Mathematical Breakthroughs in Wave Propagation

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FL22 Mathematical Breakthroughs in Wave Propagation

This funded Laureate Fellowship proposal comprises three interrelated research themes:

A. Nonlinear waves, concentrating on global solutions of the nonlinear Schrödinger equation (NLS).

B. Rough waves, concentrating on wave propagation in rough media.

C. Chaotic waves; concentrating on eigenfunctions on compact hyperbolic surfaces.

What is wave propagation (in 5 minutes)?

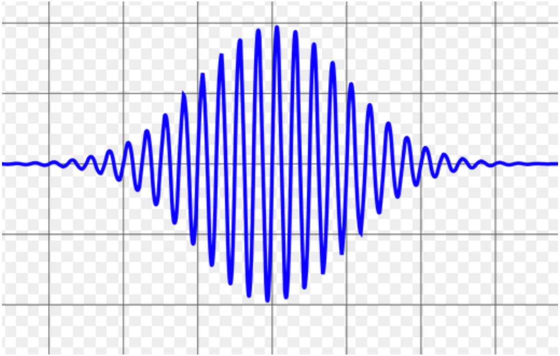
Given a wave equation, such as the time-dependent Schrödinger equation in $\mathbb{R}^{n+1} \ni (x, t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$,

$$(D_t + \Delta)u = 0, \quad D_t = -i\partial_t, \quad \Delta = \sum_{j=1}^{n} D_{x_j} D_{x_j},$$

there are 'wave packet' solutions that take the approximate form

$$\chi(x - 2\xi t)e^{i(x\cdot\xi - t|\xi|^2)},$$

where $|\xi|$ is a large parameter. The wave packet, at each time t, has a location $x(t)$ and a spatial frequency of oscillation, ξ as well as a temporal frequency τ, which in the example above is $-|\xi|^2$. Together these give a curve of points $(x, t; \xi, \tau)$ in phase space.



It turns out this point $(x, t; \xi, \tau)$ moves according to a Hamiltonian dynamical system. Which Hamiltonian? The symbol of the operator, $\tau + |\xi|^2$. We find

$$\dot{x} = 2\xi, \quad \dot{t} = 1; \quad \dot{\xi} = 0, \quad \dot{\tau} = 0.$$

Microlocal analysis takes place on phase space — in effect, we track wave packets and estimate norms of functions by decomposing them in phase space and estimating each piece.

Such decomposition is done using pseudodifferential operators, which arise from symbols defined on phase space.

Time-dependent Schrödinger equation

We work in $\mathbb{R}^{n+1}$, with coordinates (x,t), where $x \in \mathbb{R}^n$ is a spatial variable and $t \in \mathbb{R}$ is time.

- Let D_j denote $-i\partial_{x_j}$ and $D_t = -i\partial_t$. We denote by $\Delta = \sum_j D_j D_j$ the (positive) Laplacian (the opposite sign convention to the usual PDE one!)

- We are given a real-valued smooth potential function $V(x,t)$, which is compactly supported *in spacetime*.

- Let $p \geq 3$ be an odd integer.

We then consider the NLS in $\mathbb{R}^{n+1}$

$$Pu(x,t) := \big(D_t + \Delta + V(x,t)\big)u(x,t) = N[u],$$

where c is a real constant and $N[u] := -c|u|^{p-1}u$ is nonlinear for $c \neq 0$. The NLS is called defocusing when $c > 0$ and focusing when $c < 0$.

What questions should we ask about NLS?

We can get a feel for the equation by setting $c = 0$ and $V = 0$, in which case we have just

$$(D_t + \Delta)u = 0.$$

We can solve this equation by Fourier transformation: the solution is given by a superposition of pure frequencies

$$u(x,t) = \int e^{ix\cdot\xi} e^{-it|\xi|^2} f(\xi) d\xi,$$

where f is "arbitrary". Suppose we take f to be a Schwartz function, what is the structure of u? Stationary phase gives the expansion for large $|t|$:

$$u(x,t) \sim (4\pi i t)^{-n/2} e^{i|x|^2/4t} f\Big(\frac{x}{2t}\Big), \quad t \to \pm\infty.$$

