On Phases of 3d $\mathcal{N} = 2$ Chern – Simons – Matter Theories

Wei Gu

Presentation at Matrix

Based on arXiv : 2105.02247 *with D.Pei and M.Zhang*,

arXiv : 2008.04909 *with L.Mihalcea, E.Sharpe and H.Zou*,

working in progress with L.Mihalcea, E.Sharpe and H.Zou

New observations

In this talk, I will be presenting some new observations:

- Taking phases does not necessarily commute with KK reductions.
- A three-dimensional version of CY+?/LG.
- Quantum K-theories from physics. (It is expected before conceptually, however, we give a technical method to do the concrete computations of a general target which has a trivial mirror-map.)

A brief review of 2d $\mathcal{N}$=(2,2) GLSMs

Since Kentaro has reviewed the basics of 2d $\mathcal{N}$=(2,2) GLSM last week, I only mention some relevant (or new) points in my talk.

- Setup.
  - I only focus on $U(k)$ gauge group in this talk. Super gauge field V, its field strength Σ with F_{01} to be one component field.
  - A twisted superpotential $\tilde{W} = -t \cdot tr\Sigma$, $t = r - i\theta$, where r is FI-parameter, θ is the usual theta angle in a gauge theory.
  - In this talk, only consider fundamental matter fields Φ_i. A possible homogeneous gauge-invariant superpotential $W(\Phi_i)$.

A brief review of 2d $\mathcal{N}$=(2,2) GLSMs

Since Kentaro has reviewed the basics of 2d $\mathcal{N}$=(2,2) GLSM last week, I only mention some relevant (or new) points in my talk.

- Setup.
  • I only focus on $U(k)$ gauge group in this talk. Super gauge field V, its field strength Σ with F_{01} to be one component field.
  • A twisted superpotential $\tilde{W} = -t \cdot tr\Sigma$, $t = r - i\theta$, where r is FI-parameter, θ is the usual theta angle in a gauge theory.
  • In this talk, only consider fundamental matter fields Φ_i. A possible homogeneous gauge-invariant superpotential $W(\Phi_i)$.
- Phases $\simeq$ ground states in a semi-classical region. Symmetries are important.
  • Lorentz symmetry: $U(t; \phi_i, F_{01})$. Perturbative RG-flow $r = \sum_i \rho_i \log \frac{\mu}{\Lambda}$
  • SUSY: $U = 0$.
  • One-form symmetries: F_{01} was missed in the previous studies. It is important! See my work '21 on the emergent dynamical decomposition or Eric's talk.
  • If semi-classical is not trusted, then we consider the effective theory by integrating out matters:

$$\widetilde{W}_{eff} = -(t + i(k-1)\pi) \sum_{a=1}^{k} \Sigma_a - \sum_i \rho_i^a \Sigma_a \left(\log\left(\rho_i^a \Sigma_a\right) - 1\right).$$

A CY-example

NLSM on quintic can be connected to an orbifolded LG model via 2d GLSM.

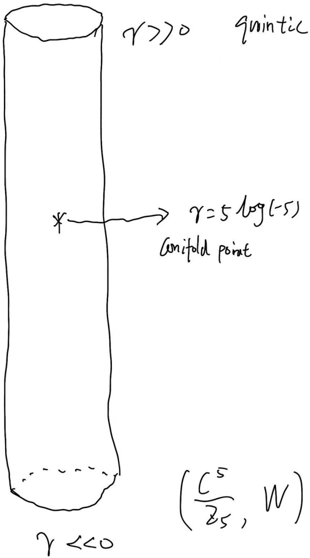

Figure: The connection between the Quintic and orbifolded LG model.

Charge-p projective space

We can consider non-CY cases.

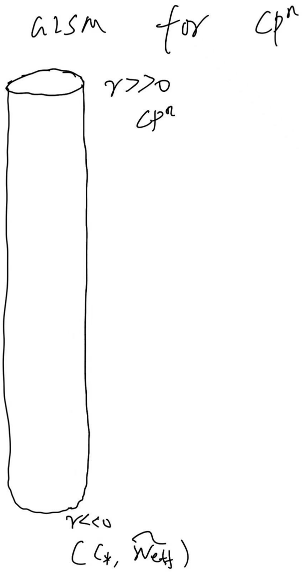

Figure: Projective Spaces.

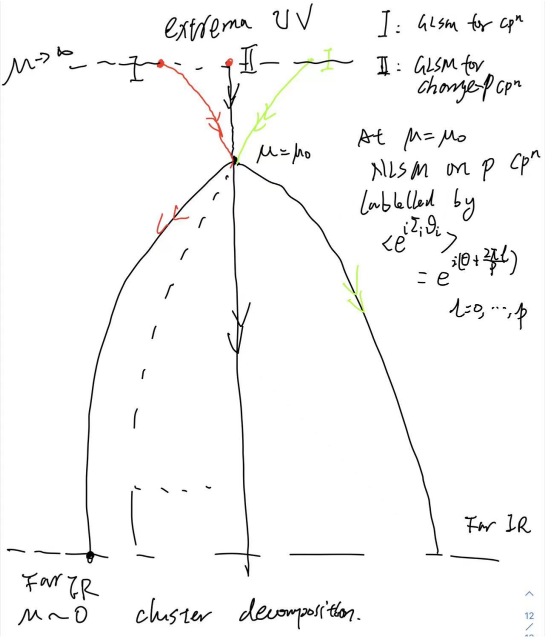

Figure: Decomposition.

3d Chern-Simons theories

For the theory without a superpotential, see Aharony, Hanany, Intriligator, Seiberg, and Strassler '97, de Boer, Hori, OZ, and Yin '97, and a review by Intriligator and Seiberg '13. The connections to 2d GLSMs, see Aganagic, Hori, Karch, and Tong '01. Other related stuff can be found in Tong et.al '99, Kapustin et.al '99 and '13, all Witten's papers, papers used supersymmetric localization,...

For the theory without a superpotential, see Aharony, Hanany, Intriligator, Seiberg, and Strassler '97, de Boer, Hori, OZ, and Yin '97, and a review by Intriligator and Seiberg '13. The connections to 2d GLSMs, see Aganagic, Hori, Karch, and Tong '01. Other related stuff can be found in Tong et.al '99, Kapustin et.al '99 and '13, all Witten's papers, papers used supersymmetric localization,...

Compared to 2d, we have one important difference of the RG-flow and one more ingredient in the Lagrangian. I will talk about the difference later on when we study the 3d CS-theory with a superpotential.

3d CS interaction

The 3d CS-terms provides a new phase: topological quantum field theory

$$S = \frac{k}{4\pi} \int d^3x \quad AdA.$$

The gauge-invariant of the CS-theory constraint k to be an integer. Its SUSY-version:

$$S = -\frac{k}{4\pi} \int d^3x d^4\Theta \quad V\left(\Sigma + \bar{\Sigma}\right).$$

Can be defined for a general compact Lie group. There are other Chern-Simons terms such as gauge-R-symmetry mixing one. We will not talk about them in this talk.

Phases of 3d N=2 CS theories

Let us first consider a $U(1)$ example without superpotential as a warm-up, the potential energy is

$$U_{s.c} = \frac{e_{eff}^2}{2}\left(\sum_i Q_i|\phi_i|^2 - F(\sigma)\right)^2 + \sum_i (Q_i\sigma)^2|\phi_i|^2,$$

where $F(\sigma) = \zeta + k_{eff}\sigma = \zeta + k\sigma + \frac{1}{2}\sum_i Q_i \mid Q_i\sigma \mid$ and $k_{eff} = k + \frac{1}{2}\sum Q_i^2 sign(Q_i\sigma)$.

