A diffeomorphism f of a closed manifold M is *partially hyperbolic* if there is a Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that

$$|Dfv^s| < 1 < |Dfv^u| \quad \text{and} \quad |Dfv^s| < |Dfv^c| < |Dfv^u|$$

for all points $p \in M$ and unit vectors $v^s \in E^s(p)$, $v^c \in E^c(p)$, and $v^u \in E^u(p)$.

Assumption. In this mini-course, E^s, E^c, and E^u are all one-dimensional and oriented, and Df preserves these orientations.

There are true foliations W^s and W^u tangent to E^s and E^u. There may or may not be foliations tangent to $E^c, E^{cs} = E^c \oplus E^s$, and $E^{cu} = E^c \oplus E^u$.
A *branching foliation* is a collection $\mathscr{F}$ of immersed surfaces $i : X \to M$ such that

(1) each surface is complete and without boundary,
(2) there is a plane field $E \subset TM$ such that each surface is tangent to E,
(3) for every point $p \in M$, there is at least one surface passing through p, and
(4) no two surfaces topologically cross (see figure).

Theorem (Brin–Burago–Ivanov)**.** If f is partially hyperbolic (under the above assumptions), then there are invariant branching foliations tangent to E^{cs} and E^{cu}.

Theorem (Brin–Burago–Ivanov)**.** A branching foliation can be approximated by a true foliation.

A cs-surface is an immersed surface $i : X \to M$ tangent to E^{cs}.
In general, cs-surfaces can be incomplete, can have boundary, and can be non-injective.
A cs*-curve* is a C^1 curve tangent to E^{cs} and transverse to E^s.

Proposition. If α is a cs-curve, then $W^s(\alpha) = \bigcup_{p \in \alpha} W^s(p)$ is a cs-surface.

An *ordered collection of cs-surfaces* is a set of cs-surfaces such that if $i_X : X \to M$ and $i_Y : Y \to M$ are in the collection then the following properties hold:

(1) if $x \in X$ and $y \in Y$ are such that $i_X(x) = i_Y(y) = p \in M$, then exactly one of $x < y$, $x = y$, or $x > y$ holds;
(2) at each point $p \in M$, this relation is transitive;
(3) the ordering is maintained along paths: if $\alpha_X : [0,1] \to X$ and $\alpha_Y : [0,1] \to Y$ are C^1 curves and $i_X \circ \alpha_X = i_Y \circ \alpha_Y$, then $\alpha_X(0) < \alpha_Y(0)$ implies $\alpha_X(1) < \alpha_Y(1)$.
(4) if i_Y "goes above" i_X at the point $i_X(x) = i_Y(y) = p \in M$, then $x < y$ (see figure).

Two cs-curves $\alpha : I \to M$ and $\beta : J \to M$ are *equivalent* if, up to reparameterization, $\alpha(t)$ and $\beta(t)$ are on the same stable leaf for all $t \in I = J$.

Proposition. If $\alpha : [0,\infty) \to M$ and $\beta : [0,\infty) \to M$ are cs-curves starting on the same stable leaf, then one of four behaviour occurs (see figure):

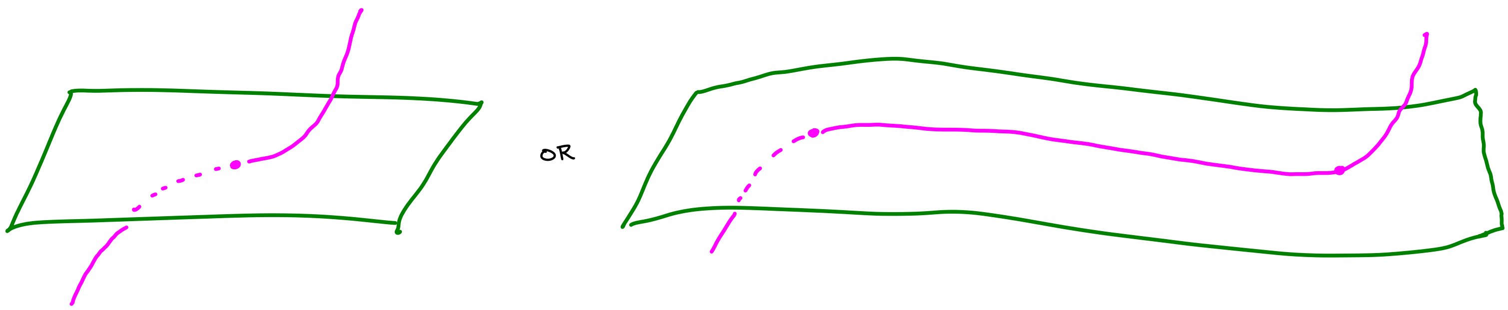
(1) the curves are equivalent,
(2) one curve is equivalent to a subcurve of the other,
(3) one curve "ascends above" the other, or
(4) the curves "split sideways".

Proposition. If $\mathscr{A}$ is a collection of cs-curves, all starting at the same stable leaf and such that no curve ascends above another, then the cs-surfaces $\{W^s(\alpha) : \alpha \in \mathscr{A}\}$ can be joined into a single cs-surface.

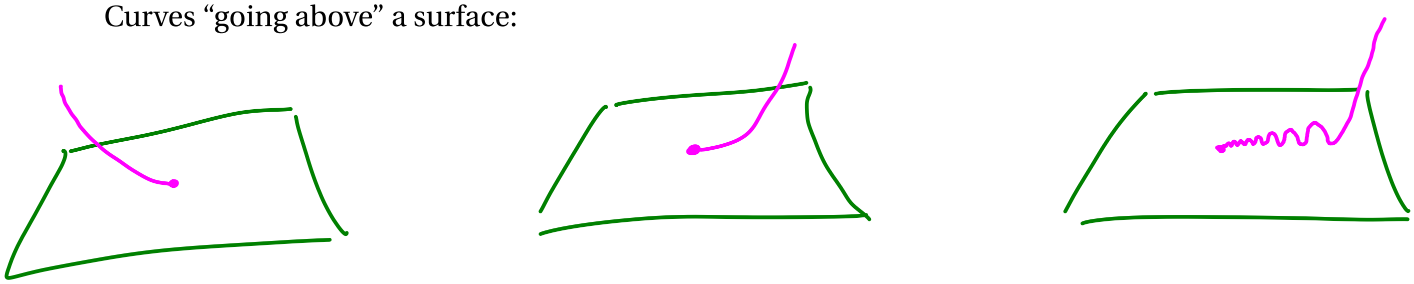
A *transverse surface* is an immersed surface $\rho : R \to M$ transverse to both E^s and E^{cs}.
A *transverse rectangle* is a transverse surface $\rho : [-1,1] \times [-1,1] \to M$ such that $\rho(\{t\} \times [-1,1])$ is transverse to E^{cs} for all $t \in [-1,1]$.

FIGURES

A curve topologically crossing a surface:



Curves "going above" a surface:



The four behaviours of curves starting on the same stable leaf: