Workshop on

Groups and Geometries

November 29 – December 3 2021

Matrix Institute

© 2021, the organizers:
Alice Devillers
James Parkinson
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Anne Thomas
https://www.matrix-inst.org.au/events/groups-and-geometries/

Contents

1 Schedule

All times Melbourne time (GMT+11).

All talks will take place on Zoom.

Please click this URL to start or join: https://uwa.zoom.us/j/89872914039?pwd=VUZxbnJraUdEV3ZRREZjWERBQTJRQT09

Password: 532019

	Monday	Tuesday	Wednesday	Thursday	Friday
12:00 – 12:25	*Arun Ram* 10	*Stephan Tornier* 11	*Alex Bishop* 4	*Matthew Conder* 5	*Michael Giudici* 7
12:30 – 12:55	*Youming Qiao* 9	*Adam Piggott* 8	*Yeeka Yau* 12	*Alex Elzenaar* 6	workshop picture in Zoom

Problems/discussions in Sococo (program will be updated during the week).

There is a Zoom link in each room, so we can use a whiteboard, if necessary.

	Monday	Tuesday	Wednesday	Thursday	Friday
from 11:00 Room A					Youming's problem
from 14:00 Room A	Arun's problem 16	Stephan's problem 18		Stephan's problem **at 3pm**	
from 14:00 Room B	Youming's problem 15	Jeroen's problem 17		Arun's problem	Jeroen's problem
from 14:00 Room C	Tomasz's problem 14		Tomasz's problem		
16:00 Room A		Recap meeting			

2 Contributed talks

Kazhdan-Lusztig Polynomials and Bruhat Posets

Alex Bishop

University of Sydney

For the pairs of elements in a Coxeter group we may define polynomials, known as the Kazhdan-Lusztig polynomials, which encode how to obtain a 'nice' basis for a ring known as the Hecke algebra of the Coxeter group. Although there are several formulae known for computing these polynomials, the process quickly becomes computationally infeasible even for small examples. Our goal is to find explicit formulae for the Kazhdan-Lusztig polynomials of certain classes of (infinite) Coxeter groups. In this talk, I will give a quick introduction to the theory of Kazhdan-Lusztig polynomials, and describe why they are interesting and important. I will also provide some known results and conjectures on their connection with interval posets induced by Bruhat order. I will not assume any background in representation theory, and this talk will be as self-contained as possible.

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Discrete two-generated subgroups of $\mathrm{PSL}(2, \mathbb{Q}_p)$

Matthew Conder

University of Auckland

(Joint work with Jeroen Schillewaert)

The group $\mathrm{PSL}(2, \mathbb{Q}_p)$ acts by isometries on the corresponding Bruhat-Tits tree. By studying this action, we give necessary conditions for a two-generated subgroup of $\mathrm{PSL}(2, \mathbb{Q}_p)$ to be discrete. We discuss a special class of subgroups which are the only obstruction to obtaining a full classification of discrete two-generated subgroups of $\mathrm{PSL}(2, \mathbb{Q}_p)$.

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Approximating the Riley slice exterior

Alex Elzenaar

University of Auckland

(Joint work with Gaven Martin and Jeroen Schillewaert)

It has been known since at least the time of Poincaré that isometries of 3-dimensional hyperbolic space H^3 can be represented by 2×2-matrices over the complex numbers: the matrices represent fractional linear transformations on the sphere at infinity, and hyperbolic space is rigid enough that every hyperbolic motion is determined by such an action at infinity. A discrete subgroup of $SL(2,\mathbb{C})$ is called a Kleinian group; the quotient of H^3 by the action of such a group is an orbifold, and its boundary at infinity is a Riemann surface. The set of all Kleinian groups whose corresponding surface is supported on a 4-punctured sphere is called the Riley slice; it is naturally embedded in $\mathbb{C}$, and L. Keen and C. Series in the early 1990s studied this embedding via a family of polynomials which gave a foliation (local product decomposition) of the slice. We will discuss the Keen–Series theory and its extension to the 4-coned sphere together with some recent additional results (joint with Gaven Martin and Jeroen Schillewaert) of a combinatorial flavour and some applications.

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Orbits of Sylow subgroups of finite permutation groups

Michael Giudici

University of Western Australia

(Joint work with John Bamberg, Alexander Bors, Alice Devillers, Cheryl Praeger, Gordon Royle)

We say that a finite group G acting on a set Ω has *Property* $(*)_p$ for a prime p if P_ω is a Sylow p-subgroup of G_ω for all $\omega \in \Omega$ and Sylow p-subgroups P of G, that is the operations of taking point stabilisers and finding Sylow p-subgroups commute. Property $(*)_p$ arose in the recent work of Tornier (2018) on local Sylow p-subgroups of Burger-Mozes groups. I will discuss recent joint work with John Bamberg, Alexander Bors, Alice Devillers, Cheryl E. Praeger and Gordon F. Royle that studies the permutation groups with Property $(*)_p$ and which includes a complete classification of the finite 2-transitive groups with this property.

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Rewriting systems, triangles in Cayley graphs, and the isomorphism problem for plain groups

Adam Piggott

Australian National University

(Joint work with Heiko Dietrich, Murray Elder and Youming Qiao)

Presenting groups via nice rewriting systems ensures efficient solutions to the word problem. We discuss recent progress in a program to understand exactly which groups may be presented by length-reducing rewriting systems.

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Optimisation meets CAT(0) spaces

Youming Qiao

University of Technology Sydney

In this talk we will examine a novel connection between optimisation and CAT(0) spaces, studied by Hiroshi Hirai and his collaborators. Briefly speaking, a classical topic in optimisation is to study submodular functions on Boolean lattices. Only recently, the study of submodular functions on modular lattices started, where CAT(0) spaces arise naturally as the domain of convex relaxations of such functions.

One application of this approach is to solve the so-called non-commutative rank problem. This problem was proposed by P. M. Cohn in the 1970s in the study of skew fields and received quite some attention in theoretical computer science in the past few years. Now it admits three solutions, through tools and results ranging from invariant theory, quantum information, and linear algebra. The solution through submodular functions on modular lattices is the newest, with a great potential for further development and new applications.

I will conclude this talk with some questions for future study. This talk will be largely expository.

References

[1] M. Hamada, H. Hirai: Computing the nc-rank via discrete convex optimization on CAT(0) spaces, SIAM J. Appl. Algebra 5 (2021), 455–478. https://arxiv.org/pdf/2012.13651.pdf

[2] G. Ivanyos, Y. Qiao, K. V. Subrahmanyam: Constructive non-commutative rank computation is in deterministic polynomial time. Comput. Complex. 27(4): 561-593 (2018) https://arxiv.org/abs/1512.03531

[3] A. Garg, L. Gurvits, R. M. de Oliveira, A. Wigderson: Operator Scaling: Theory and Applications. Found. Comput. Math. 20(2): 223-290 (2020) https://arxiv.org/abs/1511.03730

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Transvections and Hecke algebras

Arun Ram

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(Joint work with Persi Diaconis and Mackenzie Simper)

Let V be an n dimensional vector space over the finite field $\mathbb{F}_q$. We study the walk on permutations which arises from repeated transvections acting on maximal flags of subspaces of V (the permutation is the type of the flag with respect to the standard flag). We show that, up to addition of a multiple of the identity (i.e. up to holding), the walk is equivalent to repeated mutliplication by the sum of he Hecke algebra basis elements corresponding the transpositions of the symmetric group. We provide an explicit formula which also determines the holding. The transvections walk on GL_n is studied by Hildebrand who shows that $n + c$ steps are necessary and sufficient for convergence to stationarity. The walk on permutations is shown to mix faster.

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A permutation group problem relating to groups acting on trees

Stephan Tornier

University of Newcastle

Let Ω be a finite set and let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group. Consider the action of G on $K_G := \prod_{\omega\in\Omega} G_\omega$ given by $g \cdot (g_\omega)_{\omega\in\Omega} := (gg_{g^{-1}\omega}g^{-1})_{\omega\in\Omega}$, where $G_\omega = \{g \in G \mid g\omega = \omega\} \leq G$. What are all the G-invariant subgroups of K_G? In the spirit of Burger–Mozes and T., every G-invariant subgroup K of K_G gives rise to a *generalised universal group* acting on a regular tree of degree $|\Omega|$ whose local action on balls of radius 2 around vertices relates to K. Generalised universal groups are important examples of groups acting on trees and play a role in the broader theory of the latter. The problem thus translates to finding more (families of) examples of them and thereby contribute to their classification.

