

Twisted Elliptic Genera

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An overview

For the last 10 years, there has been huge progress on the following relation web in both classification and computation.

6d (1,0) SCFTs classified in ([Heckman-Morrison-Rudelius-Vafa 15](#))

- 1 are engineered by F-theory compactified on local elliptic CY3
- 2 contain BPS strings with worldsheet theory as 2d (0,4) SCFTs
- 3 compactified on S^1 give 5d Kaluza-Klein (KK) theories

$$\mathbb{E}^{2d (0,4) \text{ SCFT}} = Z_{\mathbb{R}^4 \times T^2}^{6d (1,0) \text{ SCFT}} = Z_{\mathbb{R}^4 \times S^1}^{5d \text{ KK}} = Z_{\text{local elliptic CY3}}^{\text{ref. top.}}$$

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A simple example

- 1 6d (1,0) E-string theory
- 2 5d $SU(2) + 8\mathbf{F}$ KK theory
- 3 local half-K3 Calabi-Yau threefolds
- 4 a series of 2d (0,4) $O(k)$ gauge theories

An overview

When 6d (1,0) SCFT has a discrete global symmetry, we can do **twisted circle compactification** to a 5d KK theory (Bhardwaj-Jefferson-Kim-Tarazi-Vafa 19). There are two kinds

- 1 gauge algebra allows outer automorphism: folding vector multiplet
- 2 quiver structure has discrete symmetry: folding tensor multiplet

We focus on the **first kind** of twisted compactification. In such cases, the relations are generalized as

$$\mathbb{E}_{\text{twisted}}^{2d (0,4) \text{ SCFT}} = Z_{\mathbb{R}^4 \times S^1 \times S^1, \text{twisted}}^{6d (1,0) \text{ SCFT}} = Z_{\mathbb{R}^4 \times S^1}^{5d \text{ KK}} = Z_{\text{local genus-one fibered CY3}}^{\text{ref. top.}}$$

We are interested the **twisted elliptic genera** arising here.

- 6d (1,0) SCFTs and twisted compactification
- Twisted elliptic genera
- Twisted elliptic blowup equations
- Modular bootstrap on $\Gamma(N)$
- Spectral flow symmetry
- Summary

6d SCFTs and elliptic non-compact Calabi-Yau

Six is the highest dimension for SCFT

6d (2, 0)

- ADE classification

6d (1, 0)

- Tons
- F-theory compactified on **elliptic non-compact CY3**
- Elliptic fibration over non-compact base surface B in which all curve classes can be simultaneously shrinkable to zero volume
- Atomic classification ([Heckman-Morrison-Rudelius-Vafa 15](#))
- Generalized quiver from non-Higgsable clusters
- Tensor branch dimension called **“rank”** i.e. $H^{1,1}(B, \mathbb{Z})$

Rank One 6d (1, 0) SCFTs

Rank *one* 6d (1, 0) SCFTs are the natural elliptic lift of 4d $\mathcal{N} = 2$ and 5d $\mathcal{N} = 1$ gauge theories.

- Associated to Calabi-Yau as elliptic fibration over $\mathcal{O}(-n) \rightarrow \mathbb{P}^1$
- Kodaira singularity type gives the gauge algebra G supported on the $-n$ curve
- Global flavor symmetry F on a non-compact curve
- matters in representation \mathfrak{R} at intersection points between curves
- (n, G, F, \mathfrak{R}) are highly constrained by *Calabi-Yau condition*.

Full List of Rank One 6d (1,0) SCFTs

e.g. $n = 4$, $SO(r + 8) + r\mathbf{F}$ theories

n	G	F	$2(R_G, R_F)$
12	E_8	—	—
8	E_7	—	—
7	E_7	—	$(56, 1)$
6	E_6	—	—
6	E_7	$\mathfrak{so}(2)_{12}$	$(56, 2)$
5	F_4	—	—
5	E_6	$\mathfrak{u}(1)_6$	$27_{-1} \oplus \text{c.c.}$
5	E_7	$\mathfrak{so}(3)_{12}$	$(56, 3)$
4	$\mathfrak{so}(8)$	—	—
4	$\mathfrak{so}(N \geq 9)$	$\mathfrak{sp}(N - 8)_1$	$(N, 2(N - 8))$
4	F_4	$\mathfrak{sp}(1)_3$	$(26, 2)$
4	E_6	$\mathfrak{su}(2)_6 \times \mathfrak{u}(1)_{12}$	$(27, \bar{2})_{-1} \oplus \text{c.c.}$
4	E_7	$\mathfrak{so}(4)_{12}$	$(56, 2 \oplus 2)$
3	$\mathfrak{su}(3)$	—	—
3	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1$	$(8, 4)$
3	$\mathfrak{so}(8)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(1)_1^b \times \mathfrak{sp}(1)_1^c$	$(8_v \oplus 8_c \oplus 8_s, 2)$
3	$\mathfrak{so}(9)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(1)_2^b$	$(9, 4^a) \oplus (16, 2^b)$
3	$\mathfrak{so}(10)$	$\mathfrak{sp}(3)_1^a \times (\mathfrak{su}(1)_4 \times \mathfrak{u}(1)_4)^b$	$(10, 6^a) \oplus [(16_s)_1^b \oplus \text{c.c.}]$
3	$\mathfrak{so}(11)$	$\mathfrak{sp}(4)_1^a \times \mathfrak{lsing}^b$	$(11, 8^a) \oplus (32, 1_s^b)$
3	$\mathfrak{so}(12)$	$\mathfrak{sp}(5)_1$	$(12, 10) \oplus (32_s, 1)$
3	G_2	$\mathfrak{sp}(1)_1$	$(7, 2)$
3	F_4	$\mathfrak{sp}(2)_3$	$(26, 4)$
3	E_6	$\mathfrak{su}(3)_6 \times \mathfrak{u}(1)_{18}$	$(27, \bar{3})_{-1} \oplus \text{c.c.}$
3	E_7	$\mathfrak{so}(5)_{12}$	$(56, 5)$