More generally, for the linear equation,

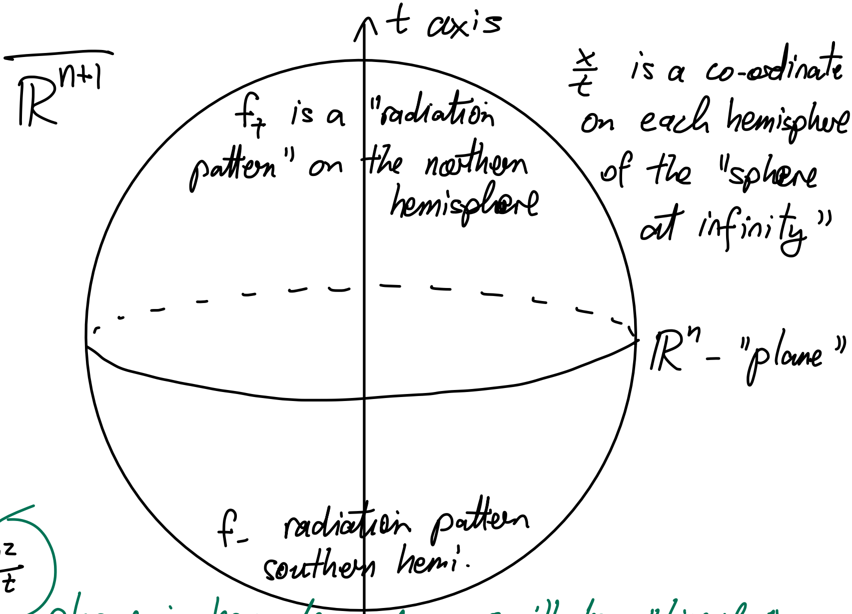
$$\big(D_t + \Delta + V(x,t)\big)u(x,t) = 0,$$

when $V \neq 0$ or there is a metric perturbation, (sufficiently nice) solutions have asymptotic expansions

$$\begin{aligned} u(x,t) &\sim (4\pi i t)^{-n/2} e^{i|x|^2/4t} f_-\big(\frac{x}{2t}\big), \quad t \to -\infty, \\ &\sim (4\pi i t)^{-n/2} e^{i|x|^2/4t} f_+\big(\frac{x}{2t}\big), \quad t \to +\infty \end{aligned}$$

where $f_\pm$ are different (they are related by the scattering operator). They are called the incoming $(-)$ and outgoing $(+)$ data.

This type of behaviour is characteristic of linear solutions and is often called radiation.



$t^{-\frac{n}{2}}$

$e^{i\frac{|x|^2}{4t}}$

phase is hom. degree 1 – oscillates "linearly".

Frequency is $\xi = \frac{x}{2t}$.

On the other hand, for the nonlinear equation with $V = 0$, and $c < 0$ (focusing case)

$$Pu(x,t) := \big(D_t + \Delta\big)u(x,t) + c|u|^{p-1}u = 0,$$

there are soliton solutions that look quite different: they are a wave that does not disperse as $t \to \pm\infty$. The simplest case is a t-harmonic solution

$$u(x,t) = Q(x)e^{iEt}, \quad (\Delta + E)Q + c|Q|^{p-1}Q = 0.$$

Solitons are a specifically nonlinear phenomenon: the focusing term acts like a gravitational attraction ($c < 0$) that can stop the wavefunction from dispersing.

Some questions (homework)

Question 1: (When) do solutions of NLS look like radiation as $t \to \pm\infty$?

Question 2: What sort of soliton behaviour can arise? Can one combine a soliton with radiation, or multiple solitons?

Question 3: How does one relate behaviour as $t \to -\infty$ with behaviour as $t \to +\infty$?

Question 4: are there completely different behaviours, unrelated to either radiation or solitons?

Preliminary results (with Gell-Redman and Gomes)

In this theorem, $\mathcal{W}^k(\mathbb{R}^n)$ is a Hilbert space where the index k simultaneously measures smoothness and decay at infinity. In fact, this space consists of functions $f(\xi)$ such that

$$A_1 \dots A_k f \in L^2(\mathbb{R}^n),$$

where the operators $A_1, \dots, A_k$ are chosen from the list

$$\xi_j, D_{\xi_j}, \xi_j D_{\xi_i} - \xi_i D_{\xi_j}. \tag{1}$$

Theorem (Linear Theorem)

Assume that $f \in \mathcal{W}^k(\mathbb{R}^n)$, $k \geq 2$. *Then there is a unique solution to* $Pu = 0$ *with prescribed (radiative) asymptotic*

$$u(x,t) \sim t^{-n/2} e^{i|x|^2/4t} f(x/t) + o(t^{-n/2}), \quad t \to -\infty. \tag{2}$$

This solution also has a radiative asymptotic as $t \to +\infty$, *and its outgoing data* f_+ *also lies in* $\mathcal{W}^k(\mathbb{R}^n)$.

