$\hbar$-opers and the geometric approach to the Bethe ansatz equations

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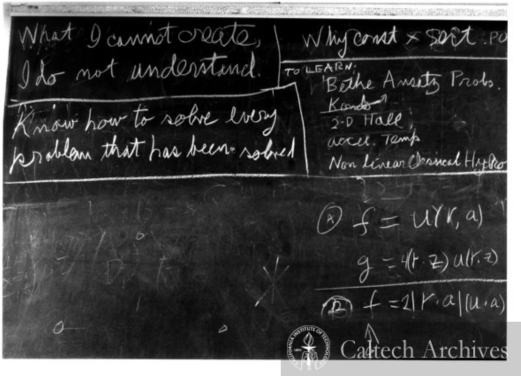
Some history of quantum integrable systems and Bethe ansatz



R.P. Feynman: "I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you dont know why. I am trying to understand all this better."

Some history of quantum integrable systems and Bethe ansatz

Exactly solvable models of statistical physics: spin chains, vertex models

- 1930s: H. Bethe: Bethe ansatz solution of Heisenberg model
- 1960-70s: R.J. Baxter, C.N. Yang: Yang-Baxter equation, Baxter operator
- 1980s: Development of QISM by Leningrad school, leading to the discovery of quantum groups by Drinfeld and Jimbo
- Since 1990s: textbook subject and an established area of mathematics and physics

- Enumerative geometry: quantum K-theory

Generalization of quantum cohomology in the early 2000s by A. Givental, Y.P. Lee and collaborators. Recently big progress in this direction by A. Okounkov and his school.

M. Aganagic, A. Okounkov, *Quasimap counts and Bethe eigenfunctions*, Mosc. Math. J., 17:4 (2017), 565-600, arXiv:1704.08746

P.Pushkar, A. Smirnov, A.Z., *Baxter Q-operator from quantum K-theory*, Adv. Math. 360 (2020) 106919, arXiv:1612.08723

P. Koroteev, P.Pushkar, A. Smirnov, A.Z., *Quantum K-theory of Quiver Varieties and Many-Body Systems*, Selecta Math. 27 (2021) 87, arXiv:1705.10419

P. Koroteev, A.Z., *qKZ/tRS Duality via Quantum K-Theoretic Counts*, Math. Res. Lett. 28(2) (2021) 435 - 470, arXiv:1802.04463

- Multiplicative connections: $(G, \hbar)$-opers

  q-deformed version of the classic example of geometric Langlands correspondence, studied in detail by B. Feigin, E. Frenkel, N. Reshetkhin: correspondence between opers (certain connections with regular singularities) and Gaudin models.

P. Koroteev, D. Sage, A. Z., *(SL(N),q) -opers, the q-Langlands correspondence, and quantum/classical duality*, Comm. Math. Phys., 381 (2021) 641-672, arXiv:1811.09937

E. Frenkel, P. Koroteev, D. Sage, A.Z., *q-opers, QQ-systems and Bethe ansatz*, to appear in J. Eur. Math. Soc., arXiv:2002.07344

P. Koroteev, A.Z.,*Toroidal q-Opers*, to appear in Journal of the Institute of Mathematics of Jussieu, in press, arXiv:2007.11786

P. Koroteev, A.Z., *q-opers, QQ-systems and Bethe ansatz II: Generalized Minors*, arXiv:2108.04184

P. Koroteev, A.Z., *3d Mirror Symmetry for Instanton Moduli Spaces*, arXiv:2105.00588

Outline

Quantum groups and Bethe ansatz

QQ-systems and Bethe ansatz. Gaudin model and opers.

$(G, \hbar)$-opers and QQ-system

$(SL(r+1), \hbar)$-opers and QQ-systems

QQ-systems and $(G, \hbar)$-opers via generalized minors

Let us consider Lie algebra $\mathfrak{g}$.

The associated *loop algebra* is $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ and t is known as *spectral parameter*.

The following representations, known as *evaluation modules* form a tensor category of $\hat{\mathfrak{g}}$:

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n),$$

where

- V_i are representations of $\mathfrak{g}$
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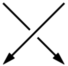
Quantum group

$$U_\hbar(\hat{\mathfrak{g}})$$

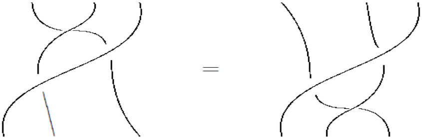
is a deformation of $U(\hat{\mathfrak{g}})$, with a nontrivial intertwiner $R_{V_1,V_2}(a_1/a_2)$:

$$V_1(a_1) \otimes V_2(a_2)$$



$$V_2(a_2) \otimes V_1(a_1)$$

which is a rational function of a_1, a_2, satisfying Yang-Baxter equation:



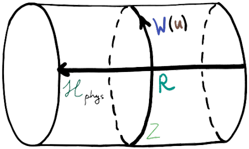
The generators of $U_\hbar(\hat{\mathfrak{g}})$ emerge as matrix elements of R-matrices (the so-called FRT construction).

Source of integrability: commuting *transfer matrices*, generating *Baxter algebra* which are weighted traces of

$$\tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{\rm phys} \to W(u) \otimes \mathcal{H}_{\rm phys}$$

over auxiliary $W(u)$ space:

$$T_{W(u)} = \mathrm{Tr}_{W(u)}\Big(M(u)\Big) = \mathrm{Tr}_{W(u)}\Big((Z \otimes 1)\ \tilde{R}_{W(u),\mathcal{H}_{phys}}\Big)$$



Here $Z \in e^{\mathfrak{h}}$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra.

