Complex Geometry and Supersymmetry.

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Sigma models 1.

A non-linear sigma model is a theory of maps from a (super) manifold Σ to a target space $\mathcal{T}$

$$X : \Sigma \to \mathcal{T}$$

$$X(x) \longmapsto X \in \mathcal{T} \ ,$$

with dynamics specified by extremising an action

$$S = \int_{\Sigma} dx \mathcal{L}(X)$$

The number of supersymmetries constrain the geometry of $\mathcal{T}$.

Sigma models 2.

In two dimensions the general bosonic sigma model action reads

$$S = \int d^2x \partial_{+\!\!+} X^\mu \left(G_{\mu\nu} + B_{\mu\nu} \right) \partial_= X^\nu =: \int d^2x \partial_{+\!\!+} X^\mu E_{\mu\nu} \partial_= X^\nu \ ,$$

The $2D$ light-cone coordinates are

$$x^{+\!\!+} = x^0 + x^1 \ , \quad x^= = x^0 - x^1 \ .$$

The B field is a gerbe connection and the action depends only on its field strength

$$H_{\mu\nu\rho} = \partial_{[\rho} B_{\mu\nu]} \ .$$

as seen directly from the field equations for X^μ or from the alternative form of the action

$$S = \int_{\partial V} d^2x \partial_{+\!\!+} X^\mu G_{\mu\nu} \partial_= X^\nu + \frac{1}{3} \int_V d^3x \epsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho H_{\mu\nu\rho} \ ,$$

The B (or H) term is called a Wess-Zumino term.

Sigma models 3.

The model is invariant under a B field gauge transformation;

$$B \to B + d\lambda \ , \quad B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\lambda_{\nu]} \ .$$

The X^μ field equations that follow from either form of the action read

$$G_{\mu\nu}\nabla^{(-)}_{\mp\mp}\partial_{=}X^\nu = G_{\mu\nu}\nabla^{(+)}_{=}\partial_{\mp\mp}X^\nu = 0 \ ,$$

where the connections now have torsion due to the inclusion of the B field;

$$\Gamma^{(\pm)\mu}_{\sigma\rho} = \Gamma^{(0)\mu}_{\sigma\rho} \pm T_{\sigma\rho}{}^{\mu} \ , \quad T_{\sigma\rho}{}^{\mu} = \tfrac{1}{2}H_{\sigma\rho\nu}G^{\nu\mu} \ .$$

Riemann geometry with torsion

Starting from $(1,1)$.

A general sigma model in $(1,1)$ is

$$S = \int d^2x D_+ D_- \Big(D_+ \Phi^i(x,\theta)(G_{ij} + B_{ij})(\Phi) D_- \Phi^j(x,\theta) \Big)_|$$

$$= \int d^2x \partial_{+\!\!+} \phi^i E_{ij}(\phi) \partial_= \phi^j + \ldots.$$

It has $(1,1)$ supersymmetry manifest by construction.
Additional supersymmetries will have the form

$$\delta\Phi^i = \epsilon^+ J^i_{(+)k} D_+ \Phi^k + \epsilon^- J^i_{(-)k} D_- \Phi^k$$

The conditions on J follow from two requirements:

- Closure of the algebra $\quad [\delta_1, \delta_2]\Phi = -i2\epsilon_1\epsilon_2 \partial\Phi$
- Invariance of the action $\quad \delta S = 0$

Additional Susy

From closure of the algebra it follows that $J^2 = -1$ and $\mathcal{N}(J) = 0$ (Nijenhuis).

From invariance of the action it follows that $J^t G J = G$ and that $\nabla^{(\pm)} J_{(\pm)} = 0$.

In terms of covariant derivatives the $(2,2)$ algebra is,

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = 2i\partial_{\underset{=}{+}} ,$$

all other (anti)commutators zero. Since we now have four θs, the multiplet contained in a general superfield can consist of 16 fields. Such a multiplet is recducible and not suitable for a sigma model description. But since the covariant deivatives anticommute with the supersymmety generators we can use them to impose constraints that will reduce the multiplet.

There are three types of constrained $(2,2)$ superfields and correspondingly three different target space geometries.

Chiral superfields Φ

$$\bar{\mathbb{D}}_{\pm}\Phi = 0$$

Twisted Chiral superfields χ

$$\bar{\mathbb{D}}_{+}\chi = 0\ , \quad \mathbb{D}_{-}\chi = 0$$

Left ℓ and Right $\mathfrak{r}$ Semichiral superfields

$$\bar{\mathbb{D}}_{+}\ell = 0\ , \quad \bar{\mathbb{D}}_{-}\mathfrak{r} = 0$$

Chiral models 1.

