$T\overline{T}$ and Supercurrent-Squared

Review Talk

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Roadmap.

The goal of this talk is to overview what is known about certain solvable, irrelevant deformations of $2D$ quantum field theories.

As with the other talks this week, I hope this will be useful for students and those who do not work directly on these questions.*

The rough outline consists of three parts:

1. Part 1: Preliminaries. What is a "solvable" irrelevant deformation? What is $T\overline{T}$ and why is it solvable? What is special about $T\overline{T}$?
2. Part 2: Supersymmetry. What more can we learn about solvable deformations in theories with SUSY? How can $T\overline{T}$ be made manifestly supersymmetric?
3. Part 3: Developments. What else is known about these deformations and what are the open questions?

* An unfortunate corollary is that it will be boring for experts.

Part 1: Preliminaries.

Studying QFTs via deformations.

An important goal is to better understand the space of all QFTs.

As we know, one fruitful strategy for doing this is to begin with a tractable (e.g. free, conformal, or exactly solvable) theory and then deform it.

If the theory has a Lagrangian description, one way to do this is by adding an integrated local operator:

$$S_0 = \int d^d x \, \mathcal{L}_0 \longrightarrow S_\lambda = S_0 + \lambda \int d^d x \, \mathcal{O}(x) \, .$$

We may then study S_λ either perturbatively in λ or exactly. Different perturbing operators $\mathcal{O}$ lead to qualitatively different deformations.

Three classes of deformations.

Recall that we classify deformations of d-dimensional field theories by the dimension Δ of the perturbing operator as relevant ($\Delta < d$), marginal ($\Delta = d$), or irrelevant ($\Delta > d$).

$$\mathcal{L}_0 = \partial^\mu \phi \partial_\mu \phi$$

irrelevant $\longrightarrow$ $\mathcal{L}_i = \mathcal{L}_0 + \lambda \left(\partial^\mu \phi \partial_\mu \phi \right)^2$

marginal $\longrightarrow$ $\mathcal{L}_m = \mathcal{L}_0 + \epsilon \, \partial^\mu \phi \partial_\mu \phi$

relevant $\longrightarrow$ $\mathcal{L}_r = \mathcal{L}_0 + m^2 \phi^2$

Irrelevant deformations are difficult.

Marginal and relevant deformations are relatively well-understood:

- Relevant operators trigger conventional renormalization group flows.
- In a conformal field theory, exactly marginal operators generate motion on the conformal manifold and parameterize the moduli space of the theory.

Adding an *irrelevant* operator defines a good effective field theory below some cutoff Λ, but it modifies the behavior of the theory in the ultraviolet. This changes the definition of the theory.

In general, one cannot add an irrelevant ("non-renormalizable") operator and uniquely recover a high energy theory; infinitely many additional couplings get turned on as one tries to flow to the UV.

A stress tensor deformation.

The prototypical example of a well-behaved irrelevant deformation in $2D$ is adding the determinant of the stress tensor $T_{\mu\nu}$ to the Lagrangian.

$$\frac{\partial \mathcal{L}_\lambda}{\partial \lambda} = \det \left(T^{(\lambda)}_{\mu\nu} \right) = \frac{1}{2} \left(\left(T^{(\lambda)\mu}{}_{\mu} \right)^2 - T^{(\lambda)\mu\nu} T^{(\lambda)}_{\mu\nu} \right) .$$

This "$T\overline{T}$" deformation has some remarkable properties, which only assume translation invariance and therefore hold in any QFT (even non-relativistic). [Zamolodchikov '04]

First, this combination actually defines a local operator by point-splitting, up to total derivatives:

$$\lim_{y \to x} \left(T^{\mu}{}_{\mu}(x) \, T^{\nu}{}_{\nu}(y) - T^{\mu\nu}(x) T_{\mu\nu}(y) \right) = \mathcal{O}_{T\overline{T}}(x) + (\text{derivs}) .$$

Factorization.

Furthermore, expectation values of the combination appearing in $T\overline{T}$ *factorize* in the following sense: if we define

$$f(x, y) = \langle T^{\mu}{}_{\mu}(x)\, T^{\nu}{}_{\nu}(y) \rangle - \langle T^{\mu\nu}(x) T_{\mu\nu}(y) \rangle \, ,$$

then one can prove that $\partial_x f = \partial_y f = 0$. So the function is actually a constant, independent of x and y!

This conclusion still holds if the expectation values are taken in any energy eigenstate $|n\rangle$.

Since f is a constant, we can evaluate it for either $y \to x$ (which defines the local operator $\mathcal{O}_{T\overline{T}}$) or for $|x - y| \to \infty$ (for which the two-point functions cluster-decompose). This implies

$$\langle \mathcal{O}_{T\overline{T}} \rangle = \langle T^{\mu}{}_{\mu} \rangle \langle T^{\nu}{}_{\nu} \rangle - \langle T^{\mu\nu} \rangle \langle T_{\mu\nu} \rangle \, .$$

A flow equation for the spectrum.