Integrability:

$$[T_{W'(u')}, T_{W(u)}] = 0$$

There are special transfer matrices is called *Baxter Q-operators*. Such operators generate all *Bethe algebra*.

Primary goal for physicists is to diagonalize $\{T_{W(u)}\}$ simultaneously.

Textbook example is XXZ Heisenberg spin chain:

$$\mathcal{H}_{XXZ} = \mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n)$$

States:

$$\uparrow\uparrow\uparrow\uparrow \downarrow \uparrow\uparrow\uparrow \downarrow \uparrow\uparrow\uparrow\uparrow \downarrow \uparrow\uparrow\uparrow\uparrow\uparrow \downarrow\downarrow \uparrow\uparrow\uparrow$$

Here $\mathbb{C}^2$ stands for 2-dimensional representation of $U_\hbar(\widehat{\mathfrak{sl}}_2)$.

Algebraic method to diagonalize transfer matrices:

Algebraic Bethe ansatz

as a part of Quantum Inverse Scattering Method developed in the 1980s.

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Bethe equations and Q-operator

The eigenvalues are generated by symmetric functions of Bethe roots $\{x_i\}$:

$$\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z\,\hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{x_i \hbar - x_j}{x_i - x_j \hbar}, \quad i = 1 \cdots k,$$

so that the eigenvalues $\mathcal{Q}(u)$ of the Q-operator are the generating functions for the elementary symmetric functions of Bethe roots:

$$\mathcal{Q}(u) = \prod_{i=1}^{k} (u - x_i)$$

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QQ-systems

Another modern view on Bethe ansatz one can find in the papers of D. Hernandez and E. Frenkel, following earlier papers by V. Bazhanov, S. Lukyanov and A. Zamolodchikov.

Extension of the category of representations of $U_\hbar(\hat{\mathfrak{g}})$ by representations of Borel subalgebra give rise to the so-called QQ-systems, which serve as the relations in the Grothendieck ring.

In the case of $U_\hbar(\widehat{\mathfrak{sl}}(2))$ the QQ-system is:

$$z\widetilde{Q}(\hbar u)Q(u) - z^{-1}Q(\hbar u)\widetilde{Q}(u) = \prod_i (u - a_i)$$

Here $Q(u)$ can be viewed as an eigenvalue of the Q-operator.

To obtain the Bethe equation calculate residues on both sides:

$$z\frac{\widetilde{Q}(\hbar u)}{Q(\hbar u)} - z^{-1}\frac{\widetilde{Q}(u)}{Q(u)} = \frac{\prod_i (u - a_i)}{Q(u)Q(\hbar u)}$$

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General form of QQ-system

For Lie algebra $\mathfrak{g}$ of rank r we have:

$$\widetilde{\xi}_i Q^i_-(u) Q^i_+(\hbar u) - \xi_i Q^i_-(\hbar u) Q^i_+(u) = \Lambda_i(u) \prod_{j \neq i} \left[\prod_{k=1}^{-a_{ij}} Q^j_+(\hbar^{b^k_{ij}} u) \right]$$

$$i = 1, \dots, r, \quad b^k_{ij} \in \mathbb{Z}$$

Here polynomials $\Lambda_i(u)$ characterize the representation $U_\hbar(\hat{\mathfrak{g}})$ and ξ_i, $\widetilde{\xi}_i$ are related to Z.

Upon certain nondegeneracy conditions there is 1-to-1 correspondence between solutions of the QQ-system and Bethe ansatz equations.

Classical limit: Gaudin model and opers

Gaudin model is a (semi)classical limit of our quantum group models (Sklyanin'89): ($u = e^{\eta v}$, $\hbar = e^{\eta}$)

$$R(u) = 1 + \eta r(v) + O(\eta^2),$$
$$M(u) = 1 + \eta L(v) + O(\eta^2),$$
$$[L^1(v_1), L^2(v_2)] = \left[r^{12}(v_1 - v_2), L^1(v_1) + L^2(v_2)\right]$$

Gaudin Hamiltonians:

$$H_k = \sum_{j \neq k} \sum_c \frac{t_k^c \otimes t_j^c}{\mathfrak{a}_k - \mathfrak{a}_j} + \langle \alpha_k^\vee, \mathcal{Z} \rangle = Res_{\mathfrak{a}_k} \mathrm{tr}\left[(L(v))^2\right]$$

Geometric description of the spectrum via G^L-oper connections (special type of connections on a principal G^L-bundle over $\mathbb{P}^1$):

Theorem (E. Frenkel'03) There is 1-to-1 correspondence between the spectrum of Gaudin model for Lie algebra $\mathfrak{g}$ and nondegenerate Miura G^L-oper connections on $\mathbb{P}^1$ with regular singularities and trivial monodromy.