Chiral superfields $\bar{\mathbb{D}}_\pm \Phi = 0$

Complex component fields:

$$\bar{\mathbb{D}}_\pm \Phi = 0 \;, \quad \Phi = \phi + \theta^\alpha \psi_\alpha + \theta^\alpha \theta_\alpha \mathcal{F} \;, \qquad \alpha = (+,-)$$

$$\phi(x) = \Phi(x,\theta)_|$$

$$\psi_\alpha(x) = \mathbb{D}_\alpha \Phi(x,\theta)_| \;,$$

$$\mathcal{F}(x) = \mathbb{D}^2 \Phi(x,\theta)_| \;,$$

$$S = \int d^2x \mathbb{D}^2 \bar{\mathbb{D}}^2 K(\Phi, \bar{\Phi})_| = \int d^2x \left\{ K_{,i\bar{j}} \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}} + ... \right\}$$

Kähler Complex geometry: $g_{i\bar{j}} = \partial^2 K / \partial \phi^i \partial \bar{\phi}^{\bar{j}}$. Potential K.

Chiral models 2.

Pushing in the spinorial derivatives and using the definition of the components we find

$$S = \int d^2x \Big[\partial_{\mp}\phi^a G_{ab} \partial_= \phi^b + i\tfrac{1}{2}(\psi_+^a \nabla_= \psi_+^b + \psi_-^a \nabla_{\mp} \psi_-^b) G_{ab}$$

$$-\tfrac{1}{4} R_{cdab} \psi_+^a \psi_+^b \psi_-^c \psi_-^d \Big]$$

after eliminating the auxiliary fields $\mathcal{F}$. Now $a = (i, \bar{i})$ etc. The geometry is Kähler

$$G_{i\bar{j}} = K_{,i\bar{j}}$$

$$\Gamma_{ij}{}^{k} = g^{k\bar{s}}\partial_i g_{j\bar{s}} = K^{k\bar{s}} K_{,ij\bar{s}} \ ,$$

$$R_{i\bar{j}k\bar{s}} = g_{m\bar{j}} \partial_{\bar{s}} (\Gamma_{ik}{}^{m}) = K_{,i\bar{j}k\bar{s}} - \Gamma_{ik}{}^{m} \Gamma^{\bar{n}}{}_{\bar{j}\bar{s}} K_{,m\bar{n}}$$

Chiral models 3.

Kähler geometry is the target space geometry of $\mathcal{N} = 1$ sigma models in 4d and for certain $(2,2)$ sigma models in 2d. The relation is $1 - 1$.

The geometry is displayed already at the $(1,1)$ level as we shall see

$$S = \int d^2x \mathbb{D}^2 \bar{\mathbb{D}}^2 K(\Phi, \bar{\Phi})_| = \int d^2x D_+ D_- \left\{ D_+ \Phi K_{,i\bar{j}} D_- \bar{\Phi} + ... \right\}$$

So to characterise the geometry $(1,1)$ suffices.

Chiral models 4.

We reduce the chiral model to $(1,1)$ as follows:

The $(1,1)$ superfields are $\Phi = \Phi_|$, where the vertical bar denotes setting half of the $(1,1)$ Fermi coordinates to zero $\theta - \bar{\theta} = 0$. The spinorial derivatives reduce as

$$\mathbb{D}_\pm = D_\pm - iQ_\pm$$
$$\bar{\mathbb{D}}_\pm = D_\pm + iQ_\pm$$

where the Ds are the $(1,1)$ derivatives and the Qs generate the non manifest second supersymmetries. The action becomes

$$S = \int d^2x \mathbb{D}^2 \bar{\mathbb{D}}^2 K(\Phi, \bar{\Phi})_| \rightarrow \int d^2x D^2 Q^2 K(\Phi, \bar{\Phi})_|$$

Chiral models 5.