Factorization implies another remarkable property of $T\overline{T}$, which concerns the spectrum of energy eigenvalues $E_n(R, \lambda)$ for a deformed theory with deformation parameter λ on a cylinder of radius R.

One can prove that these energies satisfy

$$\frac{\partial}{\partial \lambda} E_n(R, \lambda) = E_n(R, \lambda) \frac{\partial}{\partial R} E_n(R, \lambda) + \frac{P_n(R)^2}{R}.$$

where P_n is the momentum of the state. This is a PDE of inviscid Burgers' type. If $P_n = 0$, the equation becomes $\partial_\lambda E_n = E_n \partial_R E_n$, which has the implicit solution

$$E_n(R, \lambda) = E_n \left(R + \lambda E_n(R, \lambda), 0 \right) .$$

Interpretation: place the original theory (defined on a cylinder of fixed radius R) on an effective cylinder with energy-dependent radius.

Simplification in a CFT.

If the seed theory enjoys conformal invariance, then in complex coordinates $(z, \bar{z})$, $T_{\mu\nu}$ has only two components $T = T_{zz}$ and $\overline{T} = T_{\bar{z}\bar{z}}$, and

$$\det(T_{\mu\nu}) = T\overline{T} \,.$$

Note that T_{zz} has dimension $(2,0)$. For instance, $T_{zz} = (\partial\phi)^2$ for a free boson. Thus $T\overline{T}$ has dimension $(2,2)$ and is irrelevant.

Furthermore, for a CFT on a cylinder of radius R, one has

$$E_n = \frac{1}{R}\left(n + \bar{n} - \frac{c}{12}\right) \equiv \frac{c_n}{R} \,, \qquad P_n = \frac{n - \bar{n}}{R} \,.$$

If $E_n = \frac{c_n}{R}$, the implicit equation becomes

$$E_n(R, \lambda) = \frac{c_n}{R + E_n(R, \lambda)} \,,$$

whose solution is

$$E_n(R, \lambda) = \frac{R}{2\lambda}\left(\sqrt{1 + 4\lambda E_n^{(0)}} - 1\right) \,.$$

Two signs with different behaviors.

Even with $P_n \neq 0$, the energy flow equation with a CFT seed theory has a closed-form solution:

$$E_n(\lambda) = \frac{R}{2\lambda}\left(\sqrt{1 + \frac{4\lambda E_n}{R} + \frac{4\lambda^2 P_n^2}{R^2}} - 1\right).$$

In a conformal field theory on a cylinder of radius R, the ground state energy is $E_0 = -\frac{c}{12R}$. We consider two cases:

1. (Bad sign) If $\lambda < 0$, then infinitely many high-energy states are complex at any finite value of $|\lambda|$.
2. (Good sign) If $\lambda > 0$, then all energies are real when λ is small but the ground state energy becomes complex at large λ.

Torus partition function.

In terms of the modular parameter $\tau = \tau_1 + i\tau_2$ of the torus, the partition function Z satisfies a flow equation [Cardy, '18] [Datta, Jiang '18]

$$\frac{\partial Z(t,\tau,\bar{\tau})}{\partial t} = \left[\tau_2 \partial_\tau \partial_{\bar{\tau}} + \frac{1}{2}\left(\partial_{\tau_2} - \frac{1}{\tau_2}\right) t\partial_t\right] Z(t,\tau,\bar{\tau}) \, .$$

Here $t = \frac{1}{2}\pi R^2 \lambda$, where R is the radius of the spatial cylinder, is a dimensionless version of λ.*

Remarkably, although a $T\overline{T}$-deformed theory is no longer conformally invariant, its torus partition function is still modular invariant! The parameter t transforms under a modular transformation via its dependence on R. But after accounting for this, $Z(\lambda, \tau, \bar{\tau}) = Z(\lambda, \frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d})$.

Under some assumptions, $T\overline{T}$ is the only deformation with this property. [Aharony, Datta, Giveon, Jiang, Kutasov '18]

*The flow equation can be written in a diffusion-type form by changing variables. It looks like $\partial_\lambda \mathcal{Z} = (\partial_L \wedge \partial_{L'})\,\mathcal{Z}$ where $\mathcal{Z} = \frac{Z}{A}$ is a re-scaled version of Z.

Connections to string theory.

Suppose we begin with a free massless scalar $\mathcal{L}_0 = \partial_\mu \phi \partial^\mu \phi$. Solving the classical flow equation $\partial_\lambda \mathcal{L} = \det\left(T^{(\lambda)} \right)$ gives

$$\mathcal{L}(\lambda) = \frac{1}{2\lambda} \left(\sqrt{1 + \lambda \partial^\mu \phi \partial_\mu \phi} - 1 \right) .$$

This is the Nambu-Goto action in static gauge for a three-dimensional target space, suggesting a connection to string theory.

