varepsilon_1 + \cdots + \varepsilon_i}$

$n-1$ simple roots $\boxed{\alpha_i = \varepsilon_i - \varepsilon_{i+1}}$

Root lattice $L = \bigoplus_{i=1}^{n-1} \mathbb{Z}\{\alpha_i\} \subset P$

Consider <u>double lattice</u> $P^\vee \oplus P$ with skew-form $(\cdot,\cdot)$ such that P, $P^\vee$ are both isotropic, and

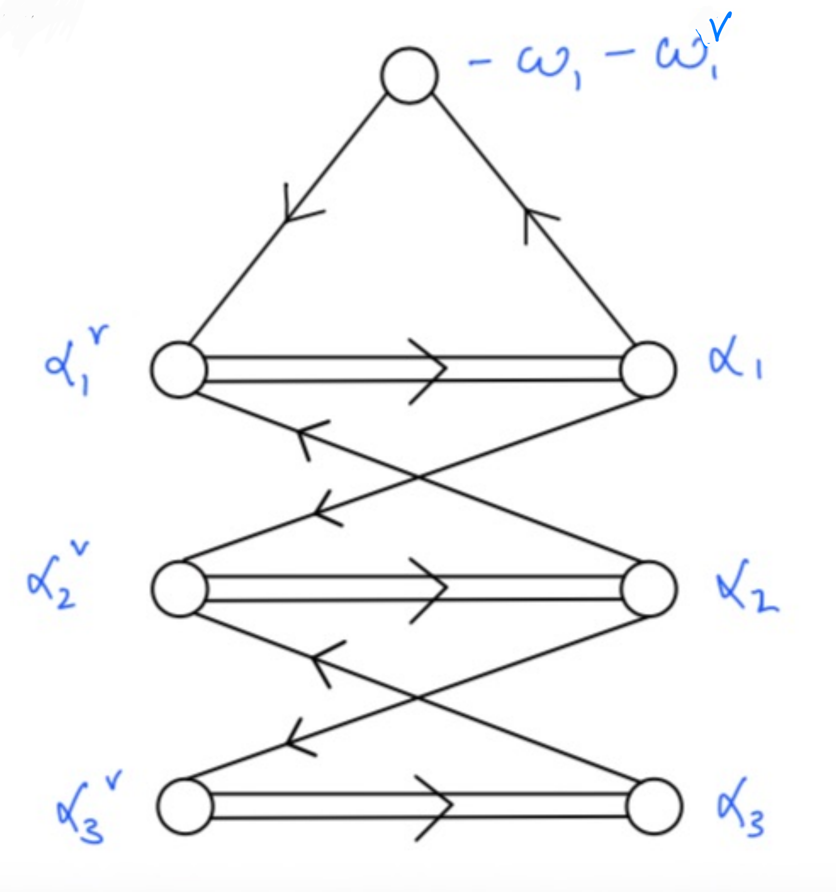
$$\boxed{(\lambda^\vee, \mu) := \langle \lambda^\vee, \mu \rangle}$$

We take

$$\Lambda = L^\vee \oplus L \oplus \mathbb{Z}\{-\omega_1 - \omega_1^\vee\} \subset P^\vee \oplus P$$

with basis $\{\alpha_i^\vee, \alpha_i\}_{i=1}^{n-1} \cup \{-\omega_1, -\omega_1^\vee\}$

<u>Quiver:</u> (GL_4)



The pair Λ, $(\cdot,\cdot)$ determines a $\boxed{\text{quantum torus algebra}}$

$$\mathcal{T} = \mathbb{Z}[q^{\pm 1}][\Lambda] \quad \text{with product}$$

$$q^{(\lambda,\mu)} X_\lambda \cdot X_\mu = X_{\lambda+\mu} .$$

We'll construct our Hamiltonians $\{H_i\}_{i=1}^n$ as elements of $\mathcal{T}$.

In our GL_n Toda example:

$\mathcal{T}$ generated by X_{α_i}, $X_{\alpha_i^\vee}$, $X_{-\omega_1-\omega_1^\vee}$.

$\mathcal{T}$ has a 1-parameter family of representations on $\mathbb{C}P \simeq \mathbb{C}[T_{GL_n}]$, labelled by $\boxed{z \in \mathbb{C}^\times}$

action of center

Explicitly:

- X_{α_i} act by multiplication: $\mathbb{C}P \ni Y_\lambda \longmapsto Y_{\lambda+\alpha_i}$
- $X_{\alpha_i^\vee}$ act by q-shift: $X_{\alpha_i^\vee} \cdot Y_\lambda = q^{\langle \alpha_i^\vee, \lambda \rangle} Y_\lambda$
- $X_{-\omega_1-\omega_1^\vee}$ acts by $q^{-1} z X_{-\omega_1} \cdot X_{-\omega_1^\vee}$.

Hilbert space structure:

$$\mathbb{C}\mathcal{P} \simeq \mathbb{C}[\mathcal{T}_{GL_n}]$$

$$\rightsquigarrow$$

$$\mathcal{H} := L_2(\mathcal{T}_{GL_n}).$$

If $|q| = 1$ and $z \in \mathbb{R}$, generators X_λ act by positive, selfadjoint operators on $\mathcal{H}$.

Idea: construct a family of $\boxed{\text{unitary}}$ operators $Q(z)$ that our commuting self-adjoint $\{H_i\}$ are recovered from the q-difference equation

$$\boxed{Q(q^2 z)\, Q(z)^{-1} = \sum_k H_k z^k}$$

(Rough analogy: unitary time evolution op. $e^{it\hat{H}}$ in QM

$\frac{1}{2\pi i}\frac{d}{dt}$

self-adjoint Hamiltonian $\hat{H}$)

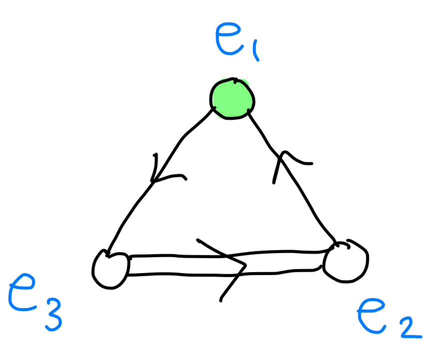
Construct $Q(z)$ via $\boxed{\text{cluster mutation}}$

<u>Combinatorially:</u> the mutation of basis $\{e_i\}$ in direction e_k is the new basis

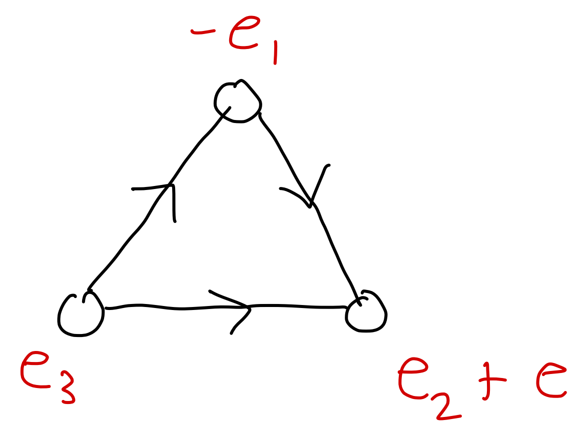
$$e_i' = \begin{cases} -e_k & i=k \\ e_i + \max(0, (e_i, e_k))\, e_k & i \neq k \end{cases}$$

(GL_2 Toda)