Full List of Rank One 6d (1,0) SCFTs

e.g. $SU(r) + 2r\mathbf{F}$ theories

n	G	F	$2(R_G, R_F)$
2	$\mathfrak{su}(1)$	$\mathfrak{su}(2)_1$	—
2	$\mathfrak{su}(2)$	$\mathfrak{so}(7)_1 \times \text{Ising}$	$(2, \mathbf{8}_s \times \mathbf{1}_s)$
2	$\mathfrak{su}(N \geq 3)$	$\mathfrak{su}(2N)_1$	$(\mathbf{N}, \overline{\mathbf{2N}}) \oplus \text{c.c.}$
2	$\mathfrak{so}(7)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(4)_1^b$	$(7, 2^a) \oplus (8, 8^b)$
2	$\mathfrak{so}(8)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(2)_1^b \times \mathfrak{sp}(2)_1^c$	$(8_v, 4^a) \oplus (8_s, 4^b) \oplus (8_c, 4^c)$
2	$\mathfrak{so}(9)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(2)_2^b$	$(9, 6^a) \oplus (16, 4^b)$
2	$\mathfrak{so}(10)$	$\mathfrak{sp}(4)_1^a \times (\mathfrak{su}(2)_4 \times \mathfrak{u}(1)_8)^b$	$(10, 8^a) \oplus [(16_s, 2^b)_1 \oplus \text{c.c.}]$
2	$\mathfrak{so}(11)$	$\mathfrak{sp}(5)_1^a \times 7^b$	$(11, 10^a) \oplus (32, 2^b)$
2	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(6)_1^a \times \mathfrak{so}(2)_8$	$(12, 12^a) \oplus (32_s, 2^b)$
2	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(6)_1^a \times \text{Ising}^b \times \text{Ising}^c$	$(12, 12^a) \oplus (32_s, 1_s^b) \oplus (32_c, 1_s^c)$
2	$\mathfrak{so}(13)$	$\mathfrak{sp}(7)_1$	$(13, 14) \oplus (64, 1)$
2	G_2	$\mathfrak{sp}(4)_1$	$(7, 8)$
2	F_4	$\mathfrak{sp}(3)_3$	$(26, 6)$
2	E_6	$\mathfrak{su}(4)_6 \times \mathfrak{u}(1)_{24}$	$(27, \overline{\mathbf{4}})_{-1} \oplus \text{c.c.}$
2	E_7	$\mathfrak{so}(6)_{12}$	$(56, 6)$

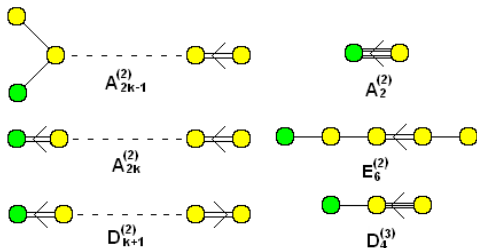
Full List of Rank One 6d (1,0) SCFTs

e.g. $SU(r) + (r + 8)\mathbf{F} + \Lambda^2$ theories, $Sp(r) + (2r + 8)\mathbf{F}$ theories

n	G	F	$2(R_G, R_F)$
1	$\mathfrak{sp}(0)$	$(E_8)_1$	—
1	$\mathfrak{sp}(N \geq 1)$	$\mathfrak{so}(4N + 16)_1$	$(2\mathbf{N}, 4\mathbf{N} + 16)$
1	$\mathfrak{su}(3)$	$\mathfrak{su}(12)_1$	$(3, \overline{12})_1 \oplus \text{c.c.}$
1	$\mathfrak{su}(4)$	$\mathfrak{su}(12)_1^a \times \mathfrak{su}(2)_1^b$	$[(4, \overline{12})_1^a \oplus \text{c.c.}] \oplus (6, 2^b)$
1	$\mathfrak{su}(N \geq 5)$	$\mathfrak{su}(N+8)_1 \times \mathfrak{u}(1)_{2N(N-1)(N+8)}$	$[(\mathbf{N}, \overline{\mathbf{N}} + 8)_{-N+4} \oplus (\Lambda^2, \mathbf{1})_{N+8}] \oplus \text{c.c.}$
1	$\mathfrak{su}(6)_*$	$\mathfrak{su}(15)_1$	$[(6, \overline{15}) \oplus \text{c.c.}] \oplus (20, \mathbf{1})$
1	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(6)_1^b$	$(7, 4^a) \oplus (8, 12^b)$
1	$\mathfrak{so}(8)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(3)_1^b \times \mathfrak{sp}(3)_1^c$	$(8_v, 6^a) \oplus (8_s, 6^b) \oplus (8_c, 6^c)$
1	$\mathfrak{so}(9)$	$\mathfrak{sp}(4)_1^a \times \mathfrak{sp}(3)_2^b$	$(9, 8^a) \oplus (16, 6^b)$
1	$\mathfrak{so}(10)$	$\mathfrak{sp}(5)_1^a \times (\mathfrak{su}(3)_4 \times \mathfrak{u}(1)_{12})^b$	$(10, 10^a) \oplus [(16_s, 3^b)_1 \oplus \text{c.c.}]$
1	$\mathfrak{so}(11)$	$\mathfrak{sp}(6)_1^a \times ?^b$	$(11, 12^a) \oplus (32, 3^b)$
1	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(7)_1^a \times \mathfrak{so}(3)_8^b$	$(12, 14^a) \oplus (32_s, 3^b)$
1	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(7)_1^a \times ?^b \times ?^c$	$(12, 14^a) \oplus (32_s, 2^b) \oplus (32_c, 1^c)$
1	G_2	$\mathfrak{sp}(7)_1$	$(7, 14)$
1	F_4	$\mathfrak{sp}(4)_3$	$(26, 8)$
1	E_6	$\mathfrak{su}(5)_6 \times \mathfrak{u}(1)_{30}$	$(27, \overline{5})_{-1} \oplus \text{c.c.}$
1	E_7	$\mathfrak{so}(7)_{12}$	$(56, 7)$