One can show that, quite generically, $T\overline{T}$-deforming a CFT gives a theory with Hagedorn density of states [Giveon, Itzhaki, Kutasov '17].

Such a deformed theory cannot be a local field theory – this density of states is again more characteristic of a string theory.

Scattering.

A final natural observable in a QFT is the S-matrix $S_0(\{p_i\})$.

The S-matrix $S(\lambda, \{p_i\})$ for the theory obtained by $T\overline{T}$-deforming a theory with S-matrix S_0 is related by a CDD factor: [Dubovsky, Gorbenko, Mirbabayi '17]

$$S_0(\{p_i\}) \longrightarrow S(\lambda, \{p_i\}) = \left(\prod_{i<j} e^{i\delta_{ij}} \right) S(\{p_i\}),$$
$$\delta_{ij} = \frac{1}{2} \lambda \epsilon_{\mu\nu} p_i^\mu p_j^\nu .$$

This is the same modification that one finds by a certain "gravitational dressing" procedure.

Although this is a healthy S-matrix, it is incompatible with a local quantum field theory and points to some non-locality.

A solvable, irrelevant deformation.

I pause here to explain what I mean by this phrase.

1. $T\overline{T}$ is an *irrelevant deformation*.

   It is irrelevant in the usual sense of power counting, but nonetheless is a well-defined local operator in any theory with a stress tensor. We expect that deforming by this operator yields a theory which exists at the quantum level, rather than merely an EFT.

2. $T\overline{T}$ is *solvable*.

   We can write equations which determine quantities in the deformed theory in terms of corresponding quantities in the undeformed theory. For instance, the energies $E_n(R, \lambda)$ are determined in terms of $E_n^{(0)}$ (similarly for the partition function, S-matrix, etc.).

Next, since supersymmetry often provides more control over QFTs, we turn to the interplay between $T\overline{T}$ and SUSY.

Part 2: Supersymmetry.

Generalities on the spectrum.

Since this conference is on $2D$ supersymmetric field theory, I don't need to convince you that adding SUSY to the game gives extra control which may be interesting in considering solvable irrelevant deformations.

First, a naïve observation: since $T\overline{T}$ affects the energy levels as

$$E_n^{(0)} \longrightarrow E_n(\lambda) = f\left(E_n^{(0)}\right) ,$$

any two states with the same undeformed energy $E_n^{(0)}$ will have equal deformed energies $E_n(\lambda)$.

So degeneracies in the spectrum, such as those implied by supersymmetry, are preserved under $T\overline{T}$. Likewise, counts of bosonic/fermionic ground states (hence the Witten index) are unchanged.

Elliptic genus.

What about other quantities in SUSY theories, like the elliptic genus?

$$\chi(\tau, \lambda) = \mathrm{Tr}_{\mathrm{RR}} \left[(-1)^{F_R} e^{i\tau_1 P - \tau_2 H(\lambda)} \right] .$$

Consider a CFT deformed by $T\overline{T}$.* Ramond ground states satisfy a BPS condition $E_n = P_n$. But from the inviscid Burgers' equation, since $E_n = \frac{c_n}{R}$, these do not flow:

$$\begin{aligned} \frac{\partial E_n}{\partial \lambda} &= E_n \frac{\partial E_n}{\partial R} + \frac{1}{R} P_n^2 \\ &= \frac{c_n}{R} \left(- \frac{c_n}{R^2} \right) + \frac{1}{R} \frac{c_n^2}{R^2} = 0 \, . \end{aligned}$$

So $H(\lambda) = H(0)$ on RR ground states and χ does not flow. [Datta, Jiang 18]

*This argument holds if the seed theory is a CFT. There is also perturbative evidence that χ does not flow if the seed theory is integrable. [Ebert, Sun, Sun 21].

Is SUSY $T\overline{T}$ trivial?

The above points might suggest that $T\overline{T}$ has a simple interaction with supersymmetry: SUSY is unbroken and perhaps indices are unchanged.

However, there are a few reasons to look more closely:

1. Although SUSY is unbroken, the action of the supercharges is corrected order-by-order in λ. Thus the SUSY transformations in the deformed theory can be quite complicated.
2. The control offered by SUSY is strongest when the symmetry is made manifest, for instance by a superspace construction.
3. With extended SUSY, more quantities become protected which makes a classical analysis more powerful (maximal SUSY is even better).
4. $T\overline{T}$ is related to strings. Is SUSY $T\overline{T}$ related to the superstring?

An $\mathcal{N} = (2,2)$ version of $T\overline{T}$?