<u>e.g.</u>



$\xrightarrow{\mu_1}$



$\Rightarrow$ get <u>new</u> quiver $\boxed{\Gamma' = \mu_k(\Gamma)}$

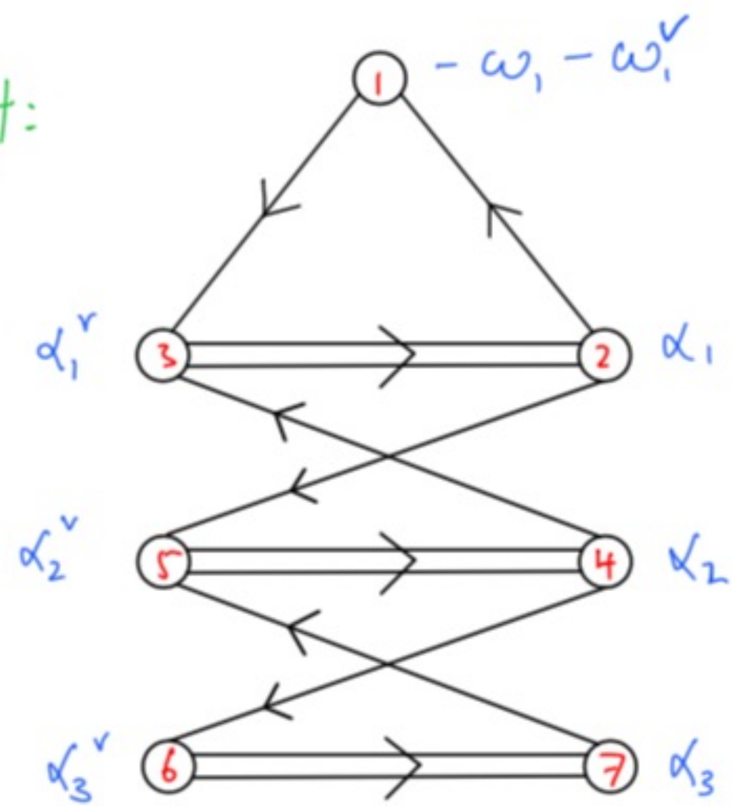
The GL_n Toda quiver has a symmetry:

a sequence of $2n-1$ mutations

$$\vec{\mu} = \mu_{\alpha_{n-1}^\vee} \mu_{\alpha_{n-1}} \cdots \mu_{\alpha_1^\vee} \mu_{\alpha_1} \mu_{-\omega_1-\omega_1^\vee} \quad \text{such that}$$

$$\boxed{\vec{\mu}(T) \simeq T}$$

$n=4$:



At the level of operators, each mutation determines a $\boxed{\text{unitary}}$ constructed using Faddeev's q-dilogarithm $\Psi(z)$: solution of

$$\Psi(q^2 z) = (1+qz)\,\Psi(z)$$

and same with

$$\boxed{e^{\pi i \hbar} = q \longleftrightarrow q^\vee = e^{\pi i/\hbar}}$$

Since $\varphi(\mathbb{R}) \subset S^1 \subset \mathbb{C}$,

X_λ self adjoint $\Longrightarrow$ $\varphi(X_\lambda)$ $\boxed{\text{unitary}}$

So to mutation in direction e_k we associate the unitary $\varphi(X_{e_k})$.

Our operator $Q_n(z) : \mathcal{H} \longrightarrow \mathcal{H}$ is the composite of unitaries for the symmetry $\vec{\mu}(\Gamma) \simeq \Gamma$ of the GL_n Toda quiver.

Then:

① $$[Q_n(z), Q_n(w)] = 0$$

② $$Q_n(q^2 z)\, Q_n(z)^{-1} = \sum_{k=0}^{n} z^k H_k$$

open Toda Hamiltonians

③ $Q_n(z)$ has a complete set of eigenfunctions given by the $U_q(\mathfrak{gl}_n,\mathbb{R})$ Whittaker functions $\{\Psi_\lambda \mid \lambda \in \mathbb{R}_+^n / S^n\}$, with eigenvalue

$$\boxed{Q_n(z)\cdot \Psi_\lambda = \left(\prod_{k=1}^{n} \varphi\left(z/\lambda_k\right)\right)\cdot \Psi_\lambda}$$

Writing $Q_n(z)$ as an integral transform,

$$Q_n(z)\,\Psi_\lambda = \prod_{k=1}^{n} \varphi(z/\lambda_k)\,\Psi_\lambda$$ becomes

a q-analog of a "Pieri" integral for <u>classical</u> Whittaker functions found by Bump & Stade.

$\boxed{\text{Moral:}}$ entire construction of q-Toda and its eigenfunctions determined by the

$\boxed{\text{Symmetry}}$ $\mu(\Gamma) \simeq \Gamma$.

Where does this symmetry come from?

