Integrable probability, special functions and combinatorics

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(Based on joint works with T. Imamura, M. Mucciconi)

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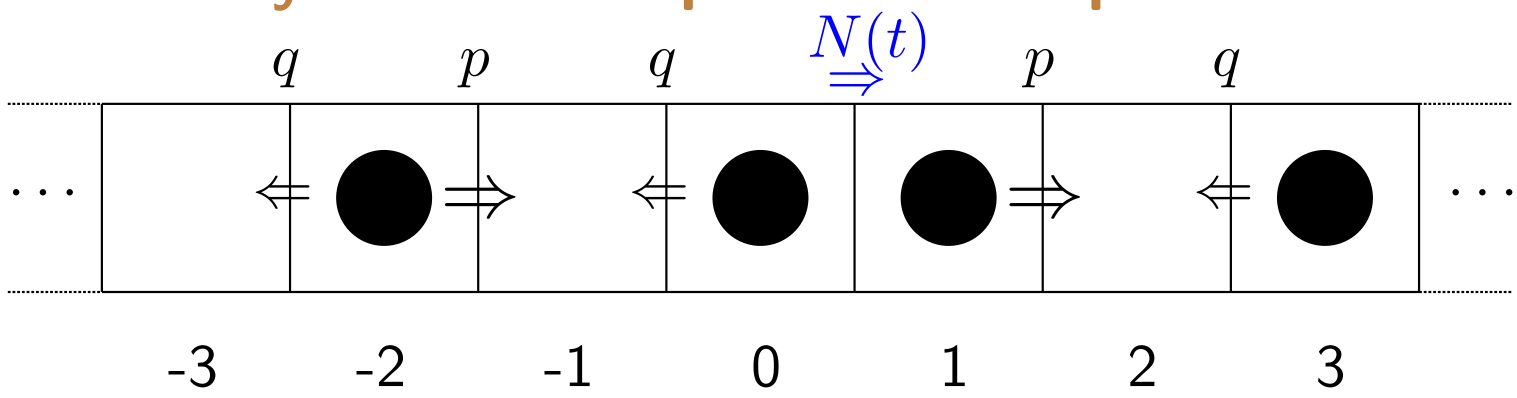
1. Introduction: ASEP

ASEP = asymmetric simple exclusion process



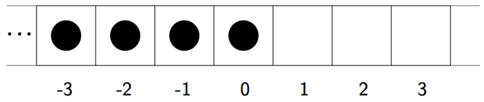
- A many particle stochastic (Markov) process. [Probability]
  Rigorous foundation established in 1970's (cf Liggett).

- A standard model in non-equilibrium statistical mechanics.
  Interested in distribution of current $N(t)$.

- Markov generator $\sim$ XXZ Hamiltonian. [Integrable]
  Methods of exactly solvable models can be applied (Baxter).

Johansson's work in 2000 for TASEP

TASEP ($p=1, q=0$) with the step i.c.



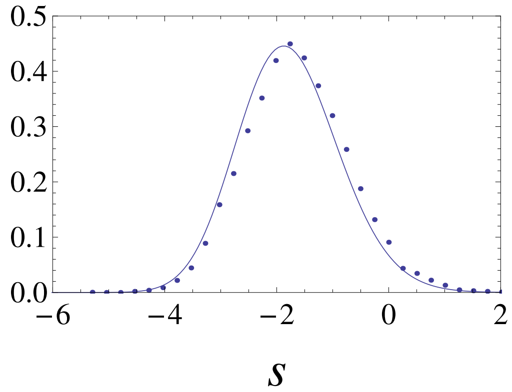
Theorem 1. Robinson-Schensted-Knuth(RSK)[Combinatorics]

$\Rightarrow$ Current is distributed as λ_1 in Schur measure [Special function]

$$\frac{1}{Z} s_\lambda(a) s_\lambda(b)$$

Theorem 2.

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{N(t)-t/4}{-2^{-4/3}t^{1/3}} \le s\right] = F_2(s)$$

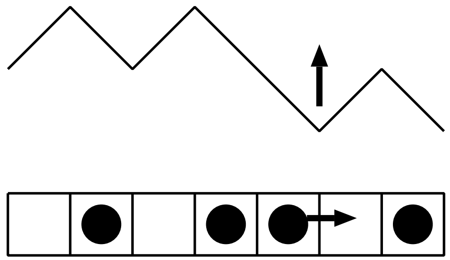


where $F_2(s)$ is the GUE Tracy-Widom distribution

$$F_2(s) = \det(1-K_{\mathrm{Ai}})_{L^2([s,\infty))}$$

from random matrix theory where K_{Ai} is the Airy kernel

$K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$



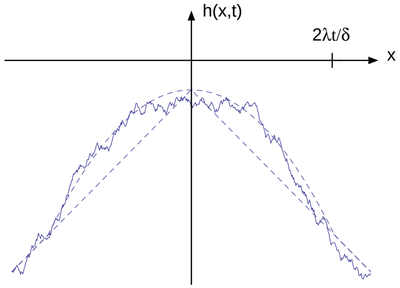
- Surface growth interpretation

KPZ equation and $T > 0$ polymer

1986 Kardar Parisi Zhang

- Height function $h = h(x,t), x \in \mathbb{R}, t \in \mathbb{R}_+$,

$$\frac{\partial}{\partial t} h = \frac{1}{2}\frac{\partial^2}{\partial x^2} h + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \eta$$

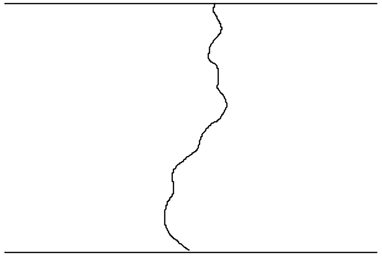


where $\eta = \eta(x,t)$ is the space time white noise.