We evaluate the action using

$$Q_\pm \Phi^a = J^a_b D_\pm \Phi^b \ , \quad J = \begin{pmatrix} i\delta^i_j & 0 \\ 0 & -i\delta^{\bar{i}}_{\bar{j}} \end{pmatrix}$$

where $J = diag(i. - i)$, which follows from the reduction of the chirality constraints.

$$0 = \bar{\mathbb{D}}_\pm \Phi^i = (D_\pm + iQ_\pm)\Phi^i \ , \qquad \Rightarrow Q_\pm \Phi^i = iD_\pm \Phi^i$$

$$\Rightarrow Q_\pm \bar{\Phi}^{\bar{i}} + = -iD_\pm \bar{\Phi}^{\bar{i}}$$

Using this and integrating by parts we find the $(1,1)$ action:

$$\int d^2x D^2 Q^2 K(\Phi, \bar{\Phi})_| = \int d^2x D^2 \Big(D_+ \Phi^i K_{i\bar{j}}(\Phi) D_- \Phi^{\bar{j}} \Big)$$

Chiral and twisted models 1.

Chiral superfields $\bar{\mathbb{D}}_{\pm}\Phi = 0$

Twisted Chiral superfields χ $\bar{\mathbb{D}}_{+}\chi = 0 \ , \quad \mathbb{D}_{-}\chi = 0$

The $(1,1)$ reduction $\mathbb{D}_{\pm} = D_{\pm} - iQ_{\pm} \ , \quad \bar{\mathbb{D}}_{\pm} = D_{\pm} + iQ_{\pm}$ gives

$$Q_{\pm}\Phi^{a} = J^{a}_{b}D_{\pm}\Phi^{b} \ , \quad Q_{\pm}\chi^{a'} = \pm J^{a'}_{b'}D_{\pm}\chi^{b'} \ .$$

Consider a model with both chiral and twisted chiral fields

$$\int d^2xD^2Q^2K(\Phi,\bar{\Phi},\chi,\bar{\chi})_| =: \int d^2xD^2Q^2K(\mathbb{X})_|$$

$$= \int d^2xD^2\Big(D_{+}\mathbb{X}^{A}(G_{AB}+B_{AB})(\mathbb{X})D_{-}\mathbb{X}^{B}\Big)$$

$$= \int d^2xD^2\Big(D_{+}\mathbb{X}^{A}E_{AB}(\mathbb{X})D_{-}\mathbb{X}^{B}\Big)$$

Chiral and twisted models 2.

The metric and B-field are given by

$$E_{AB} = \begin{pmatrix} 0 & K_{i\bar{j}} & K_{ij'} & 0 \\ K_{\bar{i}j} & 0 & 0 & K_{\bar{i}\bar{j}'} \\ -K_{i'j} & 0 & 0 & -K_{i'\bar{j}'} \\ 0 & -K_{\bar{i}'\bar{j}} & -K_{\bar{i}'j'} & 0 \end{pmatrix}$$

which leads to the following field strength

$$H_{i\bar{j}k'} = K_{,i\bar{j}k'} \;, \qquad H_{i\bar{j}k'} = -K_{,i\bar{j}\bar{k}'}$$
$$H_{i'\bar{j}'k} = -K_{,i'\bar{j}'k} \;, \quad H_{i'\bar{j}'\bar{k}} = K_{,i'\bar{j}'\bar{k}} \;.$$

Chiral and twisted models 3.

The two complex structures can be read off from the non manifest transformations of Φ and χ according to

$$\delta_{\pm}\mathbb{X} = \epsilon_{(\pm)}^{\alpha}\mathbb{J}^{(\pm)}D_{\alpha}$$

which leads to

$$\mathbb{J}^{(+)} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad \mathbb{J}^{(-)} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}.$$

It is easy to see that they commute $[\mathbb{J}^{(+)}, \mathbb{J}^{(-)}] = 0$ and a bit more effort shows that

$$\nabla_{(+)}\mathbb{J}^{(+)} = 0\,, \quad \nabla_{(-)}\mathbb{J}^{(-)}$$

where $\pm$ refers to $\pm\frac{1}{2}Hg^{-1}$ torsion. Finally, a local product structure is seen to be

$$\mathbb{K} = -\mathbb{J}^{(+)}\mathbb{J}^{(-)} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

A manifold endowed with two complex structures J_+ and J_-, a metric g and an antisymmetric B-field B carries a *bihermitian geometry* if g is hermitian with respect to both complex structures

$$J^t_\pm g J_\pm = g \ ,$$

and the two complex structures are covariantly constant with respect to two connections with torsion

$$\nabla_+ J_+ = 0 \ , \quad \nabla_- J_- = 0 \ ,$$

where the torsionfull connections are

$$\nabla_+ = \nabla_0 + T \ , \quad \nabla_- = \nabla_0 - T \ , \quad T_{ij}{}^k = \tfrac{1}{2} H_{ijn} g^{nk} \ .$$

There are two distinct cases of this geometry depending on whether the two complex structures commute or not.