An answer from physics: the quiver T_n describes cluster structure on "Coulomb branch ring" $\mathbb{C}_q[\mathcal{M}_C]$ of a 3D SUSY gauge theory with

$$G = GL_n \times GL_1$$

and matter $M = N \oplus N^\vee,$

$$N = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}).$$

Since $M = N \oplus N^\vee$, have duality

$$\begin{tikzcd}
\boxed{N \longleftrightarrow N^\vee} \arrow[r, squiggly] & \text{automorphism of } \mathbb{C}_q[\mathcal{M}_C] \arrow[d, leftrightarrow] \\
\text{"Kontsevich-Soibelman-Donaldson-Thomas transformation"} \arrow[u] \arrow[r, leftrightarrow] & \text{Symmetry } \tilde{\mu} \text{ and Baxter operator } Q
\end{tikzcd}$$

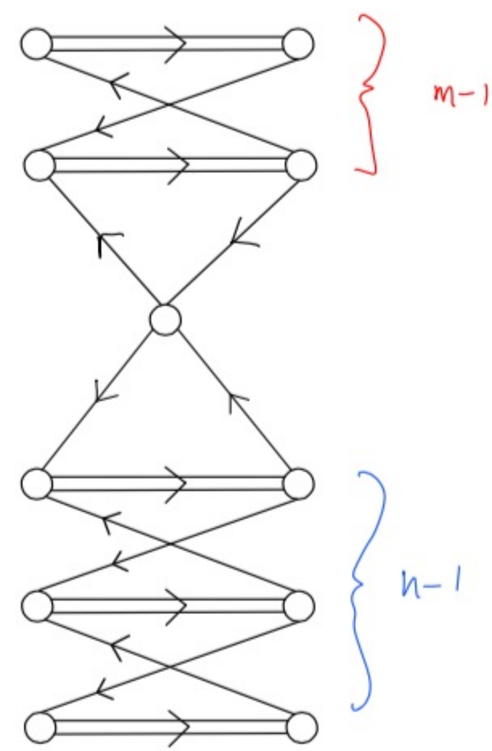
The factorization of the KS-DT transformation into q-dilogarithms encodes $\boxed{\text{BPS spectrum}}$ of theory

What about theories with

$$G = GL_n \times GL_m,$$

$$N = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m) \quad ?$$

Cluster structure on $\mathcal{M}_C \longleftrightarrow$ quiver $T_{n,m}$



and $DT(T_{n,m}) \sim$ Symmetry of $(2n-1)(2m-1)$ mutations

Original proof: using Whittaker functions.

(1) Construct a unitary $Q_{n,m}$ on $\mathcal{H} = L_2(T_{GL_n \times GL_m})$ satisfying a "rectangular" Bump-Stade identity:

$$Q_{n,m} \cdot \left(\Psi_\lambda^{GL_n} \boxtimes \Psi_\mu^{GL_m} \right) = \prod_{j,k} \varphi\left(\mu_j / \lambda_k \right) \cdot \Psi_\lambda^{GL_n} \boxtimes \Psi_\mu^{GL_m}$$

(2) Factor $\mathbb{Q}_{n,m}$ into product of q-dilogarithms and read off the corresponding mutations to obtain the period $\vec{\mu}$.

$\square$

A geometric proof:

Affine Grassmannian

$$\mathrm{Gr}_{GL_n} = \left\{ L \subset \mathbb{C}((z))^{\oplus n} \;\middle|\; \begin{array}{l} L \text{ a} \\ \mathbb{C}[[z]]\text{-lattice} \end{array} \right\} \circlearrowleft GL_n[[z]]$$

$$\uparrow$$

$$\mathrm{Gr}^{\omega_k} = \left\{ L \subset L_0 = \mathbb{C}[[z]]^{\oplus n} \;\middle|\; \begin{array}{l} zL_0 \subset L \subset L_0, \\ \dim_{\mathbb{C}}(L_0/L) = k \end{array} \right\}$$

$$\cong \mathrm{Gr}\left(k\text{-planes in } \mathbb{C}^n = \frac{L_0}{zL_0}\right).$$

w. line bundle $\mathcal{L} = \det(L_0/L)$.

