Vortex partition functions of 2D $\mathcal{N} = (2, 2)$ SYM and quantum groups

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2D SUSY Theories and Related Topics (MATRIX)

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[Based on [JEB 2107.10063, 2202.?????]**]**

Motivations

- In the last decades, several correspondences between **supersymmetric quantum field theories** and **integrable systems** have been observed. Our general goal is to study the **mathematical structure** behind these relations, and hopefully use it to improve our understanding of QFT.

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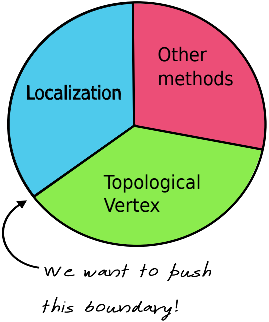
- Focus here on the algebraic structure behind **Bethe/gauge** and **BPS/CFT correspondences**.

$\Rightarrow$ The main role is played by **non-perturbative** contributions (instantons/vortices).

Methods to compute exactly the instanton corrections:



A few reasons for developing this "topological vertex" technique:

- Emphasize the role of the non-perturbative symmetry
  - $\rightsquigarrow$ Correspondences (CFT, integrability), KZ equations, blow up equation,...
- Develop a diagrammatic technique that simplifies actual calculations
  - $\rightsquigarrow$ "CFT methods" (generalization of vertex operators)
- Study of non-Lagrangian theories
- Connect with string theory realizations of QFT and brane systems
  - $\rightsquigarrow$ Description of non-perturbative dualities, RG flows,...

How?

- This technique was first developed in the context of 5D $\mathcal{N}=1$ gauge theories.

⇝ Our story starts with two observations:

i. Localization does not only give you a partition function, but also (sometimes) an algebra!

⇝ ADHM construction leads to COHA (COhomological Hall Algebra) of quiver varieties.

⇝ Many useful applications: AGT correspondence (4D $\mathcal{N}=2$/2D CFT), integrability,...

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ii. The topological vertex also comes with an algebra! **[Awata, Feigin, Shiraishi 2011]**

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toric diagram $\longleftrightarrow$ network of representations

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We will use this algebra to extend the topological vertex technique!

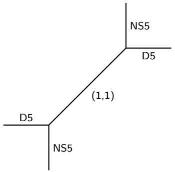
- To complete the picture and relate i. and ii., we need a third ingredient:

iii. Correspondence between topological strings and IIB string theory

toric diagram $\longleftrightarrow$ (p,q)-brane web

Recall: (p,q)-brane = bound state of p D5 and q NS5 branes

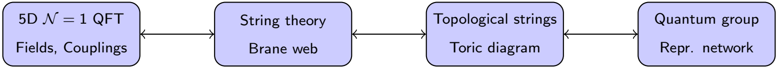
$\rightsquigarrow$ The low energy brane dynamics is described by a **5D $\mathcal{N} = 1$ gauge theory**.



	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D5	x	x	x	x	x		x			
(p,q)	x	x	x	x	x	θ	θ			

[Aharony, Hanany 1997] [Leung, Vafa 1997]

- Combining everything, we arrive at



$\Rightarrow$ Dropping "topological strings", we get to the principle of **algebraic engineering**.

Algebraic Engineering

Goal: Realize supersymmetric QFT in the representation theory of a quantum group.

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Main results of the form:

$$\mathcal{Z} = \langle \mathcal{T} \rangle$$

Partition Function

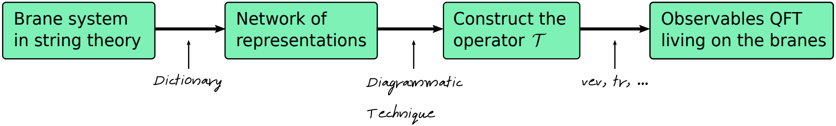
Operator acting on repr. network

$$\chi(z) = \langle X(z) \mathcal{T} \rangle$$

qq-character (BPS loop observable)