Bihermitean geometry 3

When the complex structures commute, $[J_+, J_-] = 0$, they define a third structure, a local product structure $\mathbb{K}$, by

$$\mathbb{K} := -J_+ J_- \ , \quad \Rightarrow \mathbb{K}^2 = 1 \ ,$$

This geometry is sometimes called a *BILP geometry* (for bihermitean local product).

> BILP geometry is the target space geometry of $(2,2)$ sigma models with B field and becomes manifest when the model is written in terms of chiral and twisted chiral superfields. It is a special case of Generalised Kähler geometry.

There is a generalisation to p left and q right complex structures: (p, q) Hermitean geometry.

Semichiral models 1

T. Buscher, U. L, M. Rocek Phys.Lett.B 202 (1988) 94-98

Left ℓ and Right $\mathfrak{r}$ Semichiral superfields $\bar{\mathbb{D}}_+\ell = 0, \;\; \bar{\mathbb{D}}_-\mathfrak{r} = 0$

The reduction

$$\bar{\mathbb{D}}_+\ell = (D_+ + iQ_+)\ell = 0 \;, \quad \Rightarrow Q_+L = JD_+L$$
$$\bar{\mathbb{D}}_-\mathfrak{r} = (D_- + iQ_-)\mathfrak{r} = 0 \;, \quad \Rightarrow Q_-R = JD_-R$$

where $L = (\ell, \bar{\ell})$ and $R = (\mathfrak{r}, \bar{\mathfrak{r}})$. In addition we define

$$Q_-L = \Psi_- \;, \qquad Q_+R = \Psi_+$$

These are spinorial auxiliary fields. When they are integrated out of an action they become part of the complex structures.

Semichiral models 2

A general action is

$$S = \int d^2x \mathbb{D}^2 \bar{\mathbb{D}}^2 K(L, R)_| \to \int d^2x D^2 Q^2 K(L, R)_|$$

Pushing in the Qs and using the definitions of the $(1, 1)$ components gives, after integrating out the auxiliary spinors,

$$\int d^2x D^2 \left(D_+ \mathbb{X}^A E_{AB}(\mathbb{X}) D_- \mathbb{X}^B \right)$$

where $\mathbb{X}^A = (L, R)$. Integrating out $\Psi_\pm$ results in *non-linear* relations.

The metric plus B field are

$$E_{LL} = [J, K_{LL}]K^{LR}JK_{RL}$$
$$E_{LR} = JK_{LR}J + [J, K_{LL}]K^{LR}[J, K_{RR}]$$
$$E_{RL} = -K_{RL}JK^{LR}JK_{RL}$$
$$E_{RR} = -K_{RL}JK^{LR}[J, K_{RR}]$$

while the complex structures read ($C_{LL} := [J, K_{LL}]$ etc.)

$$\mathbb{J}_+ = \begin{pmatrix} J & 0 \\ K^{RL}C_{LL} & K^{RL}JK_{LR} \end{pmatrix}, \quad \mathbb{J}_- = \begin{pmatrix} K^{LR}JK_{RL} & K^{LR}C_{RR} \\ 0 & J \end{pmatrix}$$

and will not commute in general.

This describes Bi-hermitean geometry for the symplectic case where the complex structures do not commute. The general case involves chiral, twisted chiral and semichiral fields.

$$K \to K(\Phi, \chi, L, R)$$

Generalised Complex Geometry 1: Generalised Tangent Space

N. Hitchin 2002, M. Gualtieri 2003, C.Hull...

In Generalised Complex Geometry (GCG) the tangent bundle $T\mathcal{M}$ is replaced by the sum of the tangent and cotangent bundles,

$$\mathbb{T} := T\mathcal{M} \oplus T^*\mathcal{M} ,$$

called the generalised tangent bundle. Elements of $\mathbb{X} \in \mathbb{T}$ may be written as

$$\mathbb{X} = X + \xi ,$$

where

$$X \in T\mathcal{M} , \quad \xi \in T^*\mathcal{M} .$$

Alternatively, it is often useful to write $\mathbb{X}$ as a column vector

$$\mathbb{X} = \left(\begin{array}{c} X \\ \xi \end{array} \right)$$

GCG 2. Complex Structure

A generalised almost complex structure on $\mathcal{M}$ is an endomorphism $\mathbb{J}$ of the tangent bundle which squares to minus one