(case $\mathcal{Z} = 0$)

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Roots of this correspondence

Gaudin model eigenvalue problem is a critical level limit of Knizhnik-Zamolodchikov equations:

$$(k + h^{\vee})\partial_{a_i}\Phi(a_1, a_2, \dots, a_n) = H_i\Phi(a_1, a_2, \dots, a_n),$$

$$\Phi(a_1, a_2, \dots, a_n) \in V_1(a_1) \otimes \cdots \otimes V_n(a_n)[[z]]$$

Feigin, E. Frenkel'92:

Completion of the center of $U(\hat{\mathfrak{g}})$ at the critical level is isomorphic to Gelfand-Dikii algebra associated to ${}^L\mathfrak{g}$, i.e. Poisson algebra of $Fun(Op_{{}^L\mathfrak{g}}(D^{\times}))$ (classical limit of W-algebra).

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Explicit construction of eigenvectors of KZ equation using Wakimoto modules. Obtained Bethe equations via Miura transformations, known in the generalized KdV hierarchies.

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This lead to the proposed quantum Langlands correspondence between conformal blocks (correlation functions) of W-algebra $W(^L\mathfrak{g})$ and WZW model associated to $\hat{\mathfrak{g}}$.

Correlation functions of $W(^L\mathfrak{g})$ are subject to linear differential (ΨDO in general) equations with singularities.

In a particular case of $\mathfrak{sl}_2$ ($W(\mathfrak{sl}_2)=Vir$) it is a linear Sturm-Liouville problem with prescribed singularities of second order, known as BPZ (Belavin, Polyakov, Zamolodchikov'84) equation.

In $c\to\infty$ ($k\to -h^\vee$) limit these differential operators are:

$$\partial_v^2-\sum_{i=1}^n\frac{\lambda_i(\lambda_i+2)/4}{(v-\mathfrak{a}_i)^2}-\sum_{i=1}^n\frac{c_i}{u-\mathfrak{a}_i},\quad c_i=\lambda_i\Big(\sum_{j\neq i}\frac{\lambda_i}{\mathfrak{a}_i-\mathfrak{a}_j}-\sum_{j=1}^r\frac{1}{\mathfrak{a}_i-w_j}\Big)$$

naturally appear from Miura oper connections on $\mathbb{P}^1$ with regular singularities:

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via the Drinfeld-Sokolov reduction.

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In 2017 Agangic, E. Frenkel and Okounkov introduced a q-deformed version of quantum Langlands correspondence and proved it in ADE case, explicitly identifying conformal blocks for $U_\hbar(\widehat{\mathfrak{g}})$ and $W_{q,t}({}^L\mathfrak{g})$.

M. Aganagic, E. Frenkel, A. Okounkov, *Quantum q-Langlands Correspondence*, Trans. Mosc. Math. Soc. 79, arXiv:1704.08746

Conformal blocks for $U_\hbar(\mathfrak{g})$ satisfy I. Frenkel-Reshetikhin (qKZ) equations.

Conformal blocks for $W_{q,t}({}^L\mathfrak{g})$ are satisfying some difference equations ($\hbar$-difference in $q \to 1$ limit: $t \to \hbar^{-1}$).

A natural question Igor could ask Edward:

What is the geometric meaning of the classical object behind $\hbar$-difference equations when $q \to 1$ (critical level)?
$(G, \hbar)$-opers?

Path to answer: classical QQ-system and Miura G-opers

Bethe equations of Gaudin model can be related with what we call a polynomial solution of the classical QQ-system:

$$W(q_i^-, q_i^+)(v) + \langle \alpha_i, \mathcal{Z} \rangle q_i^+(v) q_i^-(v) = \Lambda_i(v) \prod_j q_j^+(v)^{-a_{ij}},$$

for $\mathfrak{g}^L$.

Relation of E. Frenkel '03 Miura opers with regular singularities to $q_i(v)$:

$$\partial_v + \sum_i \Lambda_i(v) e_i + \sum_i \partial_v \log(q_i^+(v)) \check{\alpha}_i + \mathcal{Z}$$

Here $\{e_i, \check{\alpha}_i\}_{i=1,\ldots,r}$ are the generators of $\mathfrak{b}_+^L \subset \mathfrak{g}^L$.

B. Feigin, E. Frenkel, V. Toledano-Laredo, *Gaudin models with irregular singularities*, Adv.Math.223 (2010) 873-948

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$\hbar$-connections on $\mathbb{P}^1$

- Principal G-bundle $\mathcal{F}_G$ over $\mathbb{P}^1$
- $M_\hbar : \mathbb{P}^1 \to \mathbb{P}^1$, such that $u \mapsto \hbar u$.

  $\mathcal{F}_G^\hbar$ stands for the pullback under the map $M_\hbar$.

A meromorphic $(G, \hbar)$-connection on a principal G-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$ is a section A of $Hom_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^\hbar)$, where U is a Zariski open dense subset of $\mathbb{P}^1$.

Choose U so that the restriction $\mathcal{F}_G|_U$ of $\mathcal{F}_G$ to U is isomorphic to the trivial G-bundle.

The restriction of A to the Zariski open dense subset $U \cap M_\hbar^{-1}(U)$ is an element $A(u)$ of $G(u) \equiv G(\mathbb{C}(u))$.