For concreteness, we will attempt to realize $T\overline{T}$ in $(2,2)$ superspace.* To fix notation, supercovariant derivatives are

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - \frac{i}{2}\bar{\theta}^\pm\partial_{\pm\pm} \ , \qquad \overline{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + \frac{i}{2}\theta^\pm\partial_{\pm\pm} \ ,$$

which satisfy the algebra $\{D_\pm, \overline{D}_\pm\} = i\partial_{\pm\pm}$. The supersymmetry transformations for an $\mathcal{N} = (2,2)$ superfield $\mathcal{F}$ are given by

$$\delta_Q\mathcal{F} := i\epsilon^+\mathcal{Q}_+\mathcal{F} + i\epsilon^-\mathcal{Q}_-\mathcal{F} - i\bar{\epsilon}^+\overline{\mathcal{Q}}_+\mathcal{F} - i\bar{\epsilon}^-\overline{\mathcal{Q}}_-\mathcal{F} \ ,$$

where the action of supercharges on superfields is represented by

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + \frac{i}{2}\bar{\theta}^\pm\partial_{\pm\pm} \ , \qquad \overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - \frac{i}{2}\theta^\pm\partial_{\pm\pm} \ ,$$

satisfying $\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} = -i\partial_{\pm\pm}$ and commuting with the D's.

*The $(1,1)$, $(0,1)$, and $(0,2)$ versions will be mentioned briefly at the end of this part.

An aside on $+$'s and $-$'s.

We will exploit the fact that we are in $d = 2$ by using "bi-spinor" conventions for coordinates and vector quantities:

$$x^{\pm\pm} = \frac{1}{\sqrt{2}} \left(x^0 \pm x^1 \right) .$$

In these conventions, the number of $+$ and $-$ count the spin of an object. All contractions are with off-diagonal metrics, so scalars have the same count of $+$ of $-$. For instance,

$$T\overline{T} \sim T_{++++} T_{----} - T_{++--} T_{--++} .$$

Conservation equations look like

$$\partial_{--} T_{++++} + \partial_{++} T_{++--} = 0, \qquad \partial_{++} T_{----} + \partial_{--} T_{--++} = 0.$$

Supercurrents.

The stress tensor $T_{\mu\nu}$ for a QFT can be viewed as the Noether current for translations or as a source for the metric when the theory is coupled to gravity. We want a deformation built from a SUSY extension of $T_{\mu\nu}$.

Building on earlier work, it was shown that every $2D$ SUSY field theory admits an $\mathcal{S}$-multiplet $(\mathcal{S}_{\pm\pm}, \chi_{\pm}, \mathcal{Y}_{\pm})$ containing the stress tensor.
[Dumitrescu, Seiberg (2011)]

In some examples, the components of the $\mathcal{S}$ multiplet can be equivalently expressed using a smaller set of fields. We will see examples of this, like the FZ multiplet ($\chi_{\pm} = 0$) shortly.

Some $\mathcal{S}$-multiplet details.

The components of the $\mathcal{S}$-multiplet superfields involve the stress tensor $T_{\mu\nu}$ and a supersymmetry current $S_{\mu\alpha}$, among others. The superfields satisfy a list of constraints:

$$\overline{D}_{\pm} \mathcal{S}_{\mp\mp} = \pm \left(\chi_{\mp} + \mathcal{Y}_{\mp} \right) \, ,$$

$$\overline{D}_{\pm} \chi_{\pm} = 0 \, , \qquad \overline{D}_{\pm} \chi_{\mp} = \pm C^{(\pm)} \, , \qquad D_{+} \chi_{-} - \overline{D}_{-} \bar{\chi}_{+} = k \, ,$$

$$D_{\pm} \mathcal{Y}_{\pm} = 0 \, , \qquad \overline{D}_{\pm} \mathcal{Y}_{\mp} = \mp C^{(\pm)} \, , \qquad D_{+} \mathcal{Y}_{-} + D_{-} \mathcal{Y}_{+} = k' \, .$$

In components, these constraints enforce $\partial^{\mu} T_{\mu\nu} = 0$ and $\partial^{\mu} S_{\mu\alpha} = 0$.

The FZ-multiplet is a special case where $\chi_{\pm} = 0$, and the R-multiplet where $\mathcal{Y}_{\pm} = 0$.

Defining the $(2,2)$ deformation.

For a $\mathcal{N} = (2,2)$ theory with $\mathcal{S}$-multiplet $(\mathcal{S}_{\pm\pm}, \chi_\pm, \mathcal{Y}_\pm)$, let

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\frac{1}{8}\mathcal{T}\overline{\mathcal{T}},$$

$$\mathcal{T}\overline{\mathcal{T}} = -\int d^4\theta \, \mathcal{S}_{++}\mathcal{S}_{--} - \left(\int d\theta^- \, d\theta^+ \, \chi_+ \chi_- + \int d\bar{\theta}^- \, d\theta^+ \, \overline{\mathcal{Y}}_+ \mathcal{Y}_- + \text{c.c.} \right).$$

This is a deformation constructed out of bilinears of currents, just like the usual $T\overline{T}$. It's equivalent to $T\overline{T}$ on-shell:

$$\mathcal{T}\overline{\mathcal{T}} = T\overline{T} + (\text{eom}) + (\text{total derivatives}).$$

The on-shell equivalence guarantees that this deformation changes the spectrum in the same way as $T\overline{T}$, as it must. The extra terms are only needed for manifest SUSY.

Since $\mathcal{T}\overline{\mathcal{T}}$ involves products of supercurrents, we call it *supercurrent - squared*.

Well-definedness.

This supercurrent-squared operator is well-defined by point-splitting due to a superspace argument that is independent of the Zamolodchikov proof. If we define a point-split operator

$$\mathcal{O}(x, x', \theta) = -\mathcal{S}_{++}(x, \theta)\mathcal{S}_{--}(x', \theta) + \cdots ,$$

one can show that

$$\partial_{\pm\pm}\mathcal{O}(x, x', \theta) = 0 + (\text{eom}) + [P, \cdots] + [Q, \cdots] ,$$

inside of correlation functions. Here P and Q represent total translations and supersymmetry transformations, i.e. total superspace derivatives.

Example: free chiral superfield.

For a flavor of the calculation, let's take a chiral superfield Φ with action

$$\mathcal{L}_0 = \int d^4\theta \, \overline{\Phi}\Phi,$$

This superfield contains a complex scalar ϕ, Dirac fermions $\psi_\pm$, and an auxiliary field F:

$$\begin{aligned}\Phi = \phi + \theta^+\psi_+ + \theta^-\psi_- + \theta^+\theta^- F - i\theta^+\bar{\theta}^+\partial_{++}\phi - i\theta^-\bar{\theta}^-\partial_{--}\phi \\ - i\theta^+\theta^-\bar{\theta}^-\partial_{--}\psi_+ - i\theta^-\theta^+\bar{\theta}^+\partial_{++}\psi_- - \theta^+\theta^-\bar{\theta}^-\bar{\theta}^+\partial_{++}\partial_{--}\phi \, ,\end{aligned}$$

After doing the superspace integral, the undeformed Lagrangian is

$$\mathcal{L}_0 = \partial_{++}\bar{\phi}\partial_{--}\phi + \partial_{++}\phi\partial_{--}\bar{\phi} + i\psi_-\partial_{++}\bar{\psi}_- + i\bar{\psi}_+\partial_{--}\psi_+ + |F|^2.$$

So $F = 0$ and this is a free complex scalar with free fermions.

Obtaining the supercurrents.

The usual stress tensor $T_{\mu\nu}$ can be obtained either by a Noether procedure or as the linearized response to metric fluctuations, $T_{\mu\nu} \sim \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}$.

Likewise, the supercurrents come from either a superspace Noether procedure [Magro, Sachs, Wolf '01] or coupling to supergravity. One can use Noether for the $(1,1)$ case, but in $(2,2)$, SUGRA is more convenient.

The free chiral superfield admits an improvement of the $\mathcal{S}$ multiplet to the FZ multiplet $(\mathcal{S}_{\pm\pm} = \mathcal{J}_{\pm\pm}, \mathcal{Y}_{\pm} = D_{\pm}\mathcal{V})$. Varying the superspace action gives some linearized couplings:

$$\mathcal{L}_{\text{linear}} = \int d^4\theta \, \left(H^{++} \mathcal{J}_{++} + H^{--} \mathcal{J}_{--} \right) - \int d^2\theta \, \sigma \, \mathcal{V} - \int d^2\bar{\theta} \, \bar{\sigma} \, \overline{\mathcal{V}} \, ,$$

These are *old-minimal supergravity* prepotentials $H_{\pm\pm}$, σ.

The FZ flow equation.

When there is an FZ multiplet, the super-$T\overline{T}$ flow equation reduces to

$$\partial_\lambda \mathcal{L} = \frac{1}{8} \int d^4\theta \left(\mathcal{J}_{++} \mathcal{J}_{--} - 2 \mathcal{V} \overline{\mathcal{V}} \right) .$$

To leading order around the free Lagrangian, we find $\mathcal{J}_{\pm\pm} = 2 D_\pm \Phi \overline{D}_\pm \overline{\Phi}$, so the first order correction is

$$\mathcal{L}^{(1)} = \mathcal{L}^{(0)} + \frac{\lambda}{2} \int d^4\theta \, D_+ \Phi \overline{D}_+ \overline{\Phi} D_- \Phi \overline{D}_- \overline{\Phi}.$$

Now make an ansatz for the finite-λ result of the form

$$\mathcal{L}_\lambda = \int d^4\theta \left(\overline{\Phi} \Phi + f(\lambda, x, \bar{x}, y) D_+ \Phi \overline{D}_+ \overline{\Phi} D_- \Phi \overline{D}_- \overline{\Phi} \right) ,$$

$$x = \partial_{++} \Phi \partial_{--} \overline{\Phi} \, , \qquad y = \left(D_+ D_- \Phi \right) \left(\overline{D}_+ \overline{D}_- \overline{\Phi} \right) .$$

and try to solve for the function f.

Super-Nambu-Goto.