Phases of 3d N=2 CS theories

Let us first consider a $U(1)$ example without superpotential as a warm-up, the potential energy is

$$U_{s.c} = \frac{e_{eff}^2}{2}\left(\sum_i Q_i|\phi_i|^2 - F(\sigma)\right)^2 + \sum_i (Q_i\sigma)^2|\phi_i|^2,$$

where $F(\sigma) = \zeta + k_{eff}\sigma = \zeta + k\sigma + \frac{1}{2}\sum_i Q_i \mid Q_i\sigma \mid$ and $k_{eff} = k + \frac{1}{2}\sum Q_i^2 sign(Q_i\sigma)$.
The vacuum configuration can be obtained by solving $U_{s.c} = 0$, which says

$$\sum_i Q_i|\phi_i|^2 = F(\sigma), \qquad (Q_i\sigma)\,\phi_i = 0.$$

We have phases:
- Higgs phases: $\sum_i Q_i\langle|\phi_i|^2\rangle = \zeta$ and $\langle\sigma\rangle = 0$.
- Coulomb branch: Continuous vacua parameterized by non-compact σ. $\phi_i = \zeta = k_{eff} = 0$.
- Topological vacua: $\phi_i = F(\sigma_I) = 0$ and $k_{eff} \neq 0$. The vacuum at $\sigma_I = -\frac{\zeta}{k_{eff}}$ that is described by the effective CS theory $U(1)_{k_{eff}}$.

A similar analysis apples to nonabelian gauge theories.

Phase transition?

Now the FI-parameter space is one-dimensional, one may worry about that when we change the parameter ζ from positive to negative. The theory has a phase transition means some vacua flow to/from infinity. For example, consider the 1d gauge theory for $\mathbb{P}^N$. In the geometric phase $\zeta \gg 0$, the Witten index is $N+1$, but the Witten index is vanishing when $\zeta \ll 0$. This is called wall-crossing.

Phase transition?

Now the FI-parameter space is one-dimensional, one may worry about that when we change the parameter ζ from positive to negative. The theory has a phase transition means some vacua flow to/from infinity. For example, consider the 1d gauge theory for $\mathbb{P}^N$. In the geometric phase $\zeta \gg 0$, the Witten index is $N+1$, but the Witten index is vanishing when $\zeta \ll 0$. This is called wall-crossing.
However, we expect this does not happen in 3d. When we put the theory on three-torus to compute the Witten index, the effective theory emergence a theta angle. Let us see an example: 3d gauge theory for $\mathbb{P}^N$.
In the geometric phase $\zeta \gg 0$, we have the Witten index $N+1$. When $\zeta \ll 0$, we have $U(1)_{N+1}$ CS theory and its Witten index is $N+1$.

Quantum K-theories

After defining a 3d flat spacetime CS-theory, we can then ask what are phases of the theory on $\mathbb{R}^2 \times S^1$. Now, let us just focus on the gauge theory for $Gr(k, N)$ in this talk.

Quantum K-theories

After defining a 3d flat spacetime CS-theory, we can then ask what are phases of the theory on $\mathbb{R}^2 \times S^1$. Now, let us just focus on the gauge theory for $Gr(k, N)$ in this talk.
In a geometric phase, $\zeta \gg 0$, we have the quantum K theory for $Gr(k, N)$. While when $\zeta \ll 0$, we have the effective theory with a 3d twisted effective superpotential by integrating out KK-towers of matter fields

$$\widetilde{W}_{eff}^{3d} = \frac{2k_{SU(k)} + N}{4} \sum_a (\ln x_a)^2 + \frac{k_{U(1)} - k_{SU(k)}}{2k} \left(\sum_a \ln x_a \right)^2 + \left(\ln(-)^{k-1} q \right) \sum_a \ln x_a + N \mathrm{Li}_2(x_a).$$