Changing the trivialization is given by $\hbar$-gauge transformation:

$$A(u) \mapsto g(\hbar u)A(u)g(u)^{-1}$$

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$\hbar$-oper connections for simple simply connected Lie groups G

A $(G,\hbar)$-oper on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$:

- $\mathcal{F}_G$ is a G-bundle
- A is a meromorphic $(G,\hbar)$-connection on $\mathcal{F}_G$ over $\mathbb{P}^1$
- $\mathcal{F}_{B_-}$ is the reduction of $\mathcal{F}_{B_-}$ to B_-

$\hbar$-oper condition: there exists a Zariski open dense subset $U \subset \mathbb{P}^1$ together with a trivialization $\imath_{B_-}$ of $\mathcal{F}_{B_-}$, such that the restriction of the connection $A : \mathcal{F}_G \to \mathcal{F}_G^{\hbar}$ to $U \cap M_{\hbar}{}^{-1}(U)$, written as an element of $G(u)$ using the trivializations of $\mathcal{F}_G$ and $\mathcal{F}_G^{\hbar}$ on $U \cap M_{\hbar}{}^{-1}(U)$ induced by $\imath_{B_-}$ takes values in the Bruhat cell

$$B_-(\mathbb{C}[U \cap M_{\hbar}{}^{-1}(U)])cB_-(\mathbb{C}[U \cap M_{\hbar}{}^{-1}(U)]),$$

where c is Coxeter element: $c = \prod_i s_i$.
Locally:

$$A(u) = n'(u)\prod_i(\phi_i(u)^{\check{\alpha}_i}s_i)n(u),\ \ \phi_i(u) \in \mathbb{C}(u),\ \ n(u), n'(u) \in N(u)$$

Here $N = B/H$, $H = B/[B, B]$.

$\hbar$-Drinfeld-Sokolov reduction: Semenov-Tian-Shansky, Sevostyanov'98

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$(G, \hbar)$-Miura opers

A Miura $(G, \hbar)$-oper on $\mathbb{P}^1$ is a quadruple $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$:

- $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ is a meromorphic $(G, \hbar)$-oper on $\mathbb{P}^1$.
- $\mathcal{F}_{B_+}$ is a reduction of the G-bundle $\mathcal{F}_G$ to B_+ that is preserved by the $\hbar$-connection A.

Theorem.
i) For any Miura $(G, \hbar)$-oper on $\mathbb{P}^1$, there exists a trivialization of the underlying G-bundle $\mathcal{F}_G$ on an open dense subset of $\mathbb{P}^1$ for which the oper $\hbar$-connection has the form:

$$A(u) \in N_-(u) \prod_i (\phi_i(u)^{\check{\alpha}_i} s_i) N_-(u) \ \cap \ B_+(u).$$

ii) Any element from $N_-(u) \prod_i (\phi_i(u)^{\check{\alpha}_i} s_i) N_-(u) \ \cap \ B_+(z)$ can be written as:

$$\prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\phi_i(u) t_i(u)}{g_i(u)} e_i}$$

where each $t_i \in \mathbb{C}(u)$ is determined by the lifting of s_i.

In the following we set $t_i \equiv 1$.

$(G,\hbar)$-opers with regular singularities and Z-twisted opers

- $(G,\hbar)$-oper with regular singularities at finitely many points on $\mathbb{P}^1$:

$$A(u)=n'(u)\prod_i(\Lambda_i^{\check{\alpha}_i}(u)s_i)n(u),\ \Lambda_i(u)\in\mathbb{C}[u].$$

For any Miura $(G,\hbar)$-oper with regular singularities:

$$A(u)=\prod_i g_i^{\check{\alpha}_i}(u)e^{\frac{\Lambda_i(u)}{g_i(u)}e_i}.$$

- $(G,\hbar)$-oper is Z-twisted if it is gauge equivalent to $Z\in H$, namely

$$A(u)=v(\hbar u)Zv^{-1}(u),\ \text{where } Z=\prod_i z_i^{\check{\alpha}_i},\ v(u)\in G(u).$$

We assume Z is regular semisimple. In that case there are W_G Miura opers for a given oper.

In the extreme case $Z=1$ we have G/B Miura opers for a given oper.

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Nondegenerate Z-twisted Miura $(G,\hbar)$-opers and QQ-systems

Nondegeneracy conditions (see detailed discussion in our paper):

$$A(u)=\prod_i g_i^{\check{\alpha}_i}(u)e^{\frac{\Lambda_i(u)}{g_i(u)}e_i},\quad g_i(u)=z_i\frac{y_i(\hbar u)}{y_i(u)}$$

Each $y_i(u)$ is a polynomial, and for all i,j,k with $i\neq j$ and $a_{ik}\neq 0, a_{jk}\neq 0$, the zeros of $y_i(u)$ and $y_j(u)$ are $\hbar$-distinct from each other and from the zeros of $\Lambda_k(u)$.

Explicit formula for $v(u)$, such that

$$A(u)=v(u\hbar)Zv(u)^{-1}$$

is:

$$v(u)=\prod_{i=1}^r y_i(u)^{\check{\alpha}_i}\prod_{i=1}^r e^{-\frac{Q_-^j(u)}{Q_+^j(u)}e_i}\dots,$$

where the dots stand for the exponentials of higher commutator terms.