There is an issue of well-definedness (cf Hairer).

- Cole-Hopf transformation: $Z = Z(x,t) = e^{h(x,t)}$

$$\frac{\partial}{\partial t} Z = \frac{1}{2}\frac{\partial^2}{\partial x^2} Z + \eta Z$$



Directed polymer at a finite temperature $(T > 0)$

Fredholm determinant with $T > 0$ kernel

Theorem. 2010 TS-Spohn, Amir-Corwin-Quastel

Laplace transform of $Z(0,t)$ with i.c. $Z(x,0) = \delta(x)$

$$\mathbb{E}[\exp(-Z(0,t)e^{\frac{t}{24}-(t/2)^{1/3}s})] = \det(1 - K_t)_{L^2(\mathbb{R}_+)}$$

$$K_t(x,y) = \int_{\mathbb{R}} \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{1+e^{(t/2)^{1/3}(s-\lambda)}} d\lambda$$

- Can be derived by Bethe ansatz. Generalization to many discrete KPZ models associated with Macdonald measure (2012 Borodin-Corwin) $\frac{1}{Z} P_\lambda(a) Q_\lambda(b)$.
- The kernel with Fermi-Dirac factor $1/(1+e^{\beta(\epsilon-\mu)})$ for KPZ eq. suggests connection to free fermion at finite temperature.
- We found a connection by a generalization of RSK.

Free fermion and its correlation kernel

- A free fermion is a quantum many (infinite) particle system for which each one particle state $\phi_n(x)$ ($n \geq 1$, energy ϵ_n) can be either occupied or empty (Pauli principle).

- At $T = 0$, for N particles, the ground state filling $n = 1, \ldots, N$ is realized. The probability density is given by
$$\frac{1}{Z}\left(\det(\phi_n(x_m))_{n,m=1}^{N}\right)^2$$
Correlations are described by the kernel
$K(x, y) = \sum_{n=1}^{N} \phi_n(x)\phi_n(y)$.

- For $T > 0$, state n is filled with prob $\frac{1}{1+e^{\beta(\mu-\epsilon_n)}}$, $\beta = \frac{1}{k_B T}$.
The correlation kernel is $K(x, y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{1+e^{\beta(\mu-\epsilon_n)}}$.

- Both cases are determinantal point process (DPP).

Cauchy identities and our refinement

Schur
$$\sum_{\lambda \in \mathcal{P}} s_\lambda(a) s_\lambda(b) = \prod_{i=1}^{n} \prod_{j=1}^{n} \frac{1}{1 - a_i b_j}$$

q-Whittaker
$$\sum_{\mu \in \mathcal{P}} P_\mu(a) Q_\mu(b) = \prod_{i=1}^{n} \prod_{j=1}^{n} \frac{1}{(a_i b_j; q)_\infty}$$

Skew Schur
$$\sum_{\substack{\lambda, \rho \in \mathcal{P} \\ \rho \subset \lambda}} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b) = \frac{1}{(q;q)_\infty} \prod_{i=1}^{n} \prod_{j=1}^{n} \frac{1}{(a_i b_j; q)_\infty}$$

We have found the first bijective proof of the Cauchy identity for q-Whittaker polynomials. It also leads to the following refinement.

Theorem. With $b_\mu(q) = \prod_{i \geq 1} \frac{1}{(q;q)_{\mu_i - \mu_{i+1}}}$,

$$\sum_{\ell=0}^{n} \frac{q^\ell}{(q;q)_\ell} \sum_{\mu : \mu_1 = n - \ell} b_\mu(q) P_\mu(a) P_\mu(b) = \sum_{\lambda, \rho : \lambda_1 = n} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