Twisted 6d (1, 0) rank-one SCFTs

- When the **gauge algebra allows outer automorphism**, i.e. folding the Dynkin diagrams, we can construct **twisted** 6d theories
- Twisted circle compactification to **5d KK theories** (Bhardwaj-Jefferson-Kim-Tarazi-Vafa 19, ...).
- In **rank one**, one can fold both 6d vector and hyper multiplets.
- We denote the **truncated** gauge algebra, flavor algebra and matter representation as $\mathring{G}, \mathring{F}, \mathring{R}$.
- Upon twist, **fractional KK charges** naturally appear.



Twisted 6d (1,0) rank-one SCFTs

KK-momentum shifts of representations under twist

G	\mathring{G}	$R \rightarrow \mathring{R}$
$A_{2r}^{(2)}$	C_r	$\mathbf{Adj} \rightarrow \mathbf{Adj}_0 \oplus \mathbf{F}_{1/4} \oplus \Lambda_{1/2}^2 \oplus \mathbf{1}_{1/2} \oplus \mathbf{F}_{3/4}$ $\mathbf{F} \oplus \overline{\mathbf{F}} \rightarrow \mathbf{F}_0 \oplus \mathbf{1}_{1/4} \oplus \mathbf{F}_{1/2} \oplus \mathbf{1}_{3/4}$
$A_{2r-1}^{(2)}$	C_r	$\mathbf{Adj} \rightarrow \mathbf{Adj}_0 \oplus \Lambda_{1/2}^2$ $\mathbf{F} \oplus \overline{\mathbf{F}} \rightarrow \mathbf{F}_0 \oplus \mathbf{F}_{1/2}$ $\Lambda^2 \oplus \overline{\Lambda^2} \rightarrow \Lambda_0^2 \oplus \Lambda_{1/2}^2 \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/2}$
$D_{r+1}^{(2)}$	B_r	$\mathbf{Adj} \rightarrow \mathbf{Adj}_0 \oplus \mathbf{F}_{1/2}$ $\mathbf{V} \rightarrow \mathbf{V}_0 \oplus \mathbf{1}_{1/2}$ $\mathbf{S} \oplus \mathbf{C} \rightarrow \mathbf{S}_0 \oplus \mathbf{S}_{1/2}$
$E_6^{(2)}$	F_4	$\mathbf{Adj} \rightarrow \mathbf{Adj}_0 \oplus \mathbf{F}_{1/2}$ $\mathbf{F} \oplus \overline{\mathbf{F}} \rightarrow \mathbf{F}_0 \oplus \mathbf{F}_{1/2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/2}$
$D_4^{(3)}$	G_2	$\mathbf{Adj} \rightarrow \mathbf{Adj}_0 \oplus \mathbf{F}_{1/3} \oplus \mathbf{F}_{2/3}$ $\mathbf{V} \oplus \mathbf{S} \oplus \mathbf{C} \rightarrow \mathbf{F}_0 \oplus \mathbf{1}_0 \oplus \mathbf{F}_{1/3} \oplus \mathbf{1}_{1/3} \oplus \mathbf{F}_{2/3} \oplus \mathbf{1}_{2/3}$

Some typical twisted 6d (1, 0) rank one theories

There exist **four pure gauge twisted theories**: $n = 6, E_6^{(2)}$,
 $n = 4, D_4^{(2)}, D_4^{(3)}$ and $n = 3, A_2^{(2)}$.

n	G	\check{G}	\check{R}	\check{F}
6	$E_6^{(2)}$	F_4	–	–
4	$D_4^{(3)}$	G_2	–	–
4	$D_4^{(2)}$	B_3	–	–
4	$D_{r+4}^{(2)}$	B_{r+3}	$2r(\mathbf{V}_0 \oplus \mathbf{1}_{1/2})$	$\mathfrak{sp}(2r)$
4	$E_6^{(2)}$	F_4	$\mathbf{F}_0 \oplus \mathbf{F}_{1/2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/2}$	$\mathfrak{sp}(1)$
3	$A_2^{(2)}$	C_1	–	–
3	$D_4^{(2)}$	B_3	$\mathbf{V}_0 \oplus \mathbf{1}_{1/2} \oplus \mathbf{S}_0 \oplus \mathbf{S}_{1/2}$	$\mathfrak{sp}(1) \times \mathfrak{sp}(1)$
3	$D_4^{(3)}$	G_2	$\mathbf{F}_0 \oplus \mathbf{F}_{1/3} \oplus \mathbf{F}_{2/3} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/3} \oplus \mathbf{1}_{2/3}$	$\mathfrak{sp}(1)$
2	A_{2r}	C_r	$(2r+1)(\mathbf{F}_{1/4} \oplus \mathbf{F}_{3/4} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/2})$	$\mathfrak{so}(4r+2)$
2	A_{2r-1}	C_r	$2r(\mathbf{F}_0 \oplus \mathbf{F}_{1/2})$	$\mathfrak{so}(4r)$
2	$D_4^{(2)}$	B_3	$2(\mathbf{V}_0 \oplus \mathbf{1}_{1/2} \oplus \mathbf{S}_0 \oplus \mathbf{S}_{1/2})$	$\mathfrak{sp}(2) \times \mathfrak{sp}(2)$
2	$D_4^{(3)}$	G_2	$2(\mathbf{F}_0 \oplus \mathbf{F}_{1/3} \oplus \mathbf{F}_{2/3} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/3} \oplus \mathbf{1}_{2/3})$	$\mathfrak{sp}(2)$
2	$D_5^{(2)}$	B_4	$4(\mathbf{V}_0 \oplus \mathbf{1}_{1/2}) \oplus \mathbf{S}_0 \oplus \mathbf{S}_{1/2}$	$\mathfrak{sp}(4) \times \mathfrak{sp}(1)$
2	$D_6^{(2)}$	B_5	$6(\mathbf{V}_0 \oplus \mathbf{1}_{1/2}) \oplus \frac{1}{2}\mathbf{S}_0 \oplus \frac{1}{2}\mathbf{S}_{1/2}$	$\mathfrak{sp}(6)$
2	$E_6^{(2)}$	F_4	$2(\mathbf{F}_0 \oplus \mathbf{F}_{1/2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/2})$	$\mathfrak{sp}(2)$
1	$A_2^{(2)}$	C_1	$6(\mathbf{F}_0 \oplus \mathbf{F}_{1/2} \oplus \mathbf{1}_{1/4} \oplus \mathbf{1}_{3/4})$	$\mathfrak{so}(12)$
1	$A_3^{(2)}$	C_2	$6(\mathbf{F}_0 \oplus \mathbf{F}_{1/2}) \oplus \mathbf{1}_0 \oplus \Lambda_{1/2}^2$	$\mathfrak{so}(12) \times \mathfrak{sp}(1)$
1	$D_4^{(2)}$	B_3	$3(\mathbf{V}_0 \oplus \mathbf{1}_{1/2} \oplus \mathbf{S}_0 \oplus \mathbf{S}_{1/2})$	$\mathfrak{sp}(3) \times \mathfrak{sp}(3)$
1	$D_4^{(3)}$	G_2	$3(\mathbf{F}_0 \oplus \mathbf{F}_{1/3} \oplus \mathbf{F}_{2/3} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/3} \oplus \mathbf{1}_{2/3})$	$\mathfrak{sp}(3)$