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Cone Types, Automata and Regular Partitions of Coxeter Groups

Yeeka Yau

University of Sydney

(Joint work with James Parkinson)

Coxeter groups were famously proven to be automatic by Brink and Howlett in 1993 and the automaticity of these groups has been an area of continued interest since. In this talk, we give a brief history and summary of recent developments in this area, and we introduce the theory of Regular Partitions of Coxeter groups.

We show that Regular Partitions are essentially equivalent to the class of automata (not necessarily finite state) recognising the language of reduced words in the Coxeter group and explain how it gives a fundamentally free construction of automata. As a further application, we prove that each cone type in a Coxeter group has a unique minimal length representative. This result can be seen as an analogue of Shi's classical result that each component of the Shi arrangement of an affine Coxeter group has a unique minimal length element.

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3 Proposed problems

Small Generalised Quadrangles

Tomasz Popiel

Auckland University

The finite generalised quadrangles (GQs) of order (s,t) are classified only when $s = 2$ or $s = 3$ (or, dually, $t = 2$ or $t = 3$). Even for $s = 4$, only partial results are known, with perhaps the main open question being: do there exist GQs of order $(s,t) = (4,11)$ or $(4,12)$? Some partial results are known about putative examples in these cases, e.g. a GQ of order $(4,12)$ would need to be point-intransitive. I have previously discussed the problem of determining the existence of GQs of these orders with a few of the participants, and a good amount of progress has been made using an elementary combinatorial approach mimicking arguments used in other 'small' cases in e.g. the book of Payne and Thas. These investigations have hit a dead end, but given the concentration of expertise at this meeting, it seems like as good an opportunity as any to give the problem(s) one more try. We could start with the existing partial results (which I could circulate if there is interest), or start from scratch and try a different approach. Another option would be to look at GQs of order $(5,7)$ or $(6,6)$, which are the next smallest (and arguably more interesting) open cases.

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Optimisation meets $\mathrm{CAT}(0)$ spaces

Youming Qiao

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Submodular functions is a major topic in optimisation. Such functions are defined with respect to the Boolean lattices. Recently, submodular functions on modular lattices are studied, which naturally leads to $\mathrm{CAT}(0)$ spaces and optimisation over such spaces.

This approach was recently developed and used successfully to attack the so-called non-commutative rank problem by Hiroshi Hirai and his collaborators. The non-commutative rank problem was proposed by P. M. Cohn in the 1970's, and it received quite some attention in theoretical computer science recently. Now there are three solutions for this problem (one of them by me and my collaborators).

However, I believe that this approach, namely submodular functions on modular lattices + optimisation over $\mathrm{CAT}(0)$ spaces, should be more versatile and deserves further study. I also have some candidate problems from group theory and linear algebra that may be solved by this approach.

I will be happy to introduce the background in more detail, explain my (probably superficial) understanding of this approach, and propose some target problems that may be amenable to this approach. We can then start to understand Hirai and his collaborators' papers better (see below) and explore developing this approach and finding applications.

Reference: M. Hamada, H. Hirai: Computing the nc-rank via discrete convex optimization on $\mathrm{CAT}(0)$ spaces, SIAM Journal on Applied Geometry and Algebra 5 (2021), 455–478, `https://arxiv.org/pdf/2012.13651.pdf`

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Transvections and Hecke algebras

Arun Ram

University of Melbourne

Let λ be a partition and let J_λ be the unipotent matrix in Jordan normal form with block sizes given by λ. Let C_λ be the sum of the elements in the conjugacy class of J_λ in $G = GL_n(\mathbb{F}_q)$ (an element of the group algebra).

The action of G on the maximal flags (maximal chains in the lattice of subspaces of $\mathbb{F}_q^n$) gives a representation 1_B^G of G. There is a unique element Z_λ in the Hecke algebra which acts on 1_B^g in the same way that C_λ does. The elements Z_λ give a basis of the centre of the Hecke algebra.

