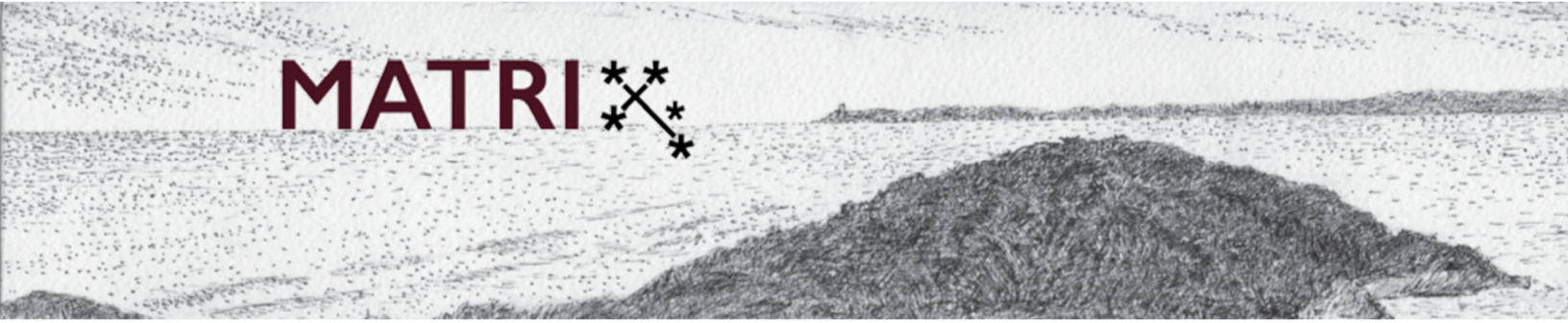

MATRIX Research Program:

Symmetry for Group Actions in Differential Geometry

The hyperbolic disc in wycinanka łowicka style by Katarzyna Nurowska

Talk Titles and Abstracts

23 – 27 May 2022

MATRIX, Creswick

MATRIX meeting 23–27 May 2022: Titles

Monday 10am Rod Gover
Conformal submanifolds, distinguished submanifolds, and integrability

Monday 2pm Yuri Nikolayevsky
Einstein hypersurfaces in symmetric spaces

Tuesday 9am Artem Pulemotov
Mountain pass approach to the prescribed Ricci curvature problem

Tuesday 2pm Tim Buttsworth
U(2)-invariant Ricci flows on four-dimensional manifolds

Wednesday 9am Holger Dullin
Momentum Polytopes related to geodesic flows on spheres

Wednesday 2pm Ian Marquette
Superintegrable Hamiltonians with higher order symmetries and Painlevé transcendents

Wednesday 3pm Sean Gasiorek
Minkowski billiards on the hyperboloid of one sheet

Thursday 9am Peter van der Kamp
Integrable curvature evolution

Thursday 2pm Yang Shi
Coxeter groups and connections between integrable systems

MATRIX meeting 23–27 May 2022: Abstracts

Monday 10am Rod Gover

For conformal geometries of Riemannian signature, we provide a comprehensive treatment of the basic local theory of embedded submanifolds of arbitrary codimension. We provide three distinct, but equivalent, fundamental tractor-valued invariants of submanifolds. For any one of these "normal objects" its derivative by the pullback of the conformal tractor connection recovers the data of the tractor second fundamental form. The vanishing of this, so the normal object being parallel, determines a notion of distinguished submanifold. For the case of curves this exactly characterises conformal circles, while for hypersurfaces it is the totally umbilic condition. So, for other codimensions, this unifying notion interpolates between these extremes, and we prove that this coincides with a property of ambient conformal circles remaining in the submanifold, that we term weakly conformally circular. We prove that submanifolds are strongly conformally circular if and only also a second tractor invariant vanishes, this encodes a Fialkow type tensor invariant. We prove that the property of being distinguished is also captured by a type of moving incidence relation. This second characterisation is used to provide a very general theory and construction of quantities that are necessarily conserved along the submanifold. The formalism thus leads to a conformal submanifold first integral theory that generalises the ideas available for curves . We prove that any normal solution to an equation from the class of first BGG equations can yield such a conserved quantity, and we show that it is easy to provide explicit formulae for these. For some equations the condition of normality is not required. We also show that for normal solutions of conformal Killing-Yano equations, a certain zero locus of the solution is necessarily a distinguished submanifold.