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Z-twisted Miura $(SL(2),\hbar)$-opers with regular singularities

$$\hbar-\text{connection}:\quad A(u)=\begin{pmatrix} g(u) & \Lambda(u)\\ 0 & g(u)^{-1}\end{pmatrix},$$

$$A(u)=v(\hbar u)Zv(u)^{-1},\qquad Z=\begin{pmatrix} z & 0\\ 0 & z^{-1}\end{pmatrix}.$$

Trivializing element:

$$v(u)=\begin{pmatrix} y(u) & 0\\ 0 & y(u)^{-1}\end{pmatrix}\begin{pmatrix} 1 & -\frac{Q_-(u)}{Q_+(u)}\\ 0 & 1\end{pmatrix}=\begin{pmatrix} y(u) & -y(u)\frac{Q_-(u)}{Q_+(u)}\\ 0 & y(u)^{-1}\end{pmatrix},$$

where $Q_+(u)$ and $Q_-(u)$ are relatively prime polynomials such that $Q_+(u)$ is a monic polynomial,

$$g(u)=zy(\hbar u)y(u)^{-1}.$$

and

$$\Lambda(u)=y(u)y(\hbar u)\left(\zeta\frac{Q_-(u)}{Q_+(u)}-\zeta^{-1}\frac{Q_-(u\hbar)}{Q_+(u\hbar)}\right).$$

Nondegeneracy: $y(u)=Q_+(u)$.

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Main theorem

That leads to the expression of Miura $(G, \hbar)$-oper connection:

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = z_i \frac{Q^i_+(\hbar u)}{Q^i_+(u)}.$$

Theorem. There is a one-to-one correspondence between the set of nondegenerate Z-twisted Miura $(G, \hbar)$-opers and the set of nondegenerate polynomial solutions of the QQ-system:

$$\widetilde{\xi}_i Q^i_-(u) Q^i_+(\hbar u) - \xi_i Q^i_-(\hbar u) Q^i_+(u) = \Lambda_i(u) \prod_{j>i} \left[Q^j_+(\hbar u)\right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(u)\right]^{-a_{ji}}, \qquad i = 1, \dots, r,$$

where $\widetilde{\xi}_i = z_i \prod_{j>i} z_j^{a_{ji}}$, $\xi_i = z_i^{-1} \prod_{j<i} z_j^{-a_{ji}}$.

In ADE case this QQ-system correspond to the Bethe ansatz equations. Beyond simply-laced case: "folded integrable models". See recent preprint of E. Frenkel, D. Hernandez, N. Reshetikhin.

Originally operators

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = z_i \frac{Q_+^i(\hbar u)}{Q_+^i(u)},$$

where $Q_\pm(u)$ are the solution of QQ-systems, were introduced by Mukhin, Varchenko'05 in the additive case with $Z = 1$.

They also introduced the following $\hbar$-gauge transformation of the $\hbar$-connection A:

$$A \mapsto A^{(i)} = e^{\mu_i(\hbar u) f_i} A(u) e^{-\mu_i(u) f_i}, \quad \text{where} \quad \mu_i(u) = \frac{\prod_{j \neq i} \left[Q_+^j(u)\right]^{-a_{ji}}}{Q_+^i(u) Q_-^i(u)}.$$

Then $A^{(i)}(u)$ can be obtained from $A(u)$ by substituting in formula for $A(u)$:

$$Q_+^j(u) \mapsto Q_+^j(u), \qquad j \neq i,$$
$$Q_+^i(u) \mapsto Q_-^i(u), \qquad Z \mapsto s_i(Z).$$

Altogether these transformation generate the "full" QQ-system.

$SL(r+1)$ opers: explicit formula

QQ-system:

$$\xi_{i+1} Q_i^+(\hbar u) Q_i^-(u) - \xi_i Q_i^+(u) Q_i^-(\hbar u) = \Lambda_i(u) Q_{i-1}^+(u) Q_{i+1}^+(\hbar u)\,, \ i = 1, \dots, r$$

$$\xi_1 = \frac{1}{z_1}, \quad \xi_2 = \frac{z_1}{z_2}, \quad \dots \quad \xi_r = \frac{z_{r-1}}{z_r}, \quad \xi_{r+1} = \frac{1}{z_r}\,,$$

Introducing notation:

$$\phi_i(u) = \frac{Q_i^-(u)}{Q_i^+(u)}\,, \qquad \rho_i(u) = \Lambda_i(u) \frac{Q_{i-1}^+(u) Q_{i+1}^+(\hbar u)}{Q_i^+(u) Q_i^+(\hbar u)}\,.$$

We have a sequence of quantum Bäcklund transformations:

$$\xi_i\, \phi_i(u) - \xi_{i+1}\, \phi_i(\hbar u) = \rho_i(u)\,, \ \ i = 1, \dots, r,$$

$$\xi_i\, \phi_{i,i+1}(u) - \xi_{i+2}\, \phi_{i,i+1}(\hbar u) = \rho_{i+1}(u)\phi_i(u)\,, \ \ i = 1, \dots, r-1,$$

$$\dots\dots$$

$$\xi_i\, \phi_{i,\dots,r-2+i}(u) - \xi_{r-2+i}\, \phi_{i,\dots,r-1+i}(\hbar u) = \rho_{r-1}(u)\phi_{i,\dots,r-3+i}(u)\,, i = 1, 2$$

$$\xi_1 \phi_{1,\dots,r}(u) - \xi_{r+1} \phi_{1,\dots,r}(\hbar u) = \rho_r(u)\phi_{1,\dots,r-1}(u)\,,$$

where

$$\phi_{i,\dots,j}(u) = \frac{Q_{i,\dots,j}^-(u)}{Q_j^+(u)}.$$

For Z-twisted oper:

$$A(u) = v^{-1}(\hbar u) Z v(u)$$

$$v(u) = \begin{pmatrix} \frac{1}{Q_1^+(u)} & \frac{Q_1^-(u)}{Q_2^+(u)} & \frac{Q_{12}^-(u)}{Q_3^+(u)} & \dots & \frac{Q_{1,\dots,r-1}^-(u)}{Q_r^+(u)} & Q_{1,\dots,r}^-(u) \\ 0 & \frac{Q_1^+(u)}{Q_2^+(u)} & \frac{Q_2^-(u)}{Q_3^+(u)} & \dots & \frac{Q_{2,\dots,r-1}^-(u)}{Q_r^+(u)} & Q_{2,\dots,r}^-(u) \\ 0 & 0 & \frac{Q_2^+(u)}{Q_3^+(u)} & \dots & \frac{Q_{3,\dots,r-1}^-(u)}{Q_r^+(u)} & Q_{3,\dots,r}^-(u) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \frac{Q_{r-1}^+(u)}{Q_r^+(u)} & Q_r^-(u) \\ 0 & \dots & \dots & \dots & 0 & Q_r^+(u) \end{pmatrix}.$$

$SL(r+1)$ opers: alternative definition

A meromorphic $(GL(r+1), \hbar)$-*oper* on $\mathbb{P}^1$ is a triple $(A, E, \mathcal{L}_\bullet)$, where E is a vector bundle of rank $r+1$ and $\mathcal{L}_\bullet$ is the corresponding complete flag of the vector bundles,

$$\mathcal{L}_{r+1} \subset ... \subset \mathcal{L}_{i+1} \subset \mathcal{L}_i \subset \mathcal{L}_{i-1} \subset ... \subset E = \mathcal{L}_1,$$

where $\mathcal{L}_{r+1}$ is a line bundle, so that $A \in Hom_{\mathcal{O}_U}(E, E^\hbar)$ satisfies the following conditions:

- $A \cdot \mathcal{L}_i \subset \mathcal{L}_{i-1}$,
- There exists Zariski open U, such that $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i+1} \to \mathcal{L}_{i-1}/\mathcal{L}_i$ is an isomorphism on $U \cap M_\hbar{}^{-1}(U)$.

An $(SL(r+1), \hbar)$-*oper* is a $(GL(r+1), \hbar)$-oper with the condition that $det(A) = 1$ on $U \cap M_\hbar{}^{-1}(U)$.

Regular singularities: $\bar{A}_i$ allowed to have zeroes at zeroes of $\Lambda_i(u)$.

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A meromorphic $(GL(r+1), \hbar)$-*oper* on $\mathbb{P}^1$ is a triple $(A, E, \mathcal{L}_\bullet)$, where E is a vector bundle of rank $r+1$ and $\mathcal{L}_\bullet$ is the corresponding complete flag of the vector bundles,

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where $\mathcal{L}_{r+1}$ is a line bundle, so that $A \in Hom_{\mathcal{O}_U}(E, E^\hbar)$ satisfies the following conditions:

- $A \cdot \mathcal{L}_i \subset \mathcal{L}_{i-1}$,
- There exists Zariski open U, such that $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i+1} \to \mathcal{L}_{i-1}/\mathcal{L}_i$ is an isomorphism on $U \cap {M_\hbar}^{-1}(U)$.

An $(SL(r+1), \hbar)$-*oper* is a $(GL(r+1), \hbar)$-oper with the condition that $det(A) = 1$ on $U \cap {M_\hbar}^{-1}(U)$.

Regular singularities: $\bar{A}_i$ allowed to have zeroes at zeroes of $\Lambda_i(u)$.

$SL(2)$ example and $\hbar$-Wronskians

A Z-twisted $(SL(2),\hbar)$-oper on $\mathbb{P}^1$ with regular singularities is a triple $(E, A, \mathcal{L})$:

- (E, A) is a $(SL(2), \hbar)$-connection
- $\mathcal{L}$ is a line subbundle so that $\bar{A} : \mathcal{L} \to (E/\mathcal{L})^{\hbar}$ is an isomorphism except for zeroes of $\Lambda(u)$.
- A is gauge equivalent to $Z \in H$

Equivalently:

$$s(\hbar u) \wedge A(u)s(u) = \Lambda(u),$$

where $s(u)$ is a section of $\mathcal{L}$.

Choosing trivialization $s(u) = \begin{pmatrix} Q_-(u) \\ Q_+(u) \end{pmatrix}$, we obtain that above condition is the QQ-system:

$$zQ_-(u)Q_+(\hbar u) - z^{-1}Q_-(\hbar u)Q_+(u) = \Lambda(u).$$

$SL(r+1)$-Wronskians and QQ-systems

More general Wronskians:

$$\mathcal{D}_k(s) = e_1 \wedge \cdots \wedge e_{r+1-k} \wedge s(u) \wedge Z^{-1} s(\hbar u) \wedge \cdots \wedge Z^{1-k} s(\hbar^{k-1} u) = \alpha_k W_k(u) \mathcal{V}_k(u)\,,$$

where

$$\mathcal{V}_k(u) = \prod_{a=1}^{r_k} (u - w_{k,a})\,,$$

and

$$W_k(s) = P_1 \cdot P_2^{(1)} \cdot P_3^{(2)} \cdots P_{k-1}^{(k-2)}, \quad P_i = \Lambda_r \Lambda_{r-1} \cdots \Lambda_{r-i+1}$$

We used the notation $f^{(j)}(u) = f(\hbar^j u)$ above.