Elliptic Genera of 6d SCFT

- Consider a elliptic non-compact Calabi-Yau X
- Compactify F-theory on $\mathbb{C}_{\epsilon_1, \epsilon_2}^2 \times T^2 \times X$
- The full partition function contains three parts

$$Z_{6d} = Z_{\text{poly}} Z_{1\text{-loop}} Z_{\text{ell}},$$

$$Z_{\text{ell}} = 1 + \mathbb{E}_1 Q_{\text{ell}} + \mathbb{E}_2 Q_{\text{ell}}^2 + \dots$$

- Z_{poly} comes from the **perturbative** part
- $Z_{1\text{-loop}}$ contains the contributions from **tensor, vector** and **hyper multiplets**
- One main goal to study 6d SCFT is to compute the d -string **elliptic genus** $\mathbb{E}_d(\tau, m_G, m_F, \epsilon_1, \epsilon_2)$ of the associated 2d (0, 4) SCFTs on the BPS strings
- \mathbb{E}_d is the natural elliptic lift the d -instanton Nekrasov partition function, also certain **Jacobi form** with $m_G, m_F, \epsilon_1, \epsilon_2$ as elliptic parameters

Twisted elliptic genera

In the twisted cases, similarly one can define **twisted elliptic genera**

$$Z_{6d}^{tw} = Z_{\text{poly}}^{tw} Z_{1\text{-loop}}^{tw} Z_{\text{ell}}^{tw}, \quad Z_{\text{ell}}^{tw} = 1 + \mathbb{E}_1^{tw} Q_{\text{ell}} + \mathbb{E}_2^{tw} Q_{\text{ell}}^2 + \dots$$

Now \mathbb{E}_d^{tw} contains **fractional q_r powers**.

- In twisted circle compactification, $Z_{6d}^{tw} = Z_{5d}^{\text{KK}}$.
- Twisted elliptic genera for a few theories have been studied in (Kim³-Lee 21, Kim³ 21)
- Now we would like go through **all twistable theories**
- We want to keep everything **elliptic** and as **6d gauge theory**, rather than geometries or 5d KK theories.

Approaches to twisted elliptic genera

In the following, I mainly talk about the **last three approaches**.

- 2d quiver gauge theories
 - $n = 4, D_4^{(2)}$, (Kim³ 21)
- Twisting from Higgsing
 - many (Kim²-Lee 19, Kim³ 21, Hayashi-Kim-Ohmori 21)
- **Twisted elliptic blowup equations**
 - $n = 3, A_2^{(2)}$, (Kim³-Lee 21)
- **Modular bootstrape**
- **Spectral flow symmetry**

Background for blowup equations

- Blowup equations are **functional equations for Nekrasov partition functions** originally for 4d $\mathcal{N} = 2$ $SU(N)$ gauge theories (Nakajima-Yoshioka 03)
- K-theoretic version – 5d $\mathcal{N} = 1$ (Nakajima-Yoshioka 05, 09, Göttsche-Nakajima-Yoshioka 06, Keller-Song 12, Kim²-Lee²-Song 19)
- further generalize to refined topological strings on local Calabi-Yau threefolds (Huang-KS-Wang 17)
- elliptic version for 6d (1, 0) SCFTs (Gu-Haghighat-Klemm-KS-Wang 18-20)
- 6d (2, 0) SCFTs, 5d $\mathcal{N} = 1^*$ (Duan-Lee-Nahmgoong-Wang 21)
- 6d twisted SCFTs and 5d KK theories (Kim³-Lee 21)

Twisted elliptic blowup equations

Let λ_0 be a coweight of G **invariant upon the twist**. If (λ_0, λ_F) is admissible for the original 6d SCFT, and let $\lambda_{\tilde{F}}$ be the reduction of λ_F , then the twisted elliptic genera $\mathbb{E}_d(\tau, m_{\tilde{G}}, m_{\tilde{F}}, \epsilon_{1,2})$ satisfy the following