Plan

1. Introduction

2. TASEP, Schur measure and $T = 0$ free fermion

3. Discrete KPZ models and q-Whittaker measure

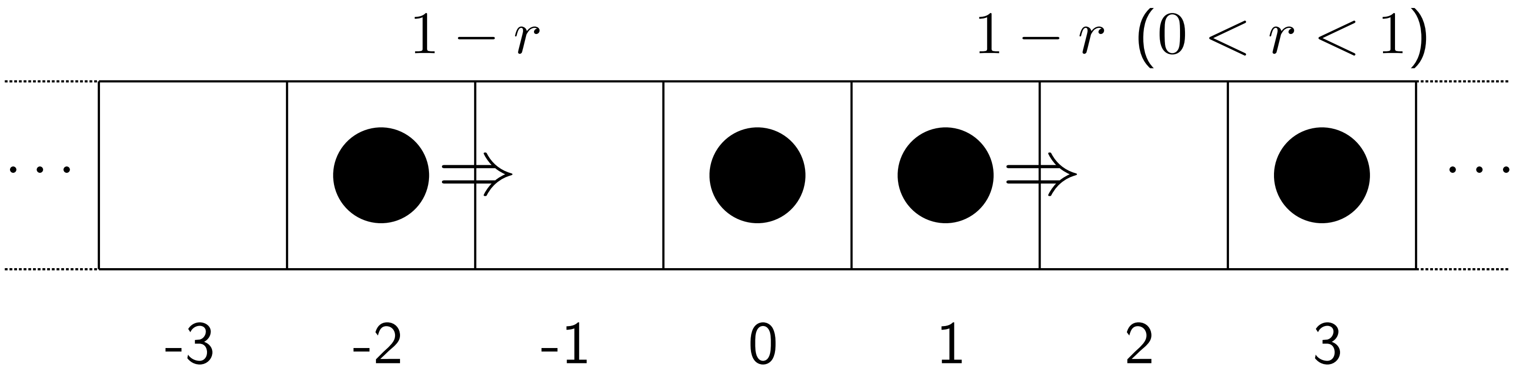
   Relation between q-Whittaker and periodic Schur measures

4. Bijection by skew RSK dynamics

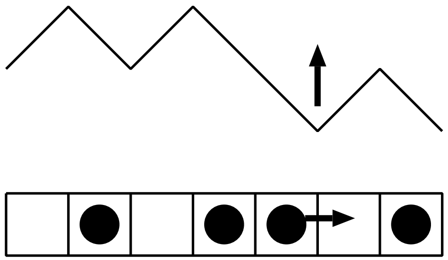
5. Applications to KPZ models: Half-space case

2. TASEP, Schur measure and $T = 0$ free fermion

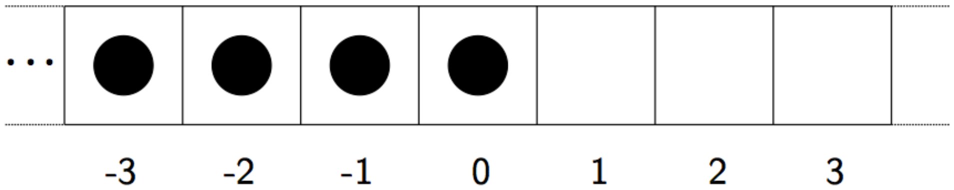
TASEP $=$ totally asymmetric simple exclusion process



$N(t)$: Integrated current at $(0,1)$ upto time t from step i.c.



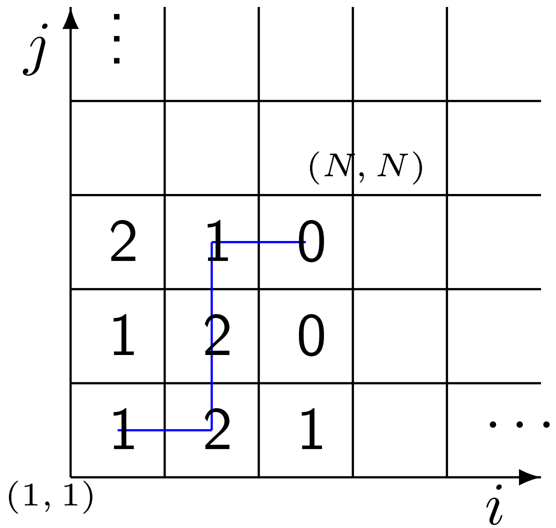
Step i.c.



$N(t) \sim h(0,t)$: height

Waiting times and $T = 0$ polymer

Waiting times: iid geo(r)

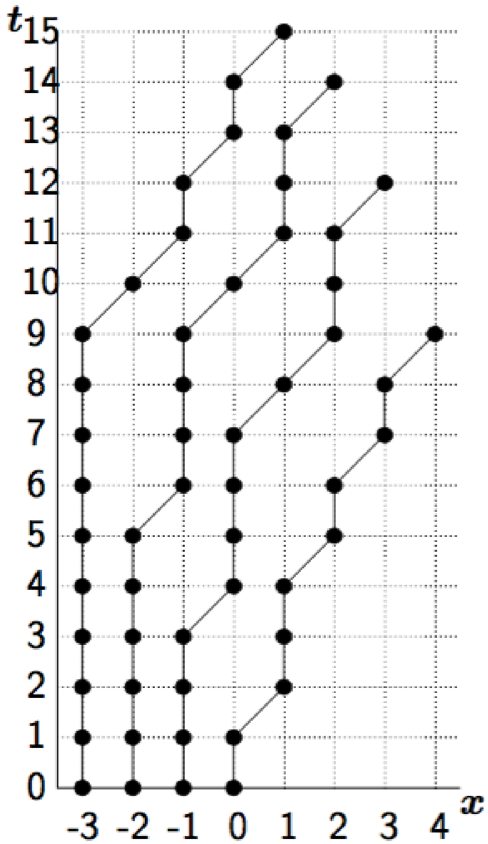


$$G_N = \max_{\substack{\text{up-right paths from} \\ (1,1)\text{to}(N,N)}} \Big(\sum_{\substack{(i,j) \\ \text{on a path}}} w_{i,j} \Big)$$