Algebra current

- This approach can be summarized as follows:



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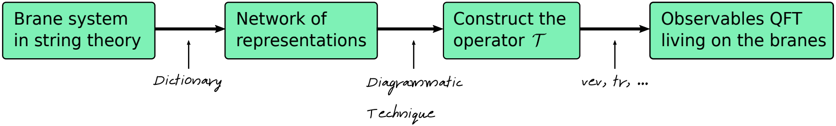
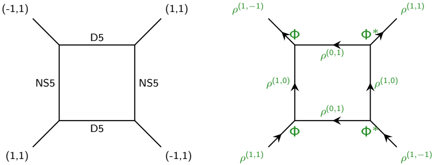


- For instance, the $\mathcal{T}$ operator corresponding to a pure $U(2)$ Super Yang-Mills theory is



$$\mathcal{T} = \sum_{\lambda_1,\lambda_2} \Phi_{\lambda_1}\Phi_{\lambda_2} \otimes \Phi^*_{\lambda_1}\Phi^*_{\lambda_2}$$

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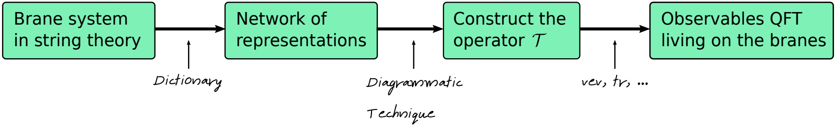

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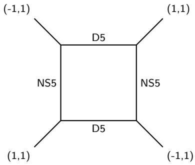

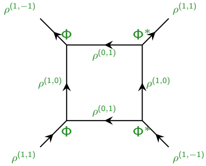

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How do we apply this program to 2D $\mathcal{N} = (2,2)$ SYM theories?!!

II. Vortex partition functions

2D $\mathcal{N} = (2,2)$ SYM theories

- We will consider a class of 2D Super Yang-Mills theories on the Euclidean background $\mathbb{C}_\epsilon$

$\rightsquigarrow$ **Omega deformation** $\Rightarrow$ supersymmetry preserving IR regulator ϵ

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- They are obtained as a deformation of **2D $\mathcal{N} = (4,4)$ SYM theories** by a **twisted mass**:

Start with a $\mathcal{N} = (4,4)$ **vector multiplet** with $U(N)$ gauge group, decompose as (Σ, Φ) with

- $\mathcal{N} = (2,2)$ twisted chiral multiplet $\Sigma = \sigma + i\sqrt{2}\theta^+\bar{\lambda}_+ - i\sqrt{2}\bar{\theta}^-\lambda_- + \cdots$
- $\mathcal{N} = (2,2)$ chiral multiplet $\Phi = \tilde{\sigma} + \sqrt{2}\theta^+\tilde{\lambda}_+ + \sqrt{2}\theta^-\tilde{\lambda}_- + \cdots$

Introduce the superpotential $\hat{W}(\Phi) = m_\Phi \Phi^2$ that breaks $\mathcal{N} = (4,4) \to \mathcal{N} = (2,2)$.

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- $\mathcal{N} = (4,4)$ hypermultiplets can be decomposed into $\mathcal{N} = (2,2)$ chiral multiplets $(Q_a, \tilde{Q}^a)$

More generally, we take here:

- $N^{(\mathrm{f})} \geq N$ fundamental chiral multiplets Q_a
- $N^{(\mathrm{af})}$ antifundamental chiral multiplets $\tilde{Q}^a$

2D or 3D?

- In fact... a little easier to study K-theoretic (q-deformed) partition function!