One can identify: $\mathcal{V}_k(u) \equiv Q_k^+(u)$ and $Q^-_{i,\dots,j}(u)$ with other minors.

The bilinear relations for the extended QQ-system are nothing but Plücker relations for minors in the $\hbar$-Wronskian matrix.

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Quantum-classical duality via $(SL(r+1), \hbar)$-opers

Take section of the line bundle $\mathcal{L}_{r+1}$ in complete flag $\mathcal{L}_\bullet$:

$$s(u) = \begin{pmatrix} s_1(u) \\ s_2(u) \\ s_3(u) \\ \vdots \\ s_r(u) \\ s_{r+1}(u) \end{pmatrix} = \begin{pmatrix} Q^-_{1,\dots,r}(u) \\ Q^-_{2,\dots,r}(u) \\ Q^-_{3,\dots,r}(u) \\ \vdots \\ Q^-_r(u) \\ Q^+_r(u) \end{pmatrix}.$$

Interesting case (XXZ chain corresponding to defining representations):

- Polynomials are of degree 1
- Only $\Lambda_1(u) = \prod_i (u - a_i)$ is nontrival

Identification:

- roots of $s_i(u)$ with momenta
- $\xi_i = z_i / z_{i-1}$ with coordinates,

Space of functions on Z-twisted Miura $(SL(r+1, \hbar)$-opers $\leftrightarrow$ space of functions on the intersection of two Lagrangian subvarieties in trigonometric Ruijsenaars-Schneider (tRS) phase space.

Bethe equations $\leftrightarrow \{H_k = f_k(\{a_i\})\}$

Here H_k are tRS Hamiltonians and f_i are elementary symmetric functions of a_i.

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$\hbar$-Opers for $\widehat{\widehat{\mathfrak{gl}}}(1)$ and Bethe ansatz

Let us "complete" Miura $(SL(r+1), \hbar)$-opers by $(\overline{GL}(\infty), \hbar)$:

$$A(u) = \prod_{i=+\infty}^{-\infty} g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = z_i \frac{Q_+^i(\hbar u)}{Q_+^i(u)}.$$

Infinite-dimensional QQ-system:

$$\xi_{i+1} Q_i^+(\hbar u) Q_i^-(u) - \xi_i Q_i^+(u) Q_i^-(\hbar u) = \Lambda_i(u) Q_{i-1}^+(u) Q_{i+1}^+(\hbar u)\,, \; i = 1, \dots, r,$$

where $\xi_i = z_i / z_{i-1}$.

Impose periodic condition: $VA(u)V^{-1} = \xi A(pu)$, where V corresponds to automorphism of Dynkin diagram $i \to i+1$.

V can be actually relized as an "infinite" Coxeter element of standard order.

That corresponds to $Q_j^{\pm}(u) = Q^{\pm}(p^j u)$, $\Lambda_j(u) = \xi^j \Lambda(u)$, $\xi_j = \xi^j$:

$$\xi Q^+(\hbar u) Q^-(u) - Q^+(u) Q^-(\hbar u) = \Lambda(u) Q^+(up^{-1}) Q^+(\hbar p u)$$

Generalized minors

Big cell in Bruhat decomposition: $G_0 = N_- H N_+$. For a given element $g \in G_0$ we can write it as $g = n_- \ h \ n_+$. Diagonal minors:

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}, \quad i = 1, \dots, r; \quad h \cdot v_{\omega_i} = [h]^{\omega_i} v_{\omega_i}$$

For $u, v \in W_G$, we define a regular function $\Delta_{u\omega_i, v\omega_i}$ on G by setting

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\tilde{u}^{-1} g \tilde{v}).$$

Action of the group element on the highest weight vector in

$$g \cdot \nu_{\omega_i} = \sum_{w \in W} \Delta_{w \cdot \omega_i, \omega_i}(g) \tilde{w} \cdot \nu_{\omega_i} + \dots,$$

where dots stand for the vectors, which do not belong to the orbit $\mathcal{O}_{W_G} = W_G \cdot \mathbb{C}\nu_{\omega_i}$.

Let $u, v \in W_G$, such that for $i \in \{1, \dots, r\}$, $\ell(uw_i) = \ell(u) + 1$, $\ell(vw_i) = \ell(v) + 1$. Then

$$\Delta_{u\cdot\omega_i, v\cdot\omega_i} \Delta_{uw_i\cdot\omega_i, vw_i\cdot\omega_i} - \Delta_{uw_i\cdot\omega_i, v\cdot\omega_i} \Delta_{u\cdot\omega_i, vw_i\cdot\omega_i} = \prod_{j \neq i} \Delta_{u\cdot\omega_j, v\cdot\omega_j}^{-a_{ji}},$$

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Miura $(G,\hbar)$-opers and generalized minors

Change notations: $B_- \to B_+$:

$$A(u) = \prod_j \left[\zeta_j \frac{Q_+^j(\hbar u)}{Q_+^j(u)}\right]^{-\check{\alpha}_j} e^{\frac{\Lambda_j(u) Q_+^j(z)}{\zeta_j Q_+^j(\hbar u)} f_j}$$

where $A(z) = v(\hbar z) Z v(z)^{-1}$ and

$$v(u) = \prod_{i=1}^r \left[Q_+^i(u)\right]^{-\check{\alpha}_i} \prod_{i=1}^r e^{-\frac{Q_-^i(u)}{Q_+^i(u)} f_i} \dots,$$

so that

$$\Delta_{w\cdot\omega_i,\omega_i}(v^{-1}(u)) = Q_+^{w,i}(u)$$

for any $w \in W$, because of Bäcklund transformations.