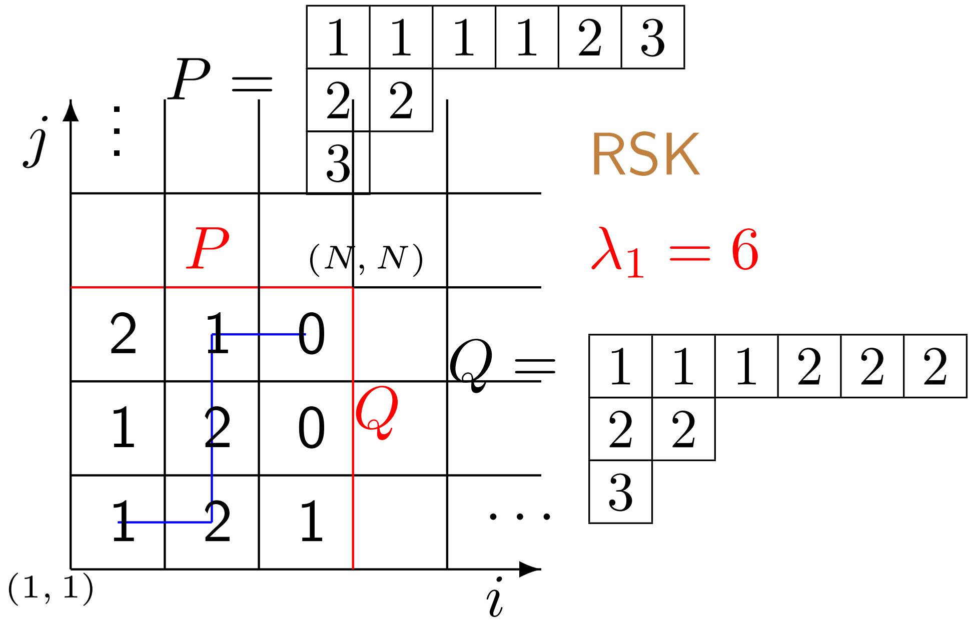
Directed polymer at $T = 0$

$$\mathbb{P}[N(t) \geq N] = \mathbb{P}[G_N \leq t]$$

Trajectories



Waiting times and $T = 0$ polymer

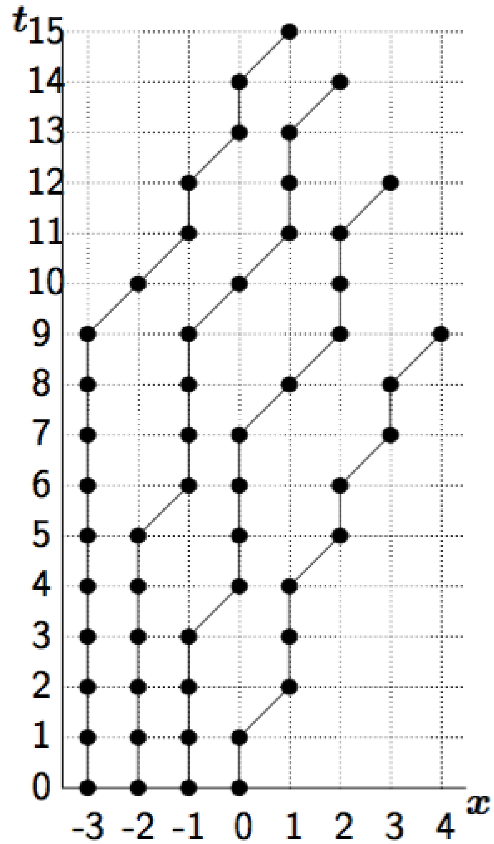


$$G_N = \max_{\substack{\text{up-right paths from}\\ (1,1)\text{to}(N,N)}} \Bigg(\sum_{\substack{(i,j)\\ \text{on a path}}} w_{i,j} \Bigg)$$

$G_3 = 6$

Generalization: geo(r) $\to$ geo$(a_i b_j)$

Trajectories



$a = (a_1, \ldots, a_N)$

Schur measure

- Schur function $s_\lambda(a) = \sum_{T \in \mathrm{SST}(\lambda)} a^T, a^T = \prod_i a_i^{\#i \text{ in } T}$, where SST is the set of semistandard tableaux.

$$\mathbb{P}[G_N \le u] = \frac{1}{Z} \sum_{\lambda, \lambda_1 \le u} s_\lambda(a) s_\lambda(b)$$

By Jacobi-Trudi formula $s_\lambda(x) = \det(\phi_n(x_m))$, the Schur measure is a DPP associated with $T = 0$ free fermion.

- 2000 Johansson

$$\lim_{t \to \infty} \mathbb{P}\left[\frac{N(t) - Jt}{ct^{1/3}} \ge -s\right] = F_2(s) = \det(1 - K_2)_{L^2(s,\infty)}$$

where F_2 is GUE Tracy-Widom distribution and kernel K_2 is

$$K_2(x, y) = \int_{\mathbb{R}_+} \mathrm{Ai}(x + \lambda)\mathrm{Ai}(y + \lambda) d\lambda$$

3. Discrete KPZ models and q-Whittaker measure

- Discrete KPZ models: ASEP, q-TASEP, sHS6VM, etc.

  2011 Borodin, Corwin

  By the branching rule of q-Whittaker function, these models are related to q-Whittaker measures.

- Geometric q-PushTASEP(2015 Matveev-Petrov) is related to the q-Whittaker measure of the form

$$\frac{1}{Z} b_\mu(q) P_\mu(a) P_\mu(b), \quad b_\mu(q) = \prod_{i \geq 1} \frac{1}{(q;q)_{\mu_i - \mu_{i+1}}}$$

  where $a = (a_1, \cdots, a_N), b = (b_1, \cdots, b_M)$.

  The Nth particle position at time M is related to μ_1 as $X_N(M) \stackrel{d}{=} \mu_1 + N$. Note: No single det formula for P_μ.

Fredholm determinants for q-Whittaker

- Standard approach (2011- Borowin, Corwin, TS, Petrov, ...)
  Markov duality + Bethe ansatz or by Macdonald operators.

$$\mathbb{E}\left[\frac{1}{(\zeta q^{-\mu_1};q)_\infty}\right] = \det(1-K)$$

NOT $T>0$ kernel. Asymptotics is rather involved.
Generalization to half-space case is difficult.

- Frobenius determinant approach (2019 Imamura, TS)

$$\mathbb{E}\left[\frac{1}{(\zeta q^{-\mu_1};q)_\infty}\right] = \det\left(1-f_\zeta K\right)_{\ell^2(\mathbb{Z})}$$

where $f_\zeta(m) = \frac{-\zeta q^m}{1-\zeta q^m}$ ($T>0$ kernel!)

Periodic Schur measure

- Periodic Schur measure (2007 Borodin, 2018 Betea-Bouttier)

$$\frac{1}{Z}\sum_{\rho\in\mathcal{P},\rho(\subset\lambda)} q^{|\rho|}s_{\lambda/\rho}(a)s_{\lambda/\rho}(b)$$

- Its shift mixed version ($\lambda_i \to \lambda_i + S$) with

$$\mathbb{P}(S=\ell) = \frac{t^\ell q^{\ell^2/2}}{(q;q)_\infty \theta(-tq^{1/2})}, \qquad \ell\in\mathbb{Z},\ \text{for } t>0$$

with $\theta(x) = (x;q)_\infty (q/x;q)_\infty$, is a DPP associated with $T>0$ free fermion and hence

$$\mathbb{P}\left(\lambda_1 + S \le n\right) = \det\left(1 - fL\right)_{\ell^2(\mathbb{Z})}$$

where $f(m) = \frac{tq^{1/2+n+m}}{1+tq^{1/2+n+m}}$ and L is K with the only difference being its contour(!). Identify $\zeta = -tq^{1/2+n}$.

Relation between q-Whittaker and periodic Schur

- We can show that the two Fredholm determinants are equivalent and this is rephrased as an identity for marginals of q-Whittaker and periodic Schur measure.

  Theorem: $\mathbb{E}\left[1/(-tq^{\frac{1}{2}+n-\mu_1};q)_\infty\right] = \mathbb{P}(\lambda_1 + S \le n)$

  Connection between q-Whittaker & periodic Schur measures

- This is equivalent to the following identity

$$\sum_{\ell=0}^{n} \frac{q^\ell}{(q;q)_\ell} \sum_{\mu:\mu_1=n-\ell} b_\mu(q) P_\mu(a) P_\mu(b) = \sum_{\lambda,\rho:\lambda_1=n} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

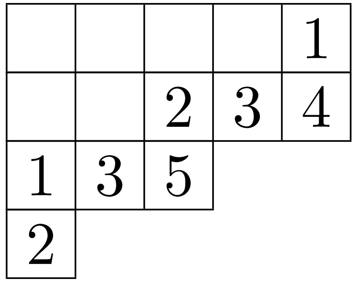
  where $b_\mu(q) = \prod_{i\ge 1} \frac{1}{(q;q)_{\mu_i-\mu_{i+1}}}$.

  More direct proof without matching Fredholm dets?

4. Bijective proof

Skew Schur function

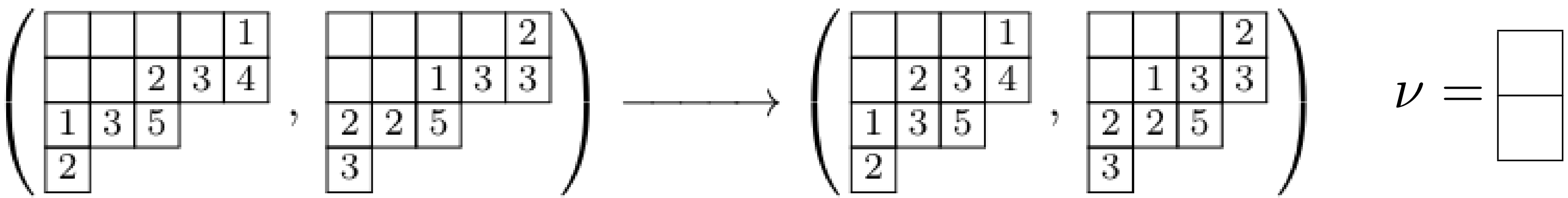
$$s_{\lambda/\rho}(x) = \sum_{T \in \mathrm{SST}(\lambda/\rho)} x^T$$



where SST is the set of skew semistandard tableaux.