  $\Rightarrow$ Study 3D $\mathcal{N} = 2$ theories on $\mathbb{C}_\epsilon \times S^1_R$, recover 2D results as $R \to 0$ ($q^2 = e^{R\epsilon} \to 1$)

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- Similarly, obtained as a massive deformation of 3D $\mathcal{N}=4$ theories:

  $\rightsquigarrow$ $\mathcal{N}=4$ vector multiplet (V, Φ), with:

  $V = (A_\mu, \sigma)$ $\mathcal{N}=2$ vector multiplet, Φ $\mathcal{N}=2$ chiral multiplet $m_\Phi \to \infty$

  $\rightsquigarrow$ $\mathcal{N}=4$ hypermultiplets $(Q_a, \tilde{Q}^a)$ broken down to:

  - $N^{(f)} \geq N$ fundamental chiral multiplets Q_a with exponentiated mass $\nu_a^{(f)} = e^{Rm_a^{(f)}}$
  - $N^{(af)}$ antifundamental chiral multiplets $\tilde{Q}^a$ with mass $\nu_a^{(af)} = e^{Rm_a^{(af)}}$

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- Choice of vacuum $\sigma = \mathrm{diag}\ m_i^{(\mathrm{f})}$, $q_i^a = \sqrt{r}\delta_i^a \rightsquigarrow \nu_i = \nu_i^{(\mathrm{f})}$ for $i = 1 \cdots N$.

  (Higgs branch with twisted mass term)

[Yoshida 2011] [Fujitsuka, Honda, Yoshida 2014]

- The partition function receives **non-perturbative contributions** from vortex configurations:

$$\mathcal{Z}_{\text{vortex}} = \sum_{k_1,k_2,\cdots,k_N \geq 0} \mathfrak{q}^{\sum_i k_i} \prod_{i,j=1}^{N} N_{k_i,k_j}(\nu_i/\nu_j)^{-1} \frac{\prod_{i=1}^{N} \prod_{a=1}^{N^{(\text{af})}} (\nu_i/\nu_a^{(\text{af})}; q^2)_{k_i}}{\prod_{i=1}^{N} \prod_{a=N+1}^{N^{(\text{f})}} (\nu_i/\nu_a^{(\text{f})}; q^2)_{k_i}},$$

with vortex charge $k = \frac{1}{2\pi} \int \text{tr}\, F$ and $\mathfrak{q} = e^{2\pi(r+i\theta)}$ (r: FI parameter, θ: theta angle).

- We have introduced a "3D Nekrasov factor"

$$N_{k,k'}(\alpha) = (\alpha q^{2k-2k'+2}; q^2)_{k'}, \quad (z,q^2)_k = \prod_{j=0}^{k-1} (1 - z q^{2j}).$$

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This is the starting point of our algebraic construction.

Vertical action

- Consider the abelian theory with $N^{(\mathrm{f})} = N = 1$, $N^{(\mathrm{af})} = 0 \rightsquigarrow$ vortex number $k \in \mathbb{Z}^{\geq 0}$

  $\Rightarrow$ The COHA acts on states $|k\rangle\rangle$ = configurations of localization fixed points.

- The algebra is defined by the action of the currents

$$X^{\pm}(z) = \sum_{n\in\mathbb{Z}} z^{-n} X_n^{\pm}, \quad \Psi^{\pm}(z) = \sum_{\pm n \geq 0} z^{-n} \Psi_n^{\pm},$$

on the states $|k\rangle\rangle$ (NB: $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$)

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We recognize the prefundamental representation of the asymptotic quantum affine $\mathfrak{sl}(2)$ algebra!

[Hernandez-Jimbo 2012]

Intermezzo: how to derive the COHA?

• In general, we need to use the techniques of algebraic geometry... But we try to give here a simpler intuitive presentation in the form

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• Matrix elements involve the Y-observables encoding variations of the Nekrasov factor,

$$\frac{N_{k,k'+1}(\nu/\nu')}{N_{k,k'}(\nu/\nu')} = Y_k(\chi'_{k'+1}),\quad \frac{N_{k+1,k'}(\nu/\nu')}{N_{k,k'}(\nu/\nu')} = Y_{k'}^*(q^2\chi_{k+1}),$$

Explicitly, $\quad Y_k(z) = 1-\nu q^{2k}/z,\quad Y_k^*(q^2z) = q^{2k}\dfrac{z-\nu q^{-2}}{z-\nu q^{2k-2}}.$

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- Recall the $\mathfrak{sl}(2)$ algebra $[X^+, X^-] = H$, $[H, X^\pm] = \pm 2X^\pm$.