Quantum K-theories

After defining a 3d flat spacetime CS-theory, we can then ask what are phases of the theory on $\mathbb{R}^2 \times S^1$. Now, let us just focus on the gauge theory for $Gr(k, N)$ in this talk.
In a geometric phase, $\zeta \gg 0$, we have the quantum K theory for $Gr(k, N)$. While when $\zeta \ll 0$, we have the effective theory with a 3d twisted effective superpotential by integrating out KK-towers of matter fields

$$\widetilde{W}^{3d}_{eff} = \frac{2k_{SU(k)} + N}{4} \sum_a (\ln x_a)^2 + \frac{k_{U(1)} - k_{SU(k)}}{2k} \left(\sum_a \ln x_a \right)^2 + \left(\ln(-)^{k-1} q \right) \sum_a \ln x_a + N \mathrm{Li}_2(x_a).$$

$x_a = e^{-2\pi R \sigma_a}$.

The topological data should be analytic continuation under the change of the FI parameter. This suggests that the ring-relations of quantum K-theories can be obtained from the vacuum equations $d\widetilde{W}^{3d}_{eff} = 0$. To get the Givental Lee's quantum K theory of a Grassmannian (Jockers.et.al '19 and Ueda and Yoshida '19), we choose

$$k_{U(1)} = -\frac{N}{2}, \qquad k_{SU(k)} = k - \frac{N}{2}.$$

We have the vacuum equations

$$(1 - x_a)^N = q(-)^{k-1}\frac{x_a^k}{x_1 \cdots x_k}.$$

But the gauge-invariant of CS-theory does not suggest the above CS-level is the only option, and a general gauge-invariant choice of levels gives the quantum K-theory with level structure defined by Ruan and Zhang.

Quantum cohomology-ring relations from physics

We can use the vacuum equations from the twisted effective superpotential to give quantum K-theory in terms of other geometric bases of target space. Let us first see how this works in 2d GLSM for $Gr(k, N)$.

Quantum cohomology-ring relations from physics

We can use the vacuum equations from the twisted effective superpotential to give quantum K-theory in terms of other geometric bases of target space. Let us first see how this works in 2d GLSM for $Gr(k, N)$. The target space quantum cohomology-ring relations can be defined from the following short exact-sequence:

$$0 \to S \to \mathbb{C}^N \to Q \to 0.$$

Let us first read of the classical cohomology-ring relation is

$$C(S) \otimes C(Q) = 1.$$

Quantum cohomology-ring relations from physics

We can use the vacuum equations from the twisted effective superpotential to give quantum K-theory in terms of other geometric bases of target space. Let us first see how this works in 2d GLSM for $Gr(k, N)$. The target space quantum cohomology-ring relations can be defined from the following short exact-sequence:

$$0 \to S \to \mathbb{C}^N \to Q \to 0.$$

Let us first read of the classical cohomology-ring relation is

$$C(S) \otimes C(Q) = 1.$$

$C(S) = 1 + c_1(S) + \ldots + c_k(S)$, where $c_i(S)$ is the Chern-class of vector bundle S, we denote the Chern-roots to be σ_a. In terms of the components we have

$$c_1(S) + c_1(Q) = 0, \ldots, c_i(s) + c_{i-1}(S)c_1(Q) + \cdots + c_i(Q) = 0, \ldots, c_k(S)c_{N-k}(Q) = 0$$

Quantum cohomology from physics

The physical vacuum equations are

$$\frac{\partial \tilde{W}_{eff}}{\partial \sigma_a} = 0,$$

which give

$$(\sigma_a)^N = (-)^{k-1} q, \quad \sigma_a \neq \sigma_b \quad \text{if} \quad a \neq b.$$

Then use the Vieta's theorem to the find the ring-relations are

$$e_1(S) + e_1(Q) = 0, \ldots, e_i(s) + e_{i-1}(S) e_1(Q) + \cdots + e_i(Q) = 0, \ldots, e_k(S) e_{N-k}(Q) = (-)^{N-k} q.$$

This suggests that the quantum cohomology-ring relation from the short exact sequence should be modified to

$$C(S) \star C(Q) = 1 + (-)^{N-k} q.$$

This was first observed by Witten '93 from a different point of view.