Have similar subvarieties $\mathrm{Gr}^{\omega_k^*} = \{ L \mid zL \subset L_0 \subset L \}$.
$\dim_{\mathbb{C}}(L/L_0) = k$

$\omega_k^* = -w_0(\omega_k)$

<u>Braverman-Finkelberg-Nakajima:</u> consider a space

$$\mathcal{R} = \left\{ (L, M, \varphi) \;\middle|\; \begin{array}{l} (L,M) \in \mathrm{Gr}_{GL_n} \times \mathrm{Gr}_{GL_m} \\ \varphi \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)[[z]] \\ \varphi(L) \subset M \end{array} \right\} \circlearrowleft G[[z]]$$

$$\downarrow$$

Gr_G , $G = GL_n \times GL_m$

$\mathcal{R}^{\omega,\omega'} :=$ fiber over $\mathrm{Gr}^{\omega,\omega'}$,

with line bundles $\mathcal{L}_1$, $\mathcal{L}_2$

BFN: $K_{G[\![z]\!]}(\mathcal{R})$ is a convolution algebra, generated by classes

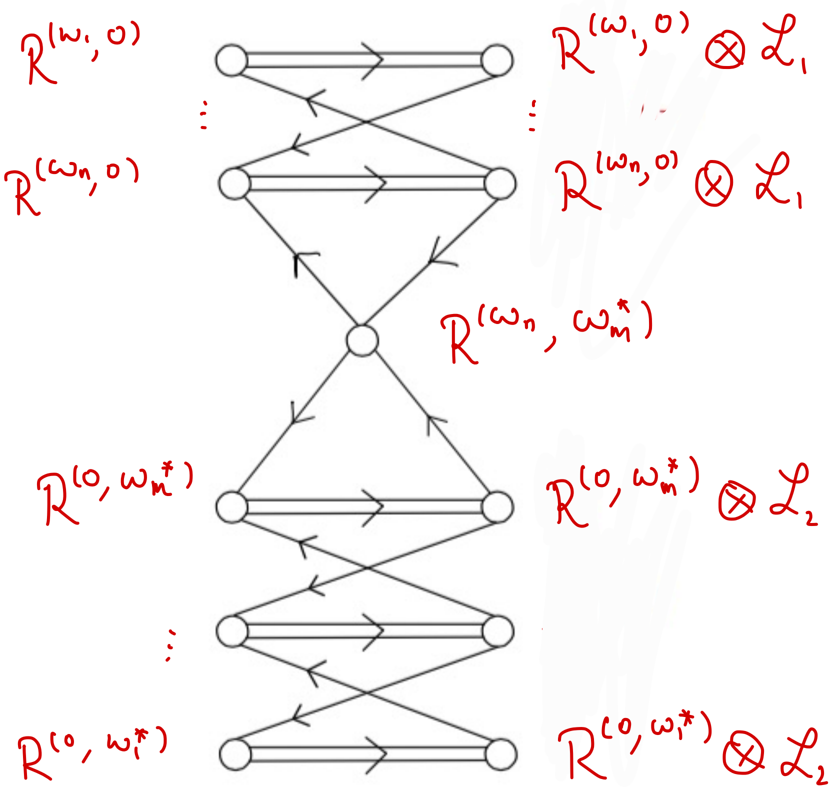
$$\boxed{\mathcal{F}^{a,b}_{\omega,\omega'} = \mathcal{O}_{\mathcal{R}^{\omega,\omega'}} \otimes \mathcal{L}_1^{a} \otimes \mathcal{L}_2^{b}}$$

"minuscule monopoles"

where $\mathcal{R}^{\omega,\omega'}$ are fibers over closed orbits in $Gr_{GL_n \times GL_m}$, i.e. ω, ω' fundamental or dual fundamental, and $a, b \in \mathbb{Z}$.

In fact, $K_{G[[z]]}(\mathcal{R})$ is a cluster algebra,

with two initial clusters (of $\mathcal{A}$-variables)



corresponding to summand

$$N \subset M = N \oplus N^\vee$$
$$N = \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$$

and the same with all $\omega \longleftrightarrow \omega^*$.

(corresponding to summand $N^\vee$)

The period $Q_{n,m}$ sends the N-cluster to $N^\vee$ one, and $\boxed{\text{all intermediate clusters}}$ consist of minuscule monopoles, i.e. classes $\mathcal{F}^{a,b}_{\omega,\omega'}$ corresponding to closed orbits.