⇝ The $\hat{\mathfrak{sl}}(2)$ Kac-Moody algebra is obtained as $\mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$.

⇝ It is generated by $X^\pm \otimes t^n \to X_n^\pm$, $H \otimes t^n \to H_n$ satisfying

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- These relation can be rewritten using currents $X^\pm(z) = \sum_n z^{-n} X_n^\pm$, $H(z) = \sum_n z^{-n} H_n$,

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- In the quantum algebra, the relations are q-deformed:

The Cartan are exponentiated $\Psi^\pm(z) = \Psi_0^\pm e^{(q-q^{-1})\sum_{\pm n>0} z^{-n} H_n}$, and

$$[X^+(z), X^-(z)] = \frac{1}{q - q^{-1}} \left(\delta(q^{-c} z/w)\Psi^+(q^{c/2} w) - \delta(q^c z/w)\Psi^-(q^{-c/2} w) \right).$$

- The **quantum group** structure follows from the presence of a **coproduct** $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$\Delta(X^+(z)) = X^+(z) \otimes 1 + \Psi^-(q^{c_{(1)}/2}z) \otimes X^+(q^{c_{(1)}}z),$$

$$\Delta(X^-(z)) = X^-(q^{c_{(2)}}z) \otimes \Psi^+(q^{c_{(2)}/2}z) + 1 \otimes X^-(z),$$

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- **Asymptotic?** Do not impose $\Psi_0^+ \Psi_0^- = 1$ to allow more general representations

$\rightsquigarrow$ Necessary to define the prefundamental rep. = limit $N \to \infty$ of spin $N/2$ rep.

Back to algebraic engineering...

- In general, we need the following ingredients:
  - A quantum group $\rightsquigarrow$ (asymptotic) quantum affine $\mathfrak{sl}(2)$ algebra
  - A vertical representation $\rightsquigarrow$ prefundamental representation ($c = 0$)
  - A horizontal representation ?
  - Intertwining operators $\Phi : V \otimes H \to H$ and $\Phi^* : H \to V \otimes H$?
  - Gluing rules?

Back to algebraic engineering...

- In general, we need the following ingredients:
  - A quantum group $\rightsquigarrow$ (asymptotic) quantum affine $\mathfrak{sl}(2)$ algebra
  - A vertical representation $\rightsquigarrow$ prefundamental representation ($c = 0$)
  - A horizontal representation ?
  - Intertwining operators $\Phi : V \otimes H \to H$ and $\Phi^* : H \to V \otimes H$?
  - Gluing rules?
- This is analogous to the conformal blocks of a 2D chiral CFT,

$\Rightarrow$ We need a vertex representation on the bosonic Fock space $\mathcal{F} = \mathbb{C}[e^{\pm Q}, J_{-1}, J_{-2}, \cdots] \, |\emptyset\rangle$.

Horizontal action

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$$\rho^{(H)}(X^+(z)) = uz^{-n}e^{Q}e^{\sum_{k>0}\frac{z^k}{k}(1-q^{2k})J_{-k}}e^{-\sum_{k>0}\frac{z^{-k}}{k}J_k}q^{-2J_0},$$

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$$\rho^{(H)}(\Psi^+(z)) = 0, \quad \rho^{(H)}(\Psi^-(z)) = e^{-\sum_{k>0}\frac{z^k}{k}(1+q^{2k})(1-q^{-2k})J_{-k}}q^{-2J_0},$$

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- Then, we compute the intertwiners

$$\rho^{(H)}\Phi = \Phi\ \left(\rho_\nu^{(V)}\otimes\iota_{P_\nu}^*\rho^{(H)}\ \Delta\right),\quad \left(\rho_\nu^{(V)}\otimes\rho^{(H)}\ \Delta'\right)\Phi^* = \Phi^*\ \iota_{P_\nu^*}\rho^{(H)}.$$

$\rightsquigarrow$ We will come back to the shift of representations ι_P, ι_P^* in the next section.