Quantum K-theories in terms of *Lambda* $-$ *y* classes

The previous picture in 2d can be applied to the 3d. We first define the classical K-theory in terms of Lambda-y classes from the short exact sequence:

$$\lambda_y(S) \otimes \lambda_y(Q) = (1+y)^N,$$

where

$$\lambda_y(S) = 1 + ye_1(x_a) + \ldots y^k e_k(x_a).$$

Following the same logic as in quantum cohomology, we first read off the quantum K-theoretic relation from $\widetilde{W}_{eff}$, then we propose the quantum version of lambda-y classes is

$$\lambda_y(S) \star \lambda_y(Q) = (1+y)^N - \frac{q}{1-q} y^{N-k} e_{N-k}(Q) \star (\lambda_y(S) - 1)\,.$$

This proposal first appeared in my paper with Leonardo, Eric, and Hao '20. Its math proof will be coming out soon.

Superpotential in 3d

The RG-flow in 3d is different from 2d. Not any gauge-invariant homogeneous superpotential is (marginally) relevant operator.

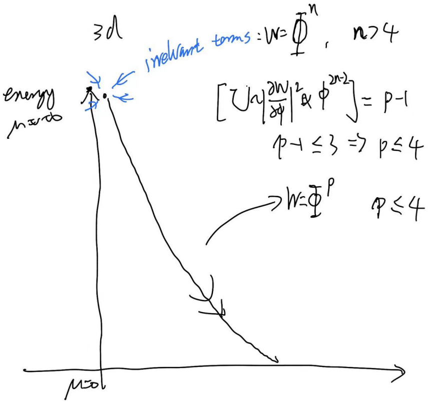

Figure: 3d RG-flow of Wess-Zomino (or Landau-Ginzburg).

3d CS theory with a superpotential

We consider 3d CS theory for a degree d hypersurface inside $\mathbb{P}^N$. We choose the bare CS-level to be $k = -\frac{N+1-d^2}{2}$. The semi-classical potential energy is

$$U_{s.c} = \frac{e_{eff}^2}{2}\left(\sum_i \mid \phi_i \mid^2 - d^2 \mid p \mid^2 - \zeta - k_{eff}\sigma\right)^2 + |G(\phi_i)|^2 + \sum_i \left| p\frac{\partial G}{\partial \phi_i}\right|^2 + \sigma^2\left(d^2 \mid p \mid^2 + \sum_i \mid \phi_i \mid^2\right).$$

The vacuum configurations

The structure of vacua is as follows:

- If $\zeta \gg 0$, we have a smooth geometric configuration

$$\sum_i \langle \mid \phi_i \mid^2 \rangle = \zeta, \quad G(\phi_i) = \sigma = p = 0.$$

If $\zeta \ll 0$, we have a $\mathbb{Z}_d$ orbifolded Landau-Ginzburg model

$$\hat{W} = \sum_i \Phi_i^d, \quad \sigma = 0, \quad \langle p \rangle = \sqrt{-\frac{\zeta}{d}}.$$

The vacuum configurations

The structure of vacua is as follows:

- If $\zeta \gg 0$, we have a smooth geometric configuration

$$\sum_i \langle | \phi_i |^2 \rangle = \zeta, \quad G(\phi_i) = \sigma = p = 0.$$

If $\zeta \ll 0$, we have a $\mathbb{Z}_d$ orbifolded Landau-Ginzburg model

$$\hat{W} = \sum_i \Phi_i^d, \quad \sigma = 0, \quad \langle p \rangle = \sqrt{-\frac{\zeta}{d}}.$$

- The topological vacua $F(\sigma) = \zeta + k(\sigma - |\sigma|) = 0$

$$\langle \sigma \rangle = -\frac{\zeta}{2k} < 0.$$

If k is positive, the topological vacua are in the geometric phase, while if k is negative, they are in the LG-point. The low energy effective theory is $U(1)_{d^2-N-1}$. Now, we see some examples in the following slides.