Problems:

(a) Show that the transition matrix between $\{Z_\lambda\}$ and Andrew Francis? basis of the centre of the Hecke algebra is triangular. Determine this transition matrix explicitly.

(b) For each λ, the action of C_λ, with respect to the standard (double coset) basis of the Hecke algebra, gives a Markov chain on permutations.Determine the convergence of this Markov chain.

In joint work with Persi Diaconis and Mackenzie Simper we have recently done this analysis when λ is the partition (21^{n-2}) (the conjugacy class of transvections).

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Proper proximality for buildings

Jeroen Schillewaert

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Proper proximality of a countable group is a tool to study rigidity properties of certain von Neumann algebras associated to groups or ergodic group actions [1]. In [2] the proper proximality of many groups acting on nonpositively curved spaces was established.

A countable group G is properly proximal if there exist a compact G-space K that does not carry any G-invariant probability measure, and a diffuse probability measure η on K, such that $\lim_{g \to \omega}((gh) \cdot \eta - g \cdot \eta) = 0$ in the weak-$*$ topology for every non-principal ultrafilter ω on G and every $h \in G$.

In [2, Theorem 4] the following was proved for **simplicial** buildings.

Theorem 1 *Let G be a countable group acting properly, minimally, non-elementarily by isometries on a locally finite thick affine building. Then G is properly proximal.*

Problem: Can we extend this result to non-discrete affine buildings introduced by Tits [6] and studied in detail by Kleiner and Leeb [3] and Parreau [5]? This requires two main lines of argument, interesting in their own right, namely the extension of (1) building theoretic arguments from the simplicial and thus more combinatorial setting to the non-discrete metric setting, and (2) extending the machinery developed by Parkinson [4] of the spherical harmonic analysis on affine buildings. This is likely the more challenging part, a good place start is the case of locally compact $\mathbb{R}$-trees, which has the added advantage of not requiring any knowledge of buildings.

References

[1] R. Boutonnet, A. Ioana, and J. Peterson. Properly proximal groups and their von Neumann algebras. arXiv:1809.01881, 2018.

[2] Camille Horbez, Jingyin Huang and Jean Lécureux, Proper proximality in non-positive curvature, https://arxiv.org/pdf/2005.08756.pdf

[3] B.Kleiner and B. Leeb, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings, *Pub. Math. I.H.E.S* **86** (1997), 115–197.

[4] J. Parkinson Spherical harmonic analysis on affine buildings. *Math. Z.* **253**, 571–606 (2006)

[5] A. Parreau, Immeubles affines: construction par les normes et étude des isométries, Crystallographic Groups and Their Generalizations (Kortrijk, 1999), *Contemporary Mathematics* **262**, American Mathematical Society, Providence (2000), 263–302.

[6] J. Tits, Immeubles de type affine, In *Buildings and the Geometry of Diagrams, Springer Lecture Notes* **1181** (Rosati ed.), Springer Verlag, 1986, pp. 159–190.

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Groups Acting on Trees With Prescribed Local Action

Stephan Tornier

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Problem. Let Ω be a finite set and let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group. Consider the action of G on $K_G := \prod_{\omega\in\Omega} G_\omega$ given by $g \cdot (g_\omega)_{\omega\in\Omega} := (g g_{g^{-1}\omega} g^{-1})_{\omega\in\Omega}$, where $G_\omega = \{g \in G \mid g\omega = \omega\} \leq G$. What are all the G-invariant subgroups of K_G?

This problem is based on and of interest to me due to [2, Proposition 3.23].

Context. Let $|\Omega| = d \in \mathbb{N}_{\geq 3}$ and consider the d-regular tree T_d. In addition, fix a finite tree $B_{d,2}$ which is isomorphic to a ball of radius 2 around a vertex in T_d. Using the notation introduced above, every G-invariant subgroup K of K_G gives rise to subgroup $\Sigma(K)$ of the automorphism group $\mathrm{Aut}(B_{d,2})$ of $B_{d,2}$. This group, in turn, gives rise to a (topological) subgroup $\mathrm{U}_2(\Sigma(K))$ of the automorphism group $\mathrm{Aut}(T_d)$ of T_d, termed a generalised universal group. Such groups are important examples of groups acting on trees and play a role in the broader theory of the latter. The problem thus translates to finding more (families of) examples of them.