The operator $\mathcal{T}$

- The operator $\mathcal{T}$ is constructed by gluing intertwiners:

$\rightsquigarrow$ horizontal gluing: product of operators

$\rightsquigarrow$ vertical gluing: scalar product $\langle\langle k \| k' \rangle\rangle = n_k \delta_{k,k'}$

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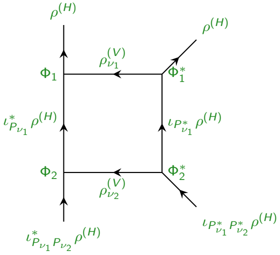
- For instance, SYM with $U(2)$ gauge group and $N^{(\mathrm{f})} = 2$, $N^{(\mathrm{af})} = 0$:



$$\mathcal{T} = \sum_{k_1,k_2\geq 0} \Phi_{k_1}\Phi_{k_2} \otimes \Phi^*_{k_1}\Phi^*_{k_2}$$

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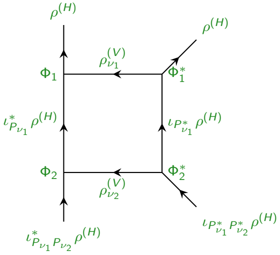

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We recover indeed $\mathcal{Z}_{\mathrm{vortex}} = \left(\langle \emptyset | \otimes \langle \emptyset | \right) \mathcal{T} \left(| \emptyset \rangle \otimes | \emptyset \rangle \right)$.

Summary

- By evaluating $\mathcal{Z}_{\text{vortex}} = \langle \mathcal{T} \rangle$, we recover the partition functions of the gauge theories.

$\rightsquigarrow$ Applies to A-type quivers with $U(N)$ gauge groups.

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c.f. **[Haouzi 2020]**

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- We still need to elucidate:
  - The relation with **brane systems**.
  - The role of **shifted** quantum algebras.

We will answer these questions in the context of 5D theories.

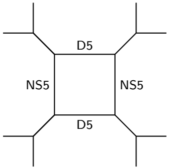
III. Higgsing, the algebraic way

Higgsing

- Our 3D $\mathcal{N}=2$ theories can be obtained from 5D $\mathcal{N}=1$ theories through Higgsing.

$\Rightarrow$ Focus here on 5D $\mathcal{N}=1$ $U(2)$ gauge theory with **four flavors**.

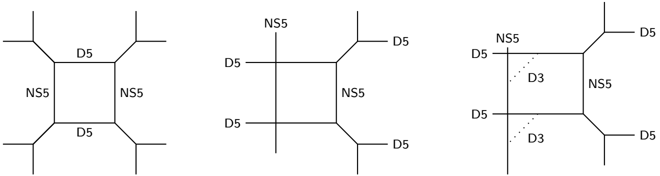
- It is realized using (p,q)-branes as follows:



	0	1	2	3	4	5	6	7	8	9
Ω-bg		ϵ_2	ϵ_2	ϵ_1	ϵ_1				ϵ_3	ϵ_3
NS5	x	x	x	x	x	x				
D5	x	x	x	x	x		x			
(p,q)	x	x	x	x	x	θ	θ			
D3	x	x	x					x		

- This is a two steps procedure:

1. Adjust the mass of two hypermultiplets.



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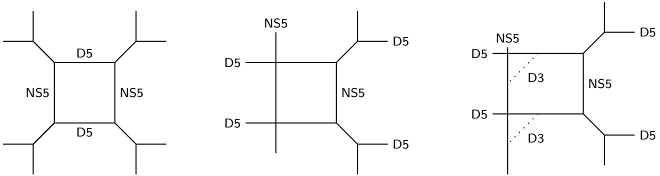

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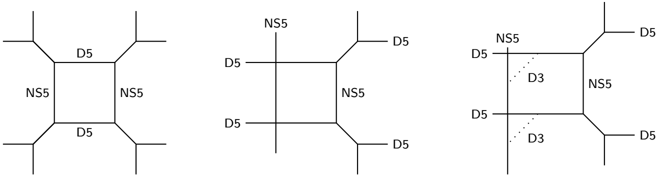

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How to understand this procedure from an algebraic point of view?