Twisted elliptic blowup equations

$$\begin{aligned} & \sum_{\lambda_{\tilde{G}} \in \phi_{\lambda_0}(Q^\vee(\tilde{G}))} (-1)^{|\lambda_{\tilde{G}}|} \theta_i^{[a]} \left(n\tau, \left(y - \frac{n}{2} \|\lambda_{\tilde{G}}\|^2 \right) (\epsilon_1 + \epsilon_2) - nd' \epsilon_1 - nd'' \epsilon_2 \right) \\ & \quad \times A_V^{\tilde{G}}(\tau, m_{\tilde{G}}, \lambda_{\tilde{G}}) A_V^{frac}(\tau, m_{\tilde{G}}, \lambda_{\tilde{G}}) A_H^{\tilde{R}}(\tau, m_{\tilde{G}}, m_{\tilde{F}}, \lambda_{\tilde{G}}, \lambda_{\tilde{F}}) \\ & \quad \times \mathbb{E}_{d'}^{tw}(\tau, m_{\tilde{G}} + \epsilon_1 \lambda_{\tilde{G}}, m_{\tilde{F}} + \epsilon_1 \lambda_{\tilde{F}}, \epsilon_1, \epsilon_2 - \epsilon_1) \\ & \quad \times \mathbb{E}_{d''}^{tw}(\tau, m_{\tilde{G}} + \epsilon_2 \lambda_{\tilde{G}}, m_{\tilde{F}} + \epsilon_2 \lambda_{\tilde{F}}, \epsilon_1 - \epsilon_2, \epsilon_2) \\ & = \Lambda(\delta) \theta_i^{[a]}(n\tau, k_F \lambda_{\tilde{F}} \cdot m_{\tilde{F}} + ny(\epsilon_1 + \epsilon_2)) \mathbb{E}_d^{tw}(\tau, m_{\tilde{G}}, m_{\tilde{F}}, \epsilon_1, \epsilon_2). \end{aligned}$$

For $\delta = 0$, $\Lambda(\delta) = 1$, we call it **unity** blowup equations. For $\delta > 0$, $\Lambda(\delta) = 0$, we call it **vanishing** blowup equations.

Twisted elliptic blowup equations

The parameters $y, \lambda_{\tilde{F}}$ of unity twisted elliptic blowup equations for rank one models. # is the number of equations with fixed characteristic a .

n	G	\tilde{G}	\tilde{F}	#	y	$\lambda_{\tilde{F}}$
6	E_6	F_4	—	1	4	\emptyset
4	$D_4^{(3)}$	G_2	—	1	2	\emptyset
4	$D_{r+4}^{(2)}$	B_{r+3}	$\mathfrak{sp}(2r)$	2^{2r}	$r+2$	$(0 \dots 01)$
4	$E_6^{(2)}$	F_4	$\mathfrak{sp}(1)$	2	5	(1)
3	$A_2^{(2)}$	C_1	—	1	1	\emptyset
3	$D_4^{(2)}$	B_3	$\mathfrak{sp}(1) \times \mathfrak{sp}(1)$	4	$5/2$	(1),(1)
3	$D_4^{(3)}$	G_2	$\mathfrak{sp}(1)$	2	$5/2$	(1)
2	A_{2r}	C_r	$\mathfrak{so}(4r+2)$	2^{2r+1}	$r + \frac{1}{2}$	$(0 \dots 01)$
2	A_{2r-1}	C_r	$\mathfrak{so}(4r)$	2^{2r}	r	$(0 \dots 01)$
2	$D_4^{(2)}$	B_3	$\mathfrak{sp}(2) \times \mathfrak{sp}(2)$	16	3	$(01),(01)$
2	$D_4^{(3)}$	G_2	$\mathfrak{sp}(2)$	4	3	(01)
2	$D_5^{(2)}$	B_4	$\mathfrak{sp}(4) \times \mathfrak{sp}(1)$	32	4	$(0001), (1)$
2	$E_6^{(2)}$	F_4	$\mathfrak{sp}(2)$	4	6	(01)
1	$A_1^{(2)}$	C_1	$\mathfrak{so}(10)$	64	$3/2$	$(0 \dots 01)$
1	$A_2^{(2)}$	C_1	$\mathfrak{so}(12)$	64	2	$(0 \dots 01)$
1	$A_3^{(2)}$	C_2	$\mathfrak{so}(12) \times \mathfrak{sp}(1)$	128	$5/2$	$(0 \dots 01), (1)$
1	$D_4^{(2)}$	B_3	$\mathfrak{sp}(3) \times \mathfrak{sp}(3)$	64	$7/2$	$(001),(001)$
1	$D_4^{(3)}$	G_2	$\mathfrak{sp}(3)$	8	$7/2$	(001)

Universal formula for twisted elliptic genera \mathbb{E}_1

From the unity equations, we can **solve twisted elliptic genera recursively**. In particular, the twisted one-string elliptic genera for the pure gauge cases have the following **universal formula**:

$$\mathbb{E}_1^{\text{tw}} = \sum_{\alpha \in \Delta_I^{\vee}} (-1)^{|\alpha^{\vee}|} \frac{D^{\alpha^{\vee}}}{D} \frac{\eta^4}{\theta_1(m_\alpha)\theta_1(m_\alpha - \varepsilon_{1,2})\theta_1(m_\alpha - 2\varepsilon_+)} \prod_{\substack{\beta \in \Delta \\ \alpha^{\vee} \cdot \beta = 1}} \frac{\eta}{\theta_1(m_\beta)} \prod_{\substack{\gamma \in \Delta_s \\ \alpha^{\vee} \cdot \gamma = 1}} \frac{\eta}{\theta_1^{[k]}(m_\gamma)}.$$