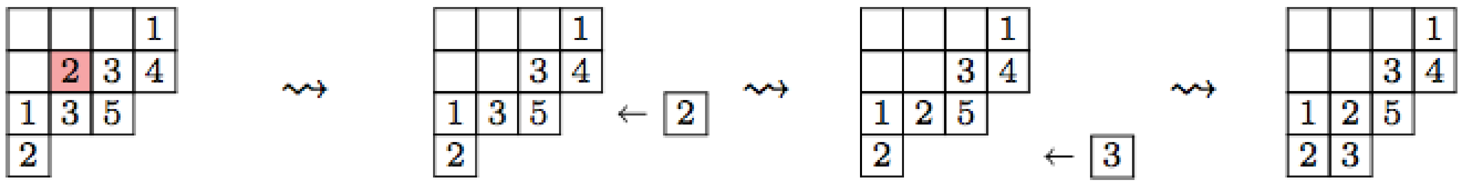
RHS of the identity is related to a pair (P, Q). Try to find a bijection from (P, Q) to something which is related to q-Whittaker function!

Sqeezing: $(P, Q) \to (P_1, Q_1)$

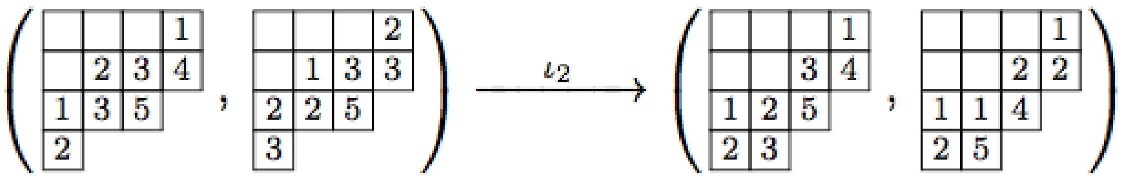


Skew RSK map

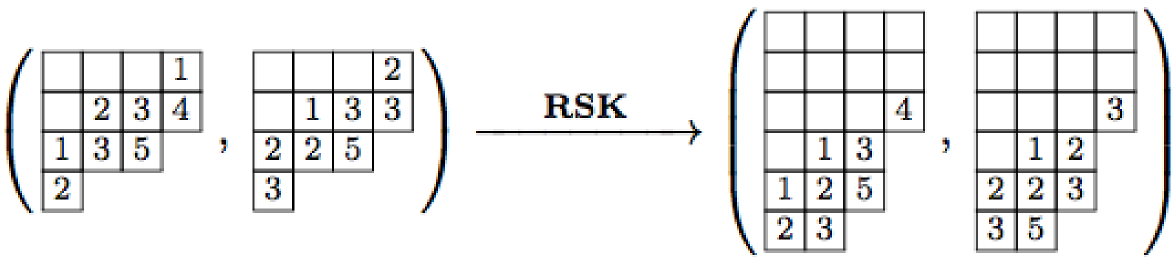
Internal insertion (Sagan-Stanley 1990)



Operation ι_2

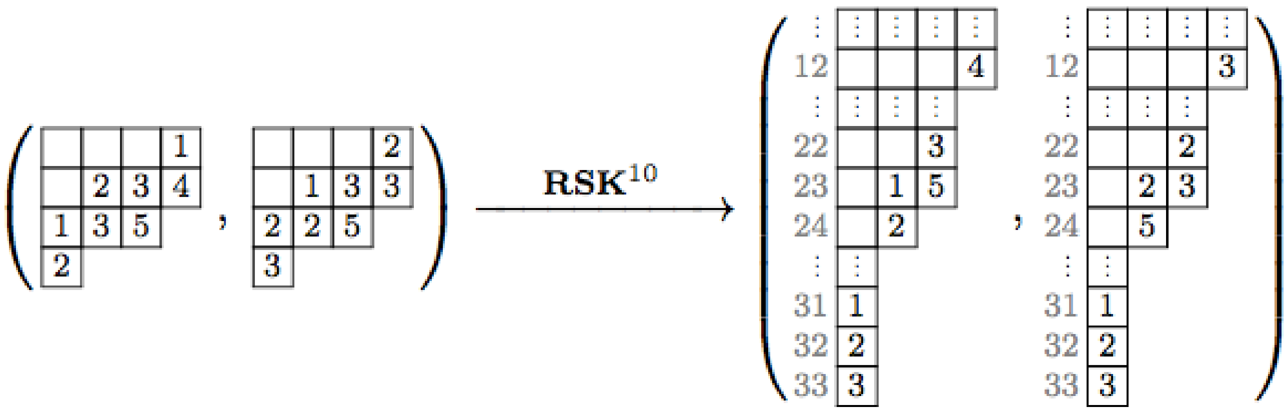


Skew RSK map: $\mathrm{RSK}(P, Q) = \iota_2^n(P, Q)$

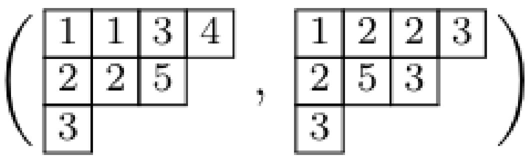


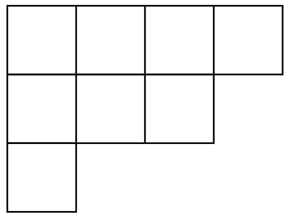
Skew RSK dynamics

Iterating skew RSK maps: $(P_{t+1}, Q_{t+1}) = \mathbf{RSK}(P_t, Q_t)$



Asymptotic tableaux and their shape

$(V, W) =$ 

$\mu =$ 

$V, W \in$ VST(μ): "vertically strict tableaux" (VST) of same shape μ with elements increasing only in each column.