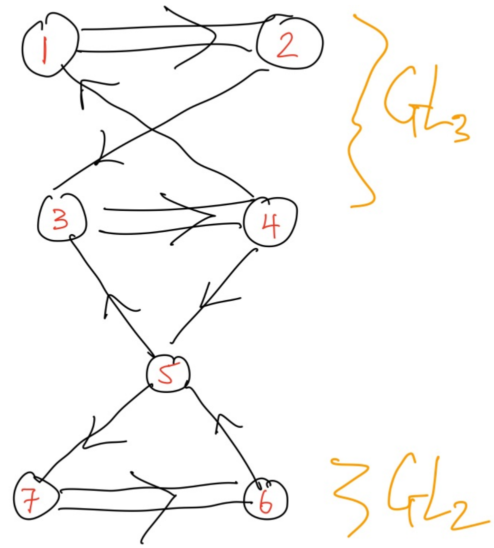
The proof is based on construction of exact triangles in $D^b Coh(R)$

e.g.

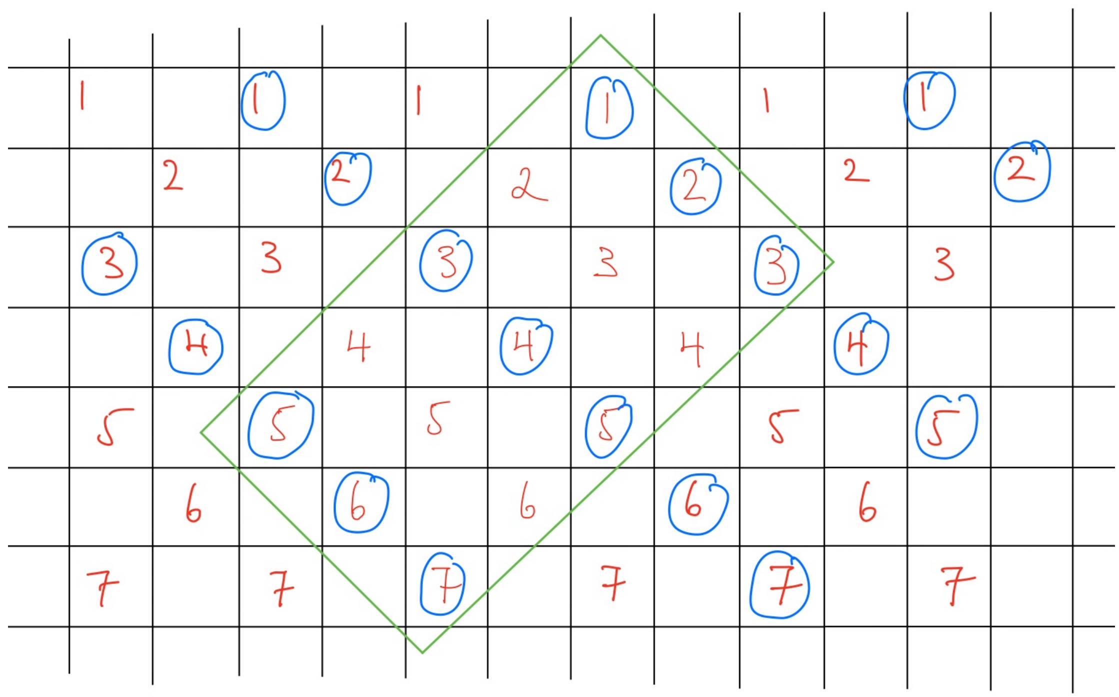
$$\boxed{F^{1,0}_{k,\ell} * F^{0,1}_{k,\ell+1} \longrightarrow F^{0,0}_{k,\ell} * F^{1,1}_{k,\ell+1} \longrightarrow F^{0,1}_{k-1,\ell} * F^{1,0}_{k+1,\ell+1},}$$

each corresponding to one of the $(2n-1)(2m-1)$ mutations in $Q_{n,m}$.

e.g. $GL_2 \times GL_3$



Take $(2n-1)\times(2m-1)$ domain



$\Rightarrow \{5, 6, 4, 7, 3, 5, 4\ 2, 6, 5, 1, 3, 2, 4, 3\}$