Engineering of 5D theories

- 5D $\mathcal{N}=1$ theories on $\mathbb{C}_{\epsilon_1}\times\mathbb{C}_{\epsilon_2}\times S^1_R$ are realized in the representation theory of the **quantum toroidal** $\mathfrak{gl}(1)$ **algebra** (= Ding-Iohara-Miki algebra).

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- In the **vertical representation** the currents act on states $|\lambda\rangle\rangle$ where the partition λ labels the configurations of instantons. $\rightsquigarrow$ D5-branes
- In the **horizontal representation**, the currents are vertex operators acting on the bosonic Fock space $\rightsquigarrow$ NS5($+n$ D5)-branes.

[Awata, Feigin, Shiraishi 2011]

[Awata, Kanno, Matsumoto, Mironov, Morozov, Morozov, Ohkubo, Zenkevich 2016]

[JEB, Fukuda, Harada, Matsuo, Zhu 2017]

Step 1. Is realized using the **shifted quantum toroidal** $\mathfrak{gl}(1)$ algebra

⇝ Main idea of shifted quantum algebra: modify the expansion of

$$\psi^{\pm}(z) = \sum_{\pm n \geq -\mu_{\pm}} z^{-n} \psi^{\pm}_{n}.$$

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The **shifted vertical representation** takes the form

$$x^+(z)\,|\lambda\rangle\rangle = P(z) \sum_{\square \in A(\lambda)} \delta(v q_1^{i-1} q_2^{j-1}/z) A_\lambda^+(\square)\,|\lambda + \square\rangle\rangle,$$

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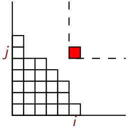
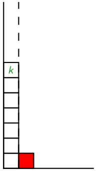
The zeros ν_a of $P(z)$ are the (exponentiated) masses of the hypermultiplets.

If $\nu_a = v q_1^{i-1} q_2^{j-1}$ for $\square = (i, j)$, then the representation becomes reducible!



- We adjust the hypermultiplets mass to $\nu_i = v_i q_1$ ($i = 1 \cdots N$).

$\Rightarrow$ Subrepresentation on Young diagrams with a single column of k boxes.



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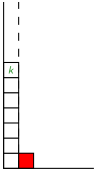

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$\Rightarrow$ Step 1. Replace the vertical representations $\rho_{v_i}^{(0,1)}$ by their shifted versions $\iota_{P_i}\rho_{v_i}^{(0,1)}$. Adjust a zero of $P(z)$ to obtain a subrepresentation on the module associated to the D3 branes.

Step 2. Take the limit $q_1 \to \infty$ (i.e. $\epsilon_1 \to \infty$) of the quantum toroidal $\mathfrak{gl}(1)$ algebra.

⇝ Formally, we find the asymptotic quantum affine $\mathfrak{sl}(2)$ algebra with $q^2 = q_2$.

⚠ Holds at the level of currents, not modes!!!

⇝ We derive the following limits for the representations:

- Shifted vertical representation $\iota_P \rho_v^{(0,1)} \longrightarrow$ prefundamental representation $\rho_v^{(V)}$.
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We recover in this way the algebraic construction of 3D $\mathcal{N} = 2$ theories!