Here $D^{\alpha^{\vee}} = \text{Det}_{\{1,0,0\}}^{\alpha^{\vee}}$ and $D = \text{Det}_{\{0,0,0\}}^{\alpha^{\vee}}$, and we define

$$\theta_{i,\{d_0,d_1,d_2\}}^{[a]} = \theta_i^{[a]}(n\tau, -nm_{\alpha^{\vee}} + (n-2)(\varepsilon_1 + \varepsilon_2) - n((d_0 + d_1)\varepsilon_1 + (d_0 + d_2)\varepsilon_2)),$$

$$\text{Det}_{\{d_0,d_1,d_2\}}^{\alpha^{\vee}} = \det \begin{pmatrix} \theta_{i,\{0,d,0\}}^{[a_1]} & \theta_{i,\{0,0,d\}}^{[a_1]} & \theta_{i,\{d_0,d_1,d_2\}}^{[a_1]} \\ \theta_{i,\{0,d,0\}}^{[a_2]} & \theta_{i,\{0,0,d\}}^{[a_2]} & \theta_{i,\{d_0,d_1,d_2\}}^{[a_2]} \\ \theta_{i,\{0,d,0\}}^{[a_3]} & \theta_{i,\{0,0,d\}}^{[a_3]} & \theta_{i,\{d_0,d_1,d_2\}}^{[a_3]} \end{pmatrix}.$$

Besides, k are the fractional KK charges for the short roots.

Structure of twisted elliptic genera

From the universal \mathbb{E}_1^{tw} formula, we find the following universal expansion for twisted pure gauge theories ($v = e^{\epsilon+}$)

$$\mathbb{E}_1^{\text{tw}} = v^{h_{\mathring{G}}^{\vee} - 1} \left(\sum_{n=0}^{\infty} \chi_{k\theta}^{\mathring{G}} v^{2n} + q_{\tau}^{\frac{1}{N}} (1 + v^2) \sum_{n=0}^{\infty} \chi_{k\theta+f}^{\mathring{G}} v^{2n} + \dots \right).$$

- q_{τ}^0 order gives **5d one \mathring{G} -instanton Hilbert series** (Benvenuti-Hanany-Mekareeya 10)
- Fractional orders $q_{\tau}^{i/N}$, $i = 1, \dots, N - 1$ all have **exact v expansion formulas**
- For example, for $D_4^{(3)}$ theory the $q_{\tau}^{2/3}$ order of \mathbb{E}_1^{tw} is

$$\sum_{n=0}^{\infty} (\chi_{2f+n\theta}^{G_2} (1 + v^2 + v^4) + \chi_{f+n\theta}^{G_2} (1 + v^2) + \chi_{n\theta}^{G_2}) v^{2n+3}.$$

Modular bootstrap on $\Gamma(N)$

We turn off all gauge and flavor fugacities to study the **modular property** of $\mathbb{E}_1(q, v)$. In many cases, this can fully determine $\mathbb{E}_1(q, v)$ with small amount of initial data.

- For untwisted theories, the **modular ansatz** was proposed in (Del Zotto-Lockhart 18) on $SL(2, \mathbb{Z})$
- For twisted cases, we propose the modular ansatz on **congruence subgroup $\Gamma(N)$** . Here N is the **twist coefficient**:

$$N = \begin{cases} 2, & A_{2r-1}^{(2)}, D_r^{(2)}, E_6^{(2)}, \\ 3, & D_4^{(3)}, \\ 4, & A_{2r}^{(2)}. \end{cases}$$

- Inspired by the modular ansatz for genus-one fibered CY3 (Cota-Klemm-Schimannek 19) (Knapp-Scheidegger-Schimannek 21)

Modular bootstrap on $\Gamma(N)$

For a theory on $(-n)$ -curve with twist coefficient $N = 2, 3, 4$, we propose modular ansatz for the reduced one-string elliptic genus:

$$\mathbb{E}_1^{\text{tw}}(\tau, \epsilon_+) = \frac{\mathcal{N}(\tau, \epsilon_+)}{\eta^{12(n-2)-4+24\delta_{n,1}} \Delta_{2N}(\frac{\tau}{N})^{\frac{s}{N}} \phi_{-2,1}(\tau, 2\epsilon_+)^{h_G^V - 1}}.$$

Here $s = \frac{N}{N-1}(c - \frac{n-2}{2} - \delta_{n,1})$. Numerator $\mathcal{N}(\tau, \epsilon_+)$ is of weight $6(n-2) + 2Ns + 12\delta_{n,1} - 2h_G^V$ and index $4(h_G^V - 1) + n - h_G^V$ and

$$\mathcal{N}(N\tau, \epsilon_+) \in M_\star(N)[\phi_{-2,1}(N\tau, \epsilon_+), \phi_{0,1}(N\tau, \epsilon_+)].$$

Δ_{2N} are certain **cuspidal forms on $\Gamma(N)$** :

$$\Delta_4(\tau) = \frac{\eta(2\tau)^{16}}{\eta(\tau)^8}, \quad \Delta_6(\tau) = \frac{\eta(3\tau)^{18}}{\eta(\tau)^6}, \quad \Delta_8(\tau) = \frac{\eta(2\tau)^8 \eta(4\tau)^{16}}{\eta(\tau)^8},$$

and $\phi_{-2,1}, \phi_{0,1}$ are **Eichler-Zagier's generators** for weak Jacobi forms.