Monday 2pm Yuri Nikolayevsky

In this talk, I will present the results of the joint paper of Jeong Hyeong Park and myself in which we give a classification of Einstein hypersurfaces in irreducible symmetric spaces. The main theorem states that there are three classes of such hypersurfaces, belonging to three very different "geometries": homogeneous geometry, Legendrian geometry and affine geometry. I will give a brief introduction into these three geometries and explain how they fit together in our classification.

Tuesday 9am Artem Pulemotov

The talk will focus on the prescribed Ricci curvature problem for homogeneous Riemannian metrics. We will discuss new results based on the variational interpretation of this problem and mountain pass techniques. Joint work with Wolfgang Ziller (The University of Pennsylvania).

Tuesday 2pm Tim Buttsworth

There are a number of interesting examples of four-dimensional Ricci flows which are foliated by Berger sphere geometries on S^3, and are therefore invariant under an action of U(2). In this talk, I will discuss some of these examples, as well as possible future avenues of study.

Wednesday 9am Holger Dullin

Separation of variables for the geodesic flows on spheres leads to a large family of integrable systems whose integrals are defined through the separation constants. Reduction by the periodic flow of the Hamiltonian leads to integrable systems on Grassmanians. Specifically for the geodesic flow on the 3-sphere S^3 the reduced system defines a family of integrable systems on $S^2 \times S^2$. We show that the image of these systems under a continuous momentum map defined through the action variables has a triangle as its image. The image is rigid and does not change when the integrable system is changed within the family. Each member of the family can be identified with a point inside a Stasheff polytope. Corners of the polytope correspond to toric systems (possibly with degenerations), edges correspond semi-toric systems (in various meanings of the word), and the face corresponds to 'generic' integrable systems. A fundamental difference of this momentum map to that of a toric or semi-toric system is that the number of tori in the preimage of a non critical point may be 1, 2, or 4. The momentum map is continuous but not smooth along the images of hyperbolic singularities. The corresponding quantum problem and generalisations to higher dimensional spheres will be discussed. This is joint work with Diana Nguyen and Sean Dawson.

Wednesday 2pm Ian Marquette

I will discuss the classification of classical and quantum superintegrable systems with higher order integrals of motion. I will point out how Painlevé transcendents and the Chazy equations play a role in solving determining equations. I will also discuss explicit results on two dimensional Euclidean space, but also on the sphere. In addition, it has been discovered that their integrals naturally connect with polynomial algebras. Those algebraic structures are interesting in their own right as they also appear in other context of mathematical physics, such as missing label problems and Racah polynomials.

Wednesday 3pm Sean Gasiorek

We give a review of Euclidean and pseudo-Euclidean billiards in the plane and in d-dimensional space. If the billiard table is bounded by confocal quadrics, periodic trajectories can be expressed in algebro-geometric terms based on work of Poncelet, Cayley, and others. Our work focuses on a billiard problem for compact domains on a hyperboloid of one sheet bounded by confocal quadrics using the pseudo-Euclidean metric and explores various geometric consequences. This is joint work with Milena Radnović (University of Sydney).

Thursday 9am Peter van der Kamp

Starting from an action of a Lie-group on a manifold the Fels-Olver moving frame method provides a set of generating invariants together with their syzygies. One of the syzygies gives us the evolution of curvature invariants if an evolution of curves is specied. Another syzygy gives us an invariant symmetry condition which can be utilised to identify integrable curvature evolution.

Thursday 2pm Yang Shi

The Kadomtsev-Petviashvili (KP) equation [9] is the master 2+1 dimension integrable equation in the sense that it degenerates to other 1+1 dimension PDEs such as the Kortewegde Vries (KdV) equation [4], or integrable ODEs such as the Painlevé equations [11][3][6]. These integrable systems and their discrete analogues have been shown to admit Coxeter groups as their discrete symmetries [1][8]. In this talk, I will describe the Coxeter groups for these systems and how the various properties of the group can be exploited to establish connections between different systems [5][10]. Here, some old geometries such as those of Menelaus [7] and Desargues [2] will be "rediscovered", and maybe some new ones along the way?

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