Remark: Similarity to Box-Ball sysytems

Combinatorial formula for q-Whittaker function

- q-Whittaker function (e.g. 2012 Schilling Tingley)

$$P_\mu(x) = \sum_{V \in \mathrm{VST}(\mu)} q^{H(V)} x^V$$

1	2	2	3
2	5	3	
3			

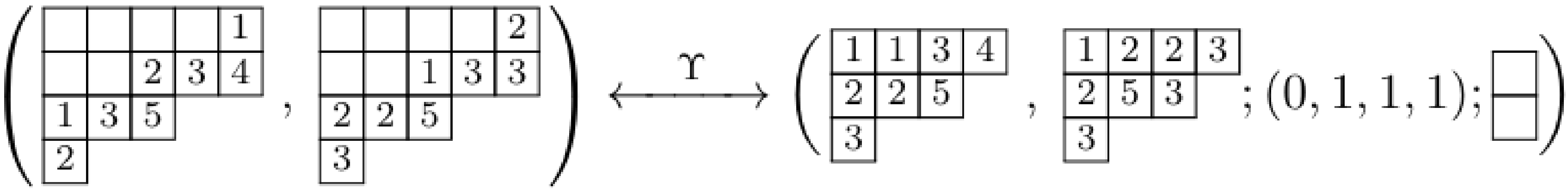
where H is the energy function (e.g. 1997 Nakayashiki Yamada). In a way $H(V)$ measures how a VST V is far away from a semistandard tableaux.

Note: P_μ tends to s_μ when $q \to 0$.

- Recall the identity

$$\sum_{\ell=0}^{n} \frac{q^\ell}{(q;q)_\ell} \sum_{\mu:\mu_1=n-\ell} b_\mu(q) P_\mu(a) P_\mu(b) = \sum_{\lambda,\rho:\lambda_1=n} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

Bijection $\Upsilon : (P, Q) \leftrightarrow (V, W, \kappa, \nu)$



(P, Q): A pair of skew SSTs with same shape λ/ρ

ν: partition obtained by "squeezing" (P, Q) to (P_1, Q_1).

(V, W): A pair of VSTs with same shape μ

$$\kappa \in \mathcal{K}(\mu) = \{\kappa = (\kappa_1, \ldots, \kappa_{\mu_1}) \in \mathbb{N}_0^{\mu_1} : \kappa_i \geq \kappa_{i+1} \text{ if } \mu'_i = \mu'_{i+1}\}$$

Theorem: There is a bijection Υ with weight preserving property

$$|\rho| = H(V) + H(W) + |\kappa| + |\nu|$$

Note $\sum_{\kappa \in \mathcal{K}(\mu)} q^{|\kappa|} = b_\mu(q)$, $\mathbb{P}[\nu_1 = \ell] = \frac{q^\ell}{(q;q)_\ell}(q;q)_\infty$.

On a proof of our bijection

- Proving properties of skew RSK dynamics based on its rules is difficult.
- Original Robinson's algorithm, which maps a permutation to a canonical one, can be understood as an application of crystal symmetry.
- We can use (affine) crystal to study skew RSK dynamics and prove our theorem (needs another seminar to explain).

5. New approach to KPZ models

- Standard approach to KPZ models with q has been to apply Markov duality & Bethe ansatz or Macdonald operator
- With our bijection, one does not need to use such methods any more. Once the mapping to (periodic) Schur measure is established, then one can simply apply the methods of DPP to get Fredholm determinants, for which asymptotic analysis by now can be done in a standard way.
- Our approach also works also for half-space models.

Half space case

Half space q-Whittaker measure (2021 Imamura-Mucciconi-TS)

$$\frac{1}{\Phi(a,z;q)} b_\mu(q;z) P_\mu(a,q^2)$$

with $\quad \Phi(a,z;q) = \prod_{i=1}^{n} \frac{1}{(a_i z;q)_\infty} \prod_{1\le i<j\le n} \frac{1}{(a_i a_j;q^2)_\infty}$

Here

$$b_\mu(q;z) = \prod_{i=2,4,6\dots} \frac{[qz^2+1]_{q^2}^{\mu_i-\mu_{i+1}}}{(q^2;q^2)_{\mu_i-\mu_{i+1}}} \prod_{i=1,3,5,\dots} \frac{z^{\mathbf{1}_{\mu_i>\mu_{i+1}}}}{(q;q)_{\mu_i-\mu_{i+1}}},$$

$$[A+B]_p^k = \sum_{j=0}^{k} A^j B^{k-j} \binom{k}{j}_p, \quad \binom{k}{j}_p = \frac{(p;p)_k}{(p;p)_j (p;p)_{k-j}}.$$

Free boundary Schur measure

- Free boundary Schur measure (2017 B-B-Nejjar-Vuletic)

$$\frac{1}{Z} z_1^{\mathsf{odd}\mu} z_2^{\mathsf{odd}\lambda} q^{|\mu|} s_{\lambda/\mu}(a) \quad \text{(Pfaffian point process)}$$