Modular bootstrap on $\Gamma(2)$: a simple example

Consider the \mathbb{Z}_2 twist of E-string theory. 4 of the 8 masses are shifted by $\frac{\pi}{2}$. The twisted one-string elliptic genus is

$$\mathbb{E}_1^{\text{tw}} = \frac{1}{2} \sum_{(r,s)=(1,4),(4,1)}^{(2,3),(3,2)} \prod_{i=1}^4 \prod_{j=5}^8 \frac{\theta_r(m_i) \theta_s(m_j)}{\eta^2}.$$

Turing off the flavor fugacities, we find the modular ansatz

$$\mathbb{E}_1^{\text{tw}} = \frac{16\Delta_4\left(\frac{\tau}{2}\right)}{\eta^8}, \quad \Delta_4(\tau) = \frac{\eta(2\tau)^{16}}{\eta(\tau)^8}.$$

More examples on modular bootstrape

- ① For pure $D_4^{(2)}$ theory on (-4) curve, **twist coefficient $N = 2$** ,

$$\mathbb{E}_1^{\text{tw}}(\tau, \epsilon_+) = \frac{\Delta_4(\tau/2)^{\frac{1}{2}} \mathcal{N}(\tau, \epsilon_+)}{\eta(\tau)^{20} \phi_{-2,1}(\tau, 2\epsilon_+)^4}.$$

Here $\mathcal{N}(\tau, \epsilon_+)$ is of weight 0 and index 14. There are 64 coefficients to fix $\mathcal{N}(2\tau, \epsilon_+)$.

- ② For pure $D_4^{(3)}$ theory on (-4) curve, **twist coefficient $N = 3$** ,

$$\mathbb{E}_1^{\text{tw}}(\tau, \epsilon_+) = \frac{\Delta_6(\tau/3)^{\frac{2}{3}} \mathcal{N}(\tau, \nu)}{\eta(\tau)^{20} \phi_{-2,1}(\tau, 2\epsilon_+)^3}.$$

Here $\mathcal{N}(\tau, \epsilon_+)$ is of weight 0 and index 10. There are 67 coefficients to fix $\mathcal{N}(3\tau, \epsilon_+)$.

- ③ For pure $A_2^{(2)}$ theory on (-3) curve, **twist coefficient $N = 4$** ,

$$\mathbb{E}_1^{\text{tw}}(\tau, \epsilon_+) = \frac{\Delta_8(\tau/4)^{\frac{1}{4}} \mathcal{N}(\tau, \epsilon_+)}{\eta(\tau)^8 \phi_{-2,1}(\tau, 2\epsilon_+)}.$$

Here $\mathcal{N}(\tau, \epsilon_+)$ is of weight 0 and index 4. There are 15 coefficients to fix $\mathcal{N}(4\tau, \epsilon_+)$

Spectral flow symmetry

For twisted one-string elliptic genus, we find the spectral flow from **R-R** sector to **NS-R** sector is induced by transformation:

$$\mathbb{E}_{\text{NS-R}}^{\dot{R}_{\text{KK}}}(\nu, q) = \pm \left(\frac{q^{1/4}}{\nu} \right)^{n-h_G^{\vee}} \mathbb{E}_{\text{R-R}}^{\dot{R}_{\text{KK}}} \left(\frac{q^{1/2}}{\nu}, q \right).$$

On the other hand, 2d (0, 4) analysis shows spectral flow shifts the KK charges of hypermultiplets by **1/2**:

$$\mathbb{E}_{\text{NS-R}}^{\dot{R}_{\text{KK}}} = \mathbb{E}_{\text{R-R}}^{\dot{R}_{\text{KK}+1/2}}.$$

These two properties together suggest interesting symmetry for the twisted elliptic genera.

Example: spectral flow symmetry for pure gauge theories

For 6d (1, 0) pure gauge $G^{(k)}$ theories on $(-n)$ -curve, we find the following **spectral flow symmetry** for the reduced twisted one-string elliptic genera:

$$\mathbb{E}_1^{G^{(k)}}\left(q, \frac{q^{1/2}}{v}\right) = \left(-\frac{q^{1/2}}{v^2}\right)^{n-3} \mathbb{E}_1^{G^{(k)}}(q, v).$$

We explicitly checked this symmetry for $A_2^{(2)}, D_4^{(2)}, D_4^{(3)}, E_6^{(2)}$.

- generalization of (Del Zotto-Lockhart 16, 18) for the untwisted cases $A_2, D_4, F_4, E_{6,7,8}$.
- relate twisted R-R elliptic genus to twisted NS-R

Spectral flow symmetry for $E_6^{(2)}$

Let us expand the twisted one-string elliptic genus as

$$\mathbb{E}_1^{\text{tw}}(q_\tau, v, m_{F_4} = 0) = v^{-3} \sum_{i,j=0}^{\infty} b_{i,j} q^i v^{2i+j}.$$

Then we find the following table for coefficients b_{ij} :

$q \setminus v$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	1	0	52	0
1/2	0	0	0	0	0	0	26	0	1079
1	0	0	0	0	0	0	0	378	0
3/2	0	0	0	0	0	0	0	0	4056
2	0	0	0	0	0	-1	0	0	0
5/2	-1	0	0	0	1	0	-26	0	0
3	0	-26	0	0	0	26	0	-378	0
7/2	-52	0	-378	0	0	0	378	0	-4004
4	0	-1079	0	-4056	0	0	0	4004	0

Spectral flow symmetry for $D_4^{(3)}$

Let us expand the twisted one-string elliptic genus as

$$\mathbb{E}_1^{\text{tw}}(q_\tau, v, m_{G_2} = 0) = v^{-1} \sum_{i,j=0}^{\infty} b_{i,j} q^i v^{2i+j}.$$

Then we find the following table for coefficients b_{ij} :

$q \backslash v$	0			2			4			6
0	0	0	0	1	0	0	14	0	0	77
1/3	0	0	0	0	7	0	0	71	0	0
2/3	0	0	0	0	0	35	0	0	301	0
1	-1	0	0	0	0	0	141	0	0	1127
4/3	0	-7	0	0	0	0	0	497	0	0
5/3	0	0	-35	0	0	0	0	0	1582	0
2	-14	0	0	-141	0	0	0	0	0	4650
7/3	0	-71	0	0	-497	0	0	0	0	0
8/3	0	0	-301	0	0	-1582	0	0	0	0
3	-77	0	0	-1127	0	0	-4650	0	0	0