The equations of motion at any λ impose that $y = 0$. This can be used to solve the flow equation for the on-shell action, yielding

$$\mathcal{L}_\lambda = \int d^4\theta \left(\bar{\Phi}\Phi + \frac{\lambda D_+\Phi \bar{D}_+\bar{\Phi} D_-\Phi \bar{D}_-\bar{\Phi}}{1 - \frac{1}{2}\lambda A + \sqrt{1 - \lambda A + \frac{1}{4}\lambda^2 B^2}} \right),$$

$$A = \partial_{++}\Phi\partial_{--}\bar{\Phi} + \partial_{++}\bar{\Phi}\partial_{--}\Phi,$$

$$B = \partial_{++}\Phi\partial_{--}\bar{\Phi} - \partial_{++}\bar{\Phi}\partial_{--}\Phi.$$

This is a supersymmetric version of the Nambu-Goto action.

We also study the case where the undeformed theory has a superpotential $W(\Phi)$. We can describe it perturbatively in λ, or give the physical potential for the boson ϕ at zero momentum, which goes like $\frac{|W'|^2}{1-\lambda|W'|^2}$ and thus develops a pole at finite λ.

Other amounts of supersymmetry.

There are similar supercurrent-squared operators for theories with $(1,1)$, $(0,1)$, or $(0,2)$ supersymmetry. For instance, in $(1,1)$, the $\mathcal{S}$-multiplet can be packaged as a single superfield $\mathcal{T}$ which is conserved in that

$$D_+ \mathcal{T}_{++-} + D_- \mathcal{T}_{---} = 0 \, , \qquad D_+ \mathcal{T}_{---} + D_- \mathcal{T}_{--+} = 0 \, .$$

The flow equation is then

$$\frac{\partial S}{\partial \lambda} = \int d^2 x \, d^2 \theta \, \left(\mathcal{T}_{+++} \mathcal{T}_{---} - \mathcal{T}_{--+} \mathcal{T}_{++-} \right) .$$

It would be interesting to study supercurrent-squared in theories with more than $(2,2)$ SUSY, but in these cases a superspace presentation is either more complicated or not possible.

Part 3: Developments.

A sampling of other directions.

There have now been several hundred papers on $T\overline{T}$ and related topics, so it is impossible to give a thorough survey.

However, I'd like to flag three other directions which I haven't mentioned so far. Each of them touches upon important open questions.

1. Part 3.1: Correlation functions. What is the fate of position-space observables in (non-local) $T\overline{T}$-deformed theories?
2. Part 3.2: Non-linearly realized symmetries. Why do these structures appear in $D = 4$, and can we find $T\overline{T}$-like operators in $D > 2$?
3. Part 3.3: Holography. How should we think about the $3D$ gravity dual to a $T\overline{T}$-deformed theory?

Part 3.1: Correlation Functions.

Do n-point functions flow in a simple way?

We have seen in the preceding parts that various quantities of interest flow in a controlled way under $T\overline{T}$ or supercurrent-squared:

1. the cylinder spectrum $E_n(R)$ follows an inviscid Burgers' equation;
2. the torus partition function satisfies a diffusion equation;
3. the S-matrix is modified by a CDD factor; ...

Is there a similar statement for correlation functions? If you know

$$f(z_i, \bar{z}_i) = \langle \mathcal{O}_1(z_1, \bar{z}_1) \, \mathcal{O}_2(z_2, \bar{z}_2) \cdots \mathcal{O}_n(z_n, \bar{z}_n) \rangle ,$$

then is the behavior f_λ of this correlator in the $T\overline{T}$ deformed theory determined universally in terms of f?

The leading correction.

There has been some progress on these position space correlation functions in $T\overline{T}$ deformed theories. [Kraus, Liu, Marolf '18] [Cardy '19]

For instance, by considering the point-split operator $T(z+\epsilon)\overline{T}(\bar{z})$ and using the conformal Ward identity, one can find the leading correction to f:

$$f_0 \longrightarrow f_0 + \lambda f_1 + \cdots \, ,$$

$$f_1 = -4 \sum_{m \neq n} \frac{1}{|\epsilon|} \log \left(\frac{|z_m - z_n|}{|\epsilon|} \right) \partial_{z_m} \partial_{\bar{z}_n} \langle \mathcal{O}_1(z_1, \bar{z}_1) \cdots \mathcal{O}_n(z_n, \bar{z}_n) \rangle \, .$$

One must then analyze the UV divergences as $\epsilon \to 0$. For instance, one can attempt to define new operators $\mathcal{O}_i^{(\lambda)}$ which have finite correlators in the deformed theory.

SUSY correlators.

Correlation functions can also be analyzed for SUSY theories using the formalism developed in Part 2.

1. In [He, Sun, Sun '20], the authors derive explicit expressions for the $\mathcal{O}(\lambda)$ corrections to n-point functions in seed theories with $(1,1)$ or $(2,2)$ superconformal symmetry using supercurrent-squared.
2. [Ebert, Sun, Sun '21] then studies the leading corrections to various correlators in $(0,2)$ SCFTs (as well as the deformation of the $\mathcal{S}$-multiplet in $(2,2)$ theories and S-matrices via TBA).