Remarks. Let $G \leq \mathrm{Sym}(\Omega)$ and $K_G = \prod_{\omega\in\Omega} G_\omega$ be as above. Clearly, the trivial subgroup of K_G as well as K_G itself are G-invariant.

- $\Gamma := \{e\} \leq K_G$,
- $\Phi := K_G$.

Assume that G is transitive. Fix $\omega_0 \in \Omega$ and let $\pi_\omega : K_G \to G_{\omega_0}$ denote the natural projection. A G-invariant subgroup K for which $\pi_{\omega_0}|_K$ is an isomorphism is called

- $\Delta \cong G_{\omega_0}$ (may or may not exist).

Theorem 2 *Let $G \leq \mathrm{Sym}(\Omega)$ be primitive and G_ω simple non-abelian for all $\omega \in \Omega$. Then Γ, Φ and Δ are the only (potential) G-invariant subgroups of K_G.*

Proof (sketch): Suppose $K \leq K_G$ is G-invariant. Let $\pi_\omega : K_G \to G_\omega$ denote the natural projection. By G-invariance, the group $\pi_\omega(K)$ is normal in G_ω. Since G_ω is simple, and G is transitive, either $\pi_\omega(K) = \{e\}$ for all $\omega \in \Omega$, or $\pi_\omega(K) = G_\omega$ for all $\omega \in \Omega$. In the first case, $K = \{e\} = \Gamma$. In the second case there is, by transitivity of G, an isomorphism $\varphi : K_G \to G_{\omega_0}^d$. By [1, Lemma 2.4], the group $\varphi(K)$ is a product of subdiagonals of $G_{\omega_0}^d$. As G is primitive, the underlying partition of this product is trivial. Hence either $K = \Phi = K_G$, or $K = \Delta$. □

There are more G-invariant subgroups of K_G when either G is imprimitive, G_ω is not simple, G_ω has non-trivial center, or G is not perfect.

Assume that G is transitive but preserves a non-trivial partition $\mathcal{P} : \Omega = \bigsqcup_{i\in I} \Omega_i$. Let $p : \Omega \to I$ be such that $\omega \in \Omega_{p(\omega)}$ for all $\omega \in \Omega$. Define

- $\Phi_{\mathcal{P}} := \{(s_\omega)_\omega \in K_G \mid \forall \omega, \omega' \in \Omega : \ p(\omega) = p(\omega') \Rightarrow s_\omega = s_{\omega'}\} \cong \prod_{i\in I} G_{\Omega_i}$.

When G is transitive and the G_ω are not simple, let $N \trianglelefteq G_{\omega_0}$. Also, let $f : \Omega \to G$, $f \mapsto f_\omega$ be a map such that $f_\omega(\omega_0) = \omega$ for all $\omega \in \Omega$. Define

- $\Phi_N := \{(f_\omega s_0^{(\omega)} f_\omega^{-1})_{\omega\in\Omega} \mid \forall\omega \in \Omega : s_0^{(\omega)} \in N\} \cong N^d$.

When G is transitive and G_{ω_0} has non-trivial center, let $C \leq Z(G_{\omega_0})$. We define

- $\Delta_C := \{(f_\omega s_0 f_\omega^{-1})_{\omega\in\Omega} \mid s_0 \in C\} \cong C$.

When G is not perfect, let $\rho : G \twoheadrightarrow A$ be a homomorphism to an abelian group. Set

- $\Pi_\rho := \{(s_\omega)_\omega \in K_G \mid \prod_{\omega\in\Omega} \rho(s_\omega) = 1\}$.

Can you think of any other general constructions with suitable assumptions on G? Can you classify the G-invariant subgroups of K_G for other classes of G?

References

[1] N. Radu, *A classification theorem for boundary 2-transitive automorphism groups of trees*, Inventiones Mathematicae, 2017, 209, 1–60.

[2] S. Tornier, *Groups Acting on Trees With Prescribed Local Action*, arXiv preprint 2002.09876v3, 2020.

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