Spectral flow symmetry for $A_2^{(2)}$

Let us expand the twisted one-string elliptic genus as

$$\mathbb{E}_1^{\text{tw}}(q, v, m_{A_1} = 0) = \sum_{i=0}^{\infty} b_{i,j} q^i v^{2i+j}.$$

$q \backslash v$	0	1	2	3	4	5
0	0	0	1	0	0	0
1/4	0	0	0	2	0	0
1/2	1	0	0	0	4	0
3/4	0	2	0	0	0	8
1	0	0	4	0	0	0
5/4	0	0	0	8	0	0
3/2	3	0	0	0	17	0
7/4	0	6	0	0	0	30
2	0	0	9	0	0	0
9/4	0	0	0	18	0	0
5/2	5	0	0	0	39	0

Example with matters

Consider the \mathbb{Z}_2 twist of $E_6 + 2\mathbf{F}$ theory on (-4) curve. Upon twist

$$\text{vector multiplet : } \mathbf{78} \rightarrow \mathbf{52}_0 + \mathbf{26}_{1/2},$$

$$\text{hyper multiplet : } 2 \cdot \mathbf{27} \rightarrow \mathbf{26}_0 + \mathbf{26}_{1/2} + \mathbf{1}_0 + \mathbf{1}_{1/2}$$

Thus the twisted theory can be written as $E_6^{(2)} + \mathbf{F}_0 + \mathbf{F}_{1/2}$.

On the other hand, the \mathbb{Z}_2 twist of $E_6 + \mathbf{F}$ theory on (-5) curve have two possibilities: $E_6^{(2)} + \mathbf{F}_0$ and $E_6^{(2)} + \mathbf{F}_{1/2}$.

The twisted elliptic genera of all these theories can be exactly computed by the recursion formula from twisted elliptic blowup equations.

Example with matters: $E_6^{(2)} + F_0 + F_{1/2}$

Let us expand the twisted one-string elliptic genus as

$$\mathbb{E}_1^{\text{tw}}(q_\tau, v, m_{F_4} = m_F = 0) = \sum_{i,j=0}^{\infty} b_{i,j} q^i v^{2i+j}.$$

Then we find the following table for coefficients b_{ij} :

$q \setminus v$	4	5	6	7	8	9	10
0	0	0	0	-1	-4	78	-754
1/2	0	0	0	2	-27	-108	2509
1	0	0	0	3	54	-512	-1736
3/2	1	-2	-3	0	86	874	-7436
2	4	27	-54	-86	0	1437	10756
5/2	-78	108	512	-874	-1437	0	16930
3	754	-2509	1736	7436	-10756	-16930	0

Orange numbers give 5d one-instanton partition function $Z_1^{F_4+F}(v)$.

Example with matters: $E_6^{(2)} + \mathbf{F}_{1/2}$ and $E_6^{(2)} + \mathbf{F}_0$

Let us expand the twisted one-string elliptic genus of $E_6^{(2)} + \mathbf{F}_{1/2}$ as

$$\mathbb{E}_1^{\text{tw}}(q_\tau, v) = q_\tau^{-4/3} \sum_{i,j=0}^{\infty} c_{ij} q_\tau^i v^j = q_\tau^{-4/3} \sum_{i,j=0}^{\infty} b_{ij} q_\tau^i v^{2i+j}.$$

We obtain the following coefficients b_{ij} :

$q \setminus v$	5	6	7	8	9	10	11	12
0	0	0	0	1	0	52	0	1053
1/2	0	0	0	0	26	-52	1079	-2106
1	0	0	0	0	-2	381	-1300	15209
3/2	0	0	0	0	-3	-46	4235	-18670
2	0	0	0	0	-2	-73	-610	38768
5/2	1	-2	1	0	-1	8	-1100	-6134
3	4	23	-50	22	2	4	578	-12778
7/2	-78	154	307	-766	441	-112	-167	10866
4	754	-2197	2610	3113	-8788	6143	-2582	-4314

Orange numbers give 5d one-instanton partition function $Z_1^{F_4}(v)$.

Example with matters: $E_6^{(2)} + \mathbf{F}_0$ and $E_6^{(2)} + \mathbf{F}_{1/2}$

Let us expand the twisted one-string elliptic genus of $E_6^{(2)} + \mathbf{F}_0$ as

$$\mathbb{E}_1^{\text{tw}}(q_\tau, \nu) = q_\tau^{-\frac{7}{12}} \sum_{i,j=0}^{\infty} c_{ij} q_\tau^i \nu^j = q_\tau^{-\frac{7}{12}} \sum_{i,j=0}^{\infty} b_{ij} q_\tau^i \nu^{2i+j}.$$

We obtain the following coefficients b_{ij} :

$q \backslash \nu$	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	1	4	-78	754
1/2	0	0	0	0	0	-2	23	154	-2197
1	0	0	0	0	0	1	-50	307	2610
3/2	1	0	0	0	0	0	22	-766	3113
2	0	26	-2	-3	-2	-1	2	441	-8788
5/2	52	-52	381	-46	-73	8	4	-112	6143
3	0	1079	-1300	4235	-610	-1100	578	-167	-2582

The twisted one-string elliptic genera of $E_6^{(2)} + \mathbf{F}_0$ and $E_6^{(2)} + \mathbf{F}_{1/2}$ theories are exactly spectral dual to each other!

Summary

- We find all twisted elliptic blowup equations for all twisted rank one 6d (1, 0) SCFTs
- We compute the twisted elliptic genera for most of them
- We determine the modular ansatz for most of the twisted elliptic genera
- We find study the spectral flow symmetry for the twisted elliptic genera
- In the cases with 2d quiver gauge description, we can directly prove the spectral flow symmetry

Thank you for listening!