The superspace presentation makes it much easier to define renormalized operators which preserves SUSY, but the behavior past leading order in λ is still mysterious.

Non-locality and correlators?

Can one make sense of $T\overline{T}$ deformed correlators at finite λ?

1. Since a $T\overline{T}$-deformed theory is not a local quantum field theory, perhaps there are no position-space correlators. Maybe the perturbation series in λ has zero radius of convergence.
2. Or perhaps correlators can be sensibly defined at large enough separation but break down when the insertion points approach the non-locality scale set by λ.
3. Should we instead consider momentum-space correlation functions, as suggested by worldsheet calculations for "single-trace" $T\overline{T}$?

The answers to these questions might shed more light on what a $T\overline{T}$-deformed theory really is!

Part 3.2: Non-linearly realized symmetries.

A shift symmetry in Dirac/Nambu-Goto.

We saw that the $T\overline{T}$ deformation of the free boson roughly looks like the Dirac Lagrangian $\mathcal{L} = \sqrt{1 + (\partial\phi)^2}$. This action possesses a non-linear shift symmetry [Aharony, Field '11], [Meineri '13]

$$\delta\phi = x^{\mu} + \phi\partial^{\mu}\phi.$$

This symmetry comes from spontaneously broken Lorentz invariance. For instance, a straight string in d dimensions breaks $ISO(d,1)$ to $ISO(1,1)$. We expect $d-1$ Nambu-Goldstone bosons, which are the fields ϕ^i.

Is this more general? For instance, does supercurrent-squared also lead to nonlinearly realized symmetries?

Supersymmetry breaking.

It was shown in $d = 2$ by [Cribiori, Farakos, von Unge] that the $\mathcal{N} = (2, 2)$ Volkov-Akulov action, which describes spontaneous supersymmetry breaking, is a supercurrent-squared deformation of a free fermionic theory.

This conclusion also holds for a purely classical version of supercurrent-squared which we describe momentarily. The KS model of the $d = 4$ VA action, looks like

$$\mathcal{L}_{KS} = -f^2 - \frac{i}{2}\left(\psi\sigma^m\partial_m\overline{\psi} - \partial_m\psi\sigma^m\overline{\psi}\right) \\ - \frac{1}{4f^2}\partial^\mu\overline{\psi}^2\partial_\mu\psi^2 - \frac{1}{16f^6}\psi^2\overline{\psi}^2\partial^2\psi^2\partial^2\overline{\psi}^2$$

where ψ is the Goldstino field and f is a constant parameter describing the scale of $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking.

This action descends from a superspace action which is a supercurrent - squared flow where the flow parameter is $\lambda \sim \frac{1}{f^2}$.

Supercurrent-squared in $d = 4$.

Consider $d = 4\,,\, \mathcal{N} = 1$ theories which possess an FZ multiplet:

$$\overline{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} = D_\alpha X,$$
$$\overline{D}_{\dot{\alpha}} X = 0.$$

We propose the following supercurrent-squared operator:

$$\mathcal{T}\overline{\mathcal{T}} = \int d^4\theta \, \left(\frac{1}{16} \mathcal{J}^{\alpha\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} - \frac{5}{8} X \overline{X} \right) .$$

This is *not* equivalent to $T\overline{T}$ on-shell:

$$\mathcal{T}\overline{\mathcal{T}} = T^{\mu\nu} T_{\mu\nu} - \frac{1}{2} \left(T^{\mu}{}_{\mu} \right)^2 + \frac{3}{8} j_\mu \partial^2 j^\mu + \frac{3}{8} \partial_\mu x \partial^\mu \bar{x} + \cdots .$$

We don't know whether this operator has nice properties at the quantum level, but we can study a classical flow with this deforming operator.

Super-Maxwell to super-BI.

Consider the undeformed theory for the field strength superfield W_α of an $\mathcal{N} = 1$ vector multiplet:

$$\mathcal{L}_0 = \frac{1}{4} \int d^2\theta W^2 + \text{c.c.}$$

Deform this theory by supercurrent-squared. On-shell, the solution can be written as

$$\mathcal{L}_\lambda = \frac{1}{4} \int d^2\theta \, W^2 + \frac{1}{4} \int d^2\bar{\theta} \overline{W}^2 + 2\lambda \int d^4\theta \, \frac{W^2 \overline{W}^2}{1 + \mathcal{A} + \sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}},$$

$$\mathcal{A} = \frac{\lambda}{2} \left(D^2 W + \overline{D}^2 \overline{W} \right) , \qquad \mathcal{B} = \frac{\lambda}{2} \left(D^2 W - \overline{D}^2 \overline{W} \right) .$$

This is an organizing principle that gives us a new way to think about the supersymmetric Born-Infeld Lagrangian.

A hidden SUSY.