$$\mathbb{J}: \ \mathbb{T}\mathcal{M} \longrightarrow \mathbb{T}\mathcal{M} \ , \quad \mathbb{J}^2 = -\mathbf{1} \ ,$$

and preserves the natural pairing metric

$$\mathbb{J}^t \eta \mathbb{J} = \eta \ , \quad \eta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

$$\mathbb{Y}^t \eta \mathbb{X} = (Y, \sigma)\eta \begin{pmatrix} X \\ \xi \end{pmatrix} = Y^\mu \xi_\mu + \sigma_\mu X^\mu \ .$$

The projection operators

$$\Pi_{\pm} := \tfrac{1}{2}\left(\mathbf{1} \pm i\mathbb{J}\right)$$

may be used to split the generalised tangent space in two parts at a point: The $+i$ eigenspace $\mathbb{L}$ and the $-i$ eigenspace $\overline{\mathbb{L}}$

$$\mathbb{T} \otimes \mathbb{C} = \mathbb{L} \oplus \overline{\mathbb{L}} = \Pi\mathbb{L} \oplus \overline{\Pi}\mathbb{L}\ .$$

In complete analogy to

$$T \otimes \mathbb{C} = T^{(0,1)} \oplus T^{(1,0)}$$

for the ordinary complex structure on $\mathcal{M}$.

GCG 3. Brackets

The Courant bracket $[\![\ ,\ ]\!]_C$ on $\mathbb{T}$ is defined by

$$[\![\mathbb{X},\mathbb{Y}]\!]_C = [\![X+\xi, Y+\eta]\!]_C = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi),$$

where $\mathbb{X},\mathbb{Y} \in C^\infty(\mathbb{T})$ and $\mathcal{L}_X$ is the Lie derivative with respect to X.

The Courant bracket is antisymmetric in its arguments but does not satisfy the Jacobi identity. The Dorfman bracket

$$[\![\mathbb{X},\mathbb{Y}]\!]_D = [X,Y] + \mathcal{L}_X d\eta - i_Y d\xi\ ,$$

does satisfy the Jacobi identity but is not antisymmetric. The relation between the brackets is

$$[\![\mathbb{X},\mathbb{Y}]\!]_D = [\![\mathbb{X},\mathbb{Y}]\!]_C + d\eta(\mathbb{X},\mathbb{Y})\ .$$

Clearly, the brackets are equal when restricted to an isotropic subspace (i.e. a subspace $\mathcal{M}$ for which $\eta(\mathbb{X},\mathbb{Y}) = 0$ for all $\mathbb{X},\mathbb{Y} \in \mathcal{M}$).

GCG 4. Integrability

Both brackets may be twisted by a closed three form H:

$$[\![\mathbb{X},\mathbb{Y}]\!] \to [\![\mathbb{X},\mathbb{Y}]\!] + i_Y i_X H.$$

Integrability of an a generalised almost complex structure $\mathbb{J}$ is defined by requiring that the subspaces defined by the projections are involutive, i.e., that

$$\Pi_{\mp}[\![\Pi_{\pm}\mathbb{X},\Pi_{\pm}\mathbb{Y}]\!]_C = 0\ , \quad \forall\, \mathbb{X},\mathbb{Y} \in C^{\infty}(\mathbb{T}).$$

This requires the vanishing of the generalised Nijenhuis tensor $\mathbb{N}$. In index free notation

$$\mathbb{N}_{\mathbb{J}}(\mathbb{X},\mathbb{Y}) = [\![\mathbb{X},\mathbb{Y}]\!]_C + \mathbb{J}[\![\mathbb{J}\mathbb{X},\mathbb{Y}]\!]_C + \mathbb{J}[\![\mathbb{X},\mathbb{J}\mathbb{Y}]\!]_C - [\![\mathbb{J}\mathbb{X},\mathbb{J}\mathbb{Y}]\!]_C = 0\ .$$

It is sometimes advantageous to define integrability with respect to the H-twisted bracket.

An integrable generalised almost complex structure is called a **generalised complex structure**.

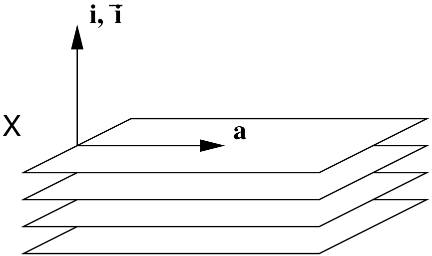
A complex geometry $\mathcal{M}, J$ or a symplectic geometry $\mathcal{M}, \omega$ are seen to be special cases of Generalised Geometry where the Generalised complex structures are

$$\mathcal{J}_J = \left(\begin{array}{cc} J & 0 \\ 0 & -J^t \end{array} \right) ,$$

and

$$\mathcal{J}_\omega = \left(\begin{array}{cc} 0 & -\omega^{-1} \\ \omega & 0 \end{array} \right) ,$$

respectively.



A generalisation of the Newlander-Nirenberg theorem shows that locally a GC manifold X looks like a foliation with $z^i, \bar{z}^{\bar{i}}$ complex and x^a Darboux coordinates.

Generalised Kähler Geometry 1.

J.Gates, C.Hull and M. Roček 1984: M.Gualtieri 2003: U,L., M. Rocek, R. von Unge, M.Zabzine 2007

Generalised Kähler Geometry is the form Bi-Hermitean takes when lifted to Generalised Complex Geometry. The additional data is that there are *two* Generalised Complex structures, $\mathcal{J}_1$ and $\mathcal{J}_2$, that *commute*, $[\mathcal{J}_1, \mathcal{J}_2] = 0$, and whose product gives an integrable local product structure $\mathcal{G}$

$$\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2 \; , \quad \mathcal{G}^2 = \mathbf{1} \; .$$

Integrability is again defined with respect to the Courant bracket or its H-twisted version. Under quite general conditions the existence on $\mathcal{M}$ of a metric g acting T, with inverse g^{-1} that acts on T^*, ensures that one can find coordinates and B-transforms such that

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \; .$$

GKG 2

Using $\mathcal{G}$ we may introduce projection operators

$$P_{\pm} := \frac{1}{2}\left(1 \pm \mathcal{G}\right) ,$$

that may be used to split the generalised tangent space $\mathbb{T}$ into the ± 1 eigenspaces of $\mathcal{G}$:

$$\mathbb{T} = \mathbb{T}_{+} \oplus \mathbb{T}_{-} ,$$

where

$$\mathbb{T}_{\pm} := P_{\pm}\mathbb{T} .$$

GKG 3. Gualtieri's map

Given the Bi-Hermitean data $(\mathcal{M}, g, H, J_+, J_-)$, we define two (non commuting) GCS's $\mathcal{J}_+$ and $\mathcal{J}_-$

$$\mathcal{J}_+ = \begin{pmatrix} J_+ & 0 \\ 0 & -J_+^t \end{pmatrix} , \quad \mathcal{J}_- = \begin{pmatrix} J_- & 0 \\ 0 & -J_-^t \end{pmatrix}$$

which commute with $P_\pm$ and are integrable wrt the H-twisted Courant bracket.

There is then a $1-1$ map between the bi-Hermitean data and Generalised Kähler geometry which reads

$$\mathcal{J}_{1/2} = \tfrac{1}{2}\Big(P_+\mathcal{J}_+ \pm P_-\mathcal{J}_-\Big) .$$

Explicitly

$$\mathcal{J}_{1/2} = \tfrac{1}{2}\begin{pmatrix} J_+ & -(\omega)^{-1}_+ \\ \omega_+ & -J^t_+ \end{pmatrix} \pm \tfrac{1}{2}\begin{pmatrix} J_- & -(\omega)^{-1}_- \\ \omega_- & -J^t_- \end{pmatrix} ,$$

where $\omega_\pm = gJ_\pm$ are the two=forms associated with $J_\pm$.

Notice that the B field only enters the definitions via H in the integrability conditions

$$\Pi^{1/2}_\mp [\![\Pi^{1/2}_\pm \mathbb{X}, \Pi^{1/2}_\pm \mathbb{Y}]\!]_H = 0 \, , \quad \forall\, \mathbb{X}, \mathbb{Y} \in C^\infty(\mathbb{T}).$$

where

$$\Pi^{1/2}_\mp := \tfrac{1}{2}\left(1 \pm i\mathcal{J}^{1/2}\right) \ .$$

Partial Summary

Susy	(1,1)	(2,2)	(2,2)	(4,4)	(4,4)
E=G+B	G, B	G	G, B	G	G, B
Geom	Riem.	Kähler	biherm.	hyperk.	bihyperc.

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