$\mathbb{P}^N[2]$, $N > 3$

We have the following phases:

- If $\zeta \gg 0$, we have NLSM on $\mathbb{P}^N[2]$, the Witten index is N if N is even and $N+1$ if N is odd.
- If $\zeta \ll 0$, A first sector is the effective $U(1)_{3-N}$ Chern-Simons theory, its Witten index is $N-3$. And the other sector is the three dimensional $\mathbb{Z}_2$ orbifolded LG-model $\widehat{W} = \sum_i \Phi_i^2$. Its vacua are

$$|\Omega\rangle \equiv d\phi_1 \wedge \cdots \wedge d\phi_{N+1}, \qquad |\mathrm{TS}\rangle_i, \quad \text{for} \quad i = 1, 2, 3.$$

  If N is even, the untwisted sector is not gauge-invariant that should be projected out. So we find that the Witten indices match at two different limits.

3d gauge theory on $\mathbb{R}^2 \times S^1$

But if we put the gauge theory first on $\mathbb{R}^2 \times S^1$, then we have the following two phases:

- If $\zeta \gg 0$, we have the NLSM on $\mathbb{P}^N[2]$, the Witten index is N if N is even and $N+1$ if N is odd.
- If $\zeta \ll 0$, A first sector can be described by $(\mathbb{C}_\star, \tilde{W}^{3d}_{eff})$, its Witten index is $N-1$. And the other sector is essentially a two dimensional $\mathbb{Z}_2$ orbifolded LG-model. Its vacua are

$$|\Omega\rangle \equiv d\phi_1 \wedge \cdots \wedge d\phi_{N+1}, \qquad |\mathrm{TS}\rangle$$

If N is even, the untwisted sector is not gauge-invariant that should be projected out. So we find that the Witten indices match at two different limits.

Taking phases does not commute KK-reduction

Here we come to our first surprise which is summarized in the following figure:

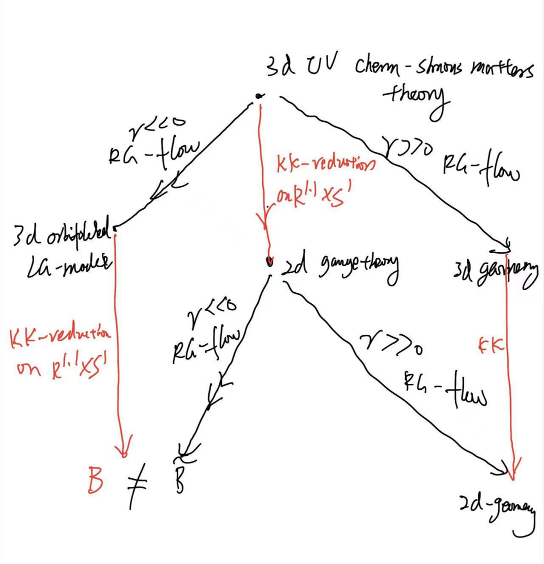

Figure: KK-reduction v.s phases

3d gauge theory for elliptic curve $\mathbb{P}^2[3]$

The bare Chern-Simons level we choose is 3. Its phase structure is the following:

- For $\zeta \gg 0$, we have two sectors. One is the NLSM on elliptic curve with vanishing Witten index, the other one is the Chern-Simons theory $U(1)_6$ with Witten index 6.
- For $\zeta \ll 0$, we have 3d $\mathbb{Z}_3$ orbifolded LG model, its Witten index is 6. See my paper w/ Pei and Zhang '21.

This 3d example is surprising because there should be no "topological vacua" in the 2d GLSM for a Calabi-Yau manifold.

Open and future directions

- Historical, people understood the 2d NLSM and LG models first, and then to the UV theory 2d GLSM. Here our logics is different, we started from 3d gauge theory first and to see what its low energy effective theories. However, these effective theories are not well-studied.
- I only discussed the Witten indices. Other quantities?
- Applications to K-GW, K-FJRW and their connections.

Thanks!