IV. Discussion

Main results

- Over the last few years, the algebraic engineering technique has been applied successfully to several supersymmetry QFT and their brane systems:

Theories	Algebra	Reference
5D $\mathcal{N}=1$	quantum toroidal $\mathfrak{gl}(1)$	**[Awata, Feigin, Shiraishi 2011]**
5D $\mathcal{N}=1$ on $\mathbb{Z}_p$-orbifold	quantum toroidal $\mathfrak{gl}(p)$	**[Awata, Kanno, et. al. 2017]**
5D $\mathcal{N}=1$ on $\mathbb{Z}_p$-orbifold	new quantum toroidal algebras!	**[JEB, Jeong 2019]**
4D $\mathcal{N}=2$	affine Yangian $\mathfrak{gl}(1)$	**[JEB, Zhang 2018]**
6D $\mathcal{N}=(1,0)$	elliptic toroidal $\mathfrak{gl}(1)$	**[Foda, Zhu 2018]**
3D $\mathcal{N}=2^*$	quantum toroidal $\mathfrak{gl}(1)$	**[Zenkevich 2018]**
3D $\mathcal{N}=2$	quantum affine $\mathfrak{sl}(2)$	**[JEB 2107.10063]**

- Other results:

⇝ D-type quiver gauge theories **[JEB, Fukuda, Matsuo, Zhu 2017]**

⇝ qq-characters observables (generating function of Wilson loops)

[JEB, Fukuda, Harada, Matsuo, Zhu 2017]

Dictionary

- The choice of algebra reflects the background of the quantum field theory:
  - Factor $\mathbb{C}_\epsilon$ $\rightsquigarrow$ **affinization**
  - Factor S^1, $S^1 \times S^1$ $\rightsquigarrow$ **trigonometric/elliptic** deformations
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- To each brane is associated a module:
  - D5/NS5 branes $\rightsquigarrow$ 2D boson Fock space $\mathcal{F}$ (equivalence $\mathcal{S}$-duality)
  - D3 $\rightsquigarrow$ $\text{Span}\{|k\rangle\rangle, k \geq 0\}$

The weights correspond to the position of the branes in the brane web.

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- Hypermultiplets can be introduced as shifted representations $\rightsquigarrow$ transverse D7-branes

Zeros of $P(z)$ correspond to the (exponentiated) masses / branes location.

Summary

In **[JEB 2107.10063]**, we present new results relevant to both mathematics and physics:

Physics:

- Algebraic construction of BPS observables for 3D $\mathcal{N}=2$ gauge theories.
- New entry in the dictionary: hypermultiplets/shifted algebras.
- Present an algebraic description of Higgsing.

Mathematics:

- New vertex representations for the shifted qu. tor. $\mathfrak{gl}(1)$ and the qu. aff. $\mathfrak{sl}(2)$ algebras.
  - $\rightsquigarrow$ Symmetric polynomials, Shuffle algebra.
- Finite dimensional highest ℓ-weight representations for shifted qu. tor. $\mathfrak{gl}(1)$ algebras.
  - $\rightsquigarrow$ Application to toroidal quantum integrable systems? **[Litvinov, Vilkoviskiy 2020]**
- Description of the limit $q_1 \to \infty$ of some representations of the qu. tor. $\mathfrak{gl}(1)$ algebra.

A few open questions...

- Application of this algebraic construction?
  - ⇝ Integrability: **[Gadde, Gukov, Putrov 2014]**? Nekrasov-Shatashvili limit $\epsilon_2 \to 0$?
  - ⇝ Finite AGT correspondence? W-algebras and twisted vertex representation?
  - ⇝ Algebraic description of mirror symmetry? (Miki's automorphism)
  - ⇝ Knot invariants?
- Toward an algebraic description of brane systems?
  - ⇝ Higher dimensional theories and higher genus quantum groups?
  - ⇝ Treat more general backgrounds by combining different quantum groups?
  - ⇝ Beyond equivariant localization? Need new algebraic structures!

    Action on JK-residues? Quiver Yangian?
  - ⇝ Other observables? Using integrability results?

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Thank you!