Theorem (Nonlinear Theorem — small data result)

Assume that the odd integer p and the space dimension n are such that

$$(p-1)\frac{n-1}{2} > 2.$$

Let k be an integer larger than $(n+3)/2$, and let $f \in \mathcal{W}^k(\mathbb{R}^n)$ have sufficiently small norm. Then the final-state problem for NLS, with nonlinearity $N[u] = \pm|u|^{p-1}u$, and with incoming data f, has a unique solution u that is small in certain Hilbert spaces. It has a radiative expansion as $t \to +\infty$, with outgoing data also in $\mathcal{W}^k$.

Conventional approach

The usual approach to solving the nonlinear Schrödinger equation (NLS) is to view it as an evolution equation in time, and specify an initial condition, i.e. $u(x, 0)$ is a given function $u_0(x)$. Long-time behaviour and scattering are then analyzed using the following sequence of steps:

- First establish local well-posedness: a solution $u(\cdot, t)$ exists for a small time interval.
- Next, show global well-posedness: the solution exists for all time.
- Finally, show (for example) scattering: as $t \to \infty$ the solution converges to a linear solution.

New approach

In the new approach that I am establishing with Jesse Gell-Redman and Sean Gomes (building on work with Jacob Shapiro and Junyong Zhang, and based on prior work by Vasy, Hintz-Vasy and Baskin-Vasy-Wunsch), we view the operator P as an operator on functions on $\mathbb{R}^{n+1}$. We find natural Hilbert spaces $\mathcal{X}_\pm$, $\mathcal{Y}_\pm$ of functions defined on $\mathbb{R}^{n+1}$ such that

$$P : \mathcal{X}_\pm \to \mathcal{Y}_\pm \text{ is invertible.}$$

There are therefore inverse maps

$$P_\pm^{-1} : \mathcal{Y}_\pm \to \mathcal{X}_\pm.$$

The sign $\pm$ indicates whether it is the advanced or retarded inverse. In the time evolution picture, we can solve either forward or backward in time.

These spaces are based on (parabolic) weighted Sobolev spaces $H^{s,r}(\mathbb{R}^{n+1})$ where s is a regularity (smoothness) index and r measures spacetime decay. The subtlety is that r needs to be have different values in different parts of phase space; in particular it is a function on phase space (and what does this even mean?) We choose two different weights $r_\pm$ so that, for $\mathcal{X}_+$, the weight r_+ allows expansions such as

$$u(x,t) \sim t^{-n/2} e^{i|x|^2/4t} f(x/t) + o(t^{-n/2}), \quad t \to \pm\infty$$

as $t \to +\infty$ but not as $t \to -\infty$. Similarly, $\mathcal{X}_-$ has a weight r_- that allows such expansions as $t \to -\infty$ but not as $t \to +\infty$. Then $\mathcal{Y}_+ = H^{s-1,r_+ +1}$ and

$$\mathcal{X}_+ = \{u \in H^{s,r_+} \mid Pu \in H^{s-1,r_+ +1}\}.$$

- Analogy with functions associated to the Laplacian on the unit disc.

Fredholm method

The invertibility is obtained by first proving a Fredholm estimate for $u \in \mathcal{X}_+$

$$\|u\|_{s,r_+} \leq C\Big(\|Pu\|_{s-1,r_++1} + \|u\|_{-M,-N}\Big),$$

with a similar estimate for $\mathcal{X}_-$. Then an abstract argument shows that P is a Fredholm operator from $\mathcal{X}_\pm$ to $\mathcal{Y}_\pm$. An additional argument is needed to show invertibility.

In more elaborate settings, invertibility may not hold and a finite dimensional kernel or cokernel may hold key geometric or analytic information.

Final state problem — using new method

Find the solution to the linear problem $Pu = 0$ with asymptotic as $t \to -\infty$ with given incoming data f.

- We form u_0, the solution to the free problem $(D_t + \Delta)u_0 = 0$ with incoming data f. The function u_0 lies in $\mathcal{X}_+ + \mathcal{X}_-$ (it has radiative expansions both as $t \to -\infty$ and $t \to +\infty$).
- We microlocally divide $u_0 = u_- + u_+$ where $u_\pm \in \mathcal{X}_\pm$, and such that u_- has the prescribed asymptotics.
- We have $Pu_- \in \mathcal{Y}_+$. In fact, $Pu_+ \in \mathcal{Y}_+$, so $Pu_- = P(u_0 - u_+) = (P_0 + V)u_0 - Pu_+ = Vu_0 - Pu_+$, and both of these terms are in $\mathcal{Y}_+$.
- We let $u = u_- - P_+^{-1} Pu_-$. This has the same asymptotics as $t \to -\infty$ as u_- since the second term is in $\mathcal{X}_+$.

Looking forward — Soliton stability problem

We can investigate the stability of a stationary soliton, i.e. a solution of the form

$$u(x,t) = Q(x)e^{iEt}, \quad (\Delta + E)Q + c|Q|^{p-1}Q = 0.$$

- In this case, the soliton will act as a potential function, but the potential is time-independent. We need different function spaces in this case, as there are now two distinguished points at 'spacetime infinity' where the soliton is concentrated (one at $t = -\infty$, one at $t = +\infty$). It is similar to a 3 body problem geometry, which my student Yilin Ma is currently working on.
- In this case, I expect the linearized problem to be Fredholm but not invertible.

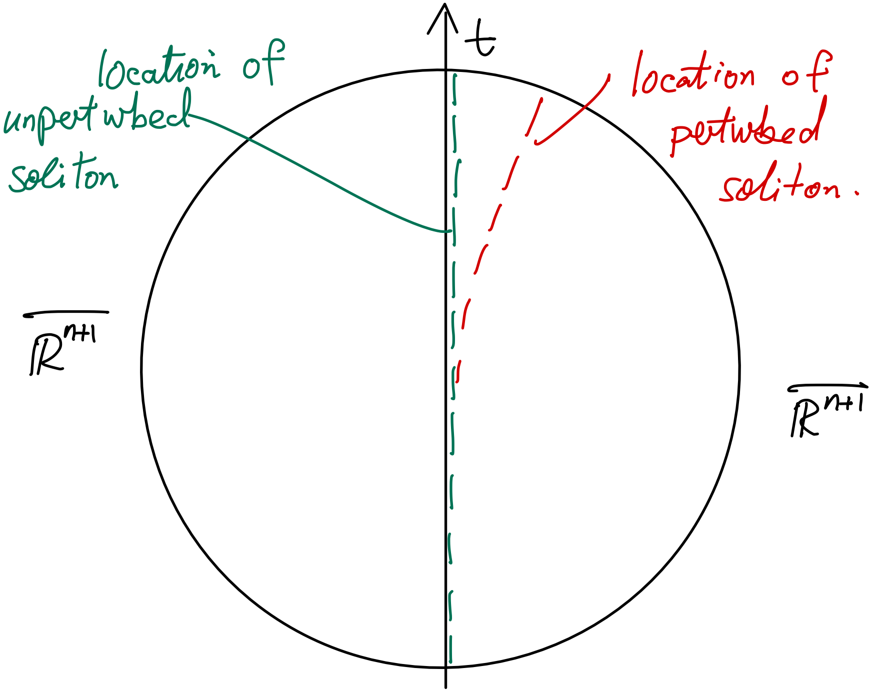
Why?

In this case, there are two phenomena that would affect invertibility of the operator from the "initial guess" for a space $\mathcal{X}_+$ to $\mathcal{Y}_+$.

- First, the interaction of the soliton with an infinitesimal perturbation would be expected to impart a velocity boost to the soliton. This corresponds to a finite-dimensional symmetry group of the equation

$$(D_t + \Delta)u - |u|^{p-1}u = 0.$$

Then the map $P : \mathcal{X}_+ \to \mathcal{Y}_+$ would have a cokernel corresponding to the dimension of this symmetry group.



t
location of unperturbed soliton
location of perturbed soliton.
$\mathbb{R}^{n+1}$
$\mathbb{R}^{n+1}$

- Second, Schlag has shown that when regarded as an evolution equation, the spatial linearization may have a finite number of unstable modes. This may require us to modify the function spaces with exponential weights in time. The choice of such weights will affect the dimension of the kernel and cokernel of P (Melrose's relative index theorem).

In either case, when a Fredholm operator has been realized, a finite-dimensional modification of the spaces may lead to invertibility. This then can be used as a basis for determining the behaviour of NLS in a neighbourhood of the soliton.

Multisolitons

It is interesting to pose a final state problem where the final state consists of two or more separated solitons together with radiation. Can we find a unique global solution with this asymptotic as $t \to -\infty$, and if so, what happens as $t \to +\infty$?

- The case $n = 1$ (which models pulses moving in an optical fibre) with $p = 3$ is integrable. In this case, special, algebraic methods can be used to prove the existence of multisoliton solutions that interact and then reform as several separated solitons.

- The problem is much more difficult in higher dimensions.

- It is the sort of problem that I hope to shed light on using the new framework, which I think is better designed to study this sort of problem compared to the conventional approach.