- Thm:Identity relating HS q-Whittaker & FB Schur measures

$$\sum_{\ell=0}^{k} g_\ell(\gamma, q) \sum_{\mu:\mu_1=k-\ell} b_\mu(q;\gamma) P_\mu(a;q^2) = \sum_{\lambda,\rho:\lambda_1=k} \gamma^{\mathrm{odd}(\lambda')+\mathrm{odd}(\rho')} q^{|\rho|} s_{\lambda/\rho}(a)$$

$$\text{with} \quad g_k(\gamma, q) = \frac{[q\gamma^2 + q^2]_{q^2}^k}{(q^2;q^2)_k}.$$

- Fredholm Pfaffian for half-space q-Whittaker measure

$$\mathbb{E}[1/(-\zeta q^{-\mu_1 - \chi}; q)_\infty] = \mathrm{Pf}(J - K)$$

where χ is a certain indep. r.v. and J is an anti-sym. kernel.

$\Rightarrow$ Log-Gamma polymer and KPZ equation in half-space

Summary

- "Integrable probability" is a relatively new field to study interacting stochastic models with integrability.

  It is not just an application of methods from integrable systems. It requires new insights and ideas and creates new developments, e.g. for combinatorics and special functions.

- By our new skew RSK dynamics, we could find a connection between q-Whittaker and periodic Schur measures.

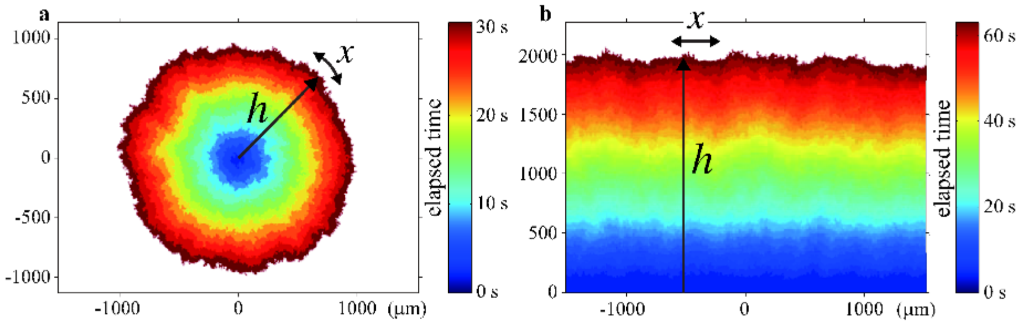
- This gives a new approach to study KPZ models by mapping them to free fermions at finite temperature.

  Our approach works also for half-space models with Pfaffian structures.

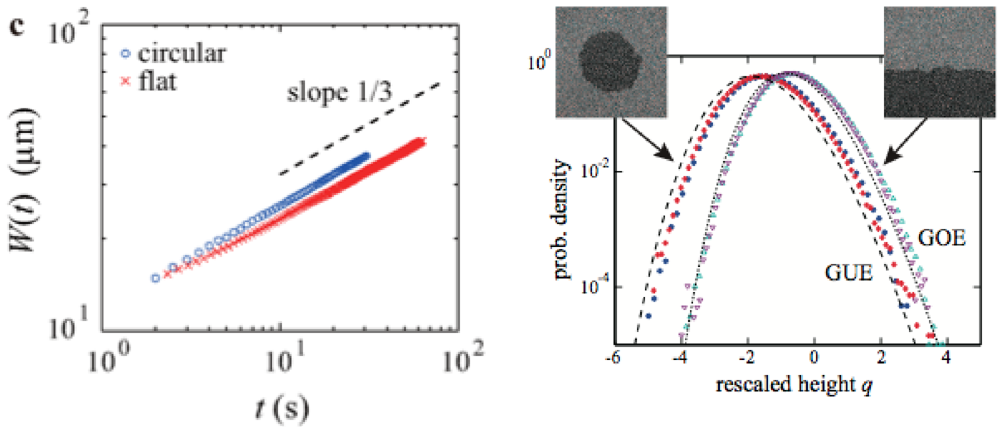
Kardar-Parisi-Zhang (KPZ) universality

Nonequilibrium statistical physics: fluctuations in surface growth.

Experiments by liquid crystal (2010 Takeuchi Sano)



Fluctuations: $O(t^{1/3})$ and scaled distributions



A remark: Cauchy identities for three polynomials

Schur

$$\sum_{\lambda\in\mathcal{P}} s_\lambda(a)s_\lambda(b) = \prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{1-a_ib_j}$$

q-Whittaker

$$\sum_{\mu\in\mathcal{P}} P_\mu(a)Q_\mu(b) = \prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(a_ib_j;q)_\infty}$$

Skew Schur

$$\sum_{\substack{\lambda,\rho\in\mathcal{P}\\ \rho\subset\lambda}} q^{|\rho|} s_{\lambda/\rho}(a)s_{\lambda/\rho}(b) = \frac{1}{(q;q)_\infty}\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{1}{(a_ib_j;q)_\infty}$$

Our bijection gives the first bijective proof of the Cauchy identity for q-Whittaker polynomials.