Although the super-BI Lagrangian is written in $\mathcal{N} = 1$ superspace, it has a hidden second supersymmetry* under which W transforms as

$$\delta W_\alpha = \eta_\alpha - \frac{1}{4} \overline{D}^2 \overline{X} \eta_\alpha - i \partial_{\alpha \dot{\alpha}} X \eta^{\dot{\alpha}},$$

where X is the chiral field determining the Lagrangian via $\mathcal{L}_\lambda = \int d^2\theta X + \int d^2\bar{\theta} \overline{X}$. Under this transformation, X transforms as

$$\delta X = 2 W^\alpha \eta_\alpha .$$

Since $W^\alpha \sim \overline{D}^2 D_\alpha V$, the Lagrangian changes by a total derivative.

Thus the supercurrent-squared deformation of a theory with $\mathcal{N} = 1$ SUSY possesses $\mathcal{N} = 2$ SUSY; we also show a similar statement for $2D$ theories.

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*In fact, one can essentially derive the super-BI Lagrangian by demanding that it have such a property [Bagger, Galperin '96].

Higher dimensions?

The (classical) $d = 4$ discussion highlights an important open question: can we define a $T\overline{T}$-like operator in $d > 2$ at the quantum level?

In any number of dimensions, there is some OPE

$$T^{\mu\nu}(x) T_{\mu\nu}(y) - \frac{1}{D-1} T^{\mu}{}_{\mu}(x) T^{\nu}{}_{\nu}(y) = \sum_{\alpha} A_{\alpha}\left(|x-y|^2\right) \mathcal{O}_{\alpha}(y),$$

In $d = 2$, the conservation equation $\partial^{\mu} T_{\mu\nu} = 0$ means that the right side defines a local operator up to derivatives. But in $d > 2$, there are not enough constraints. Since the supercurrents satisfy additional constraints by SUSY, could supercurrent-squared give a local operator?

Alternatively, could we use the principle of generating non-linearly realized symmetry to find a $d > 2$ generalization? Such symmetries are related to soft theorems for scalars – is there an amplitudes definition?

Part 3.3: Holography.

Holographic interpretations.

There have been several attempts to realize stress tensor deformations holographically. I'll mention two:

1. Cutoff AdS (double-trace, "bad sign" of λ). [McGough, Mezei, Verlinde '16]
   Conceptual challenges. Most energies are complex.

2. Linear dilaton (single-trace, "good sign"). [Giveon, Itzhaki, Kutasov '17]
   Emerges from string theory and related to little string theory. Likely well-defined.

If the dual field theory is the symmetric product orbifold $(CFT_2)^N / S_N$, then the double-trace $T\overline{T}$ and single-trace $D(x)$ are

$$T\overline{T} = \left(\sum_{i=1}^{N} T_i \right) \left(\sum_{j=1}^{N} \bar{T}_j \right) \qquad \text{v.s.} \qquad D(x) = \sum_{i=1}^{N} T_i \overline{T}_i .$$

Interpretation (2) is promising as an approach to holography beyond AdS.

The F1-NS5 solution.

The "single-trace" deformation has an interpretation in terms of little string theory and linear dilaton spacetimes.

Consider a type IIB supergravity solution with Q_1 fundamental strings and Q_5 NS5-branes:

$$ds^2 = f_1^{-1}\left(-dt^2 + dx_5^2\right) + f_5\left(dr^2 + r^2\, d\Omega_3^2\right) + \left(dx_6^2 + \cdots + dx_9^2\right),$$

$$e^{-2\Phi} = \frac{1}{g_s^2}\frac{f_1}{f_5}, \qquad H_3 = \frac{c_M}{r^3 f_1^2}\epsilon_3^{\mathcal{M}_3} + c_S \epsilon_3^{S^3},$$

$$f_1 = 1 + \frac{r_1^2}{r^2}, \qquad f_5 = 1 + \frac{r_5^2}{r^2}.$$

Here r_1 and r_5 depend on Q_1 and Q_5, and $\mathcal{M}_3 \sim (t, r, x_5)$.

In the decoupling limit $g_s \to 0$, $\alpha' \to 0$, it is well-known* that this solution looks like $\mathrm{AdS}_3 \times S^3 \times T^4$. In this region, we have $\mathrm{AdS}/\mathrm{CFT}$.

*Alternatively one can think of this limit as taking $r \ll r_1, r_5$.

An intermediate regime.

Suppose we partially decouple by taking the asymptotic g_s to zero but leaving α' finite.* The spacetime interpolates from AdS in the deep bulk to linear dilaton in the asymptotic region.

It is believed that the holographic dual to this gravity solution is a CFT deformed by a single-trace irrelevant operator which behaves like $T\overline{T}$. Most evidence comes from worldsheet calculations.

1. What plays the role of λ in the gravity theory?
2. Is the square-root formula visible directly from gravity?
3. What happens to the gravity solution when the ground state energy goes complex for large λ?

These questions will be discussed in my talk next week.

*Or drop the 1 in f_5 but not in f_1.