Generalized $(G, \hbar)$-Wronskians

(Local) definition of generalized $(G, \hbar)$-Wronskian:

$$Z^{-1}\mathscr{G}(\hbar u)\cdot \nu_{\omega_i} = \mathscr{G}(u)\cdot s_\Lambda(u)^{-1}\cdot \nu_{\omega_i},\ \ i = 1, \dots, r;$$

$$s_\Lambda(u)^{-1} = \prod_i^{\mathrm{inv}} s_i \Lambda_i^{\check{\alpha}_i}$$

The fundamental relation for $\mathscr{G}(u)$ is equivalent to the relation

$$\Delta_{w\cdot\omega_i, c^{-1}\cdot\omega_i}(\mathscr{G}(u)) = \left[\prod_j z_j^{\langle \check{\alpha}_j, w\cdot\omega_i\rangle}\right] F_i(u)\Delta_{w\cdot\omega_i,\omega_i}(\mathscr{G}(\hbar u))$$

The following relation between minors:

$$\Delta_{\omega_i,\omega_i}\Delta_{w_i\cdot\omega_i, c^{-1}\cdot\omega_i} - \Delta_{w_i\cdot\omega_i,\omega_i}\Delta_{\omega_i, c^{-1}\cdot\omega_i} = \prod_{j<i=i_l}\Delta^{-a_{ji}}_{\omega_j, c^{-1}\cdot\omega_j}\prod_{j>i=i_l}\Delta^{-a_{ji}}_{\omega_j,\omega_j}, \qquad i = 1, \dots, r,$$

where the ordering is taken with respect to decomposition of $c^{-1} = w_{i_1}\dots, w_{i_l}, \dots, w_{i_r}$, converts into the QQ-system, once we identify

$$\Delta_{\omega_i,\omega_i}(\mathscr{G}(u)) \to Q^i_+(u), \quad \Delta_{w_i\omega_i,\omega_i}(\mathscr{G}(u)) \to Q^i_-(u).$$

Relation between $(G, \hbar)$-Wronskians and $(G, \hbar)$-opers

We say that generalized $(G, \hbar)$-Wronskian with regular singularities is *nondegenerate* if $\Delta_{w\cdot\omega_i,\omega_i}(\mathscr{G}(u))$ are nonzero polynomials for all $w \in W$ and $i = 1, \ldots, r$. For all i, j, k with $i \neq j$ and $a_{ik}, a_{jk} \neq 0$, the zeros of $\Delta_{\omega_i,\omega_i}(\mathscr{G}(u))$ and $\Delta_{s_i\omega_i,\omega_i}(\mathscr{G}(u))$ are $\hbar$-distinct from each other, and also zeroes of $\Delta_{w\cdot\omega_i,\omega_i}(\mathscr{G}(u))$ are $\hbar$-distinct from the zeros of $\{\Lambda_k(u)\}_{k=1,\ldots,r}$ for all i.

Gaussian decomposition: $\mathscr{G}(u) = v(u)u(u)$, $u(u) \in N_+(u)$ defines a nondegenerate Z-twisted Miura $(G, \hbar)$-oper connection with regular singularities by the formula $A(z) = v^{-1}(\hbar u)Zv(u)$.

Theorem. There is a one-to-one correspondence between classes of nondegenerate generalized $(G, \hbar)$-Wronskians with regular singularities and nondegenerate Z-twisted Miura $(G, \hbar)$-opers with regular singularities.

Fixing an element in the class: Universal $(G, \hbar)$-Wronskian:

$$\mathscr{W}(\hbar^{k+1}u)\nu^+_{\omega_i} = Z^k\mathscr{W}(u)s^{-1}(u)s^{-1}(\hbar u)\ldots s^{-1}(\hbar^k u)\nu^+_{\omega_i},$$
$$i = 1, \ldots, r, \qquad k = 0, 1, \ldots, h/2 - 1,$$

where h is a Coxeter number and $c^{h/2} = w_0$.

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Applications

- Quantum/classical duality: duality between Bethe equations and multiparticle systems

  P. Koroteev, D. Sage, A. Z., *(SL(N),q) -opers, the q-Langlands correspondence, and quantum/classical duality*, Comm. Math. Phys., 381 (2021) 641-672, arXiv:1811.09937

- Quantum equivariant K-theory of Nakajima quiver varieties and 3D mirror symmetry

  P. Koroteev, A.Z.,*Toroidal q-Opers*, to appear in Journal of the Institute of Mathematics of Jussieu, in press, arXiv:2007.11786

  P. Koroteev, A. Z., *3d Mirror Symmetry for Instanton Moduli Spaces*, arXiv:2105.00588

- Applications to ODE/IM correspondence: affine G-opers and $(G, \hbar)$-opers

  E. Frenkel, P. Koroteev, A.Z., in progress

Thank you!