First properties of supermanifolds, their functor of points and the DeWitt topology

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Abstract In this article we show how the classification of supermanifolds to first order can be recovered through their functor of points. Subsequently, we consider the notion of L-points of a supermanifold. In analogy with the complex topology on schemes, we describe a topology on the locus of L-points of a superscheme in line with the DeWitt approach to supermanifold theory.

1 Introduction

Mathematically, supermanifolds form a class of spaces representing supercommutative rings. They originally arose as a foundational framework for new ideas in particle physics and were developed along similar lines to contemporary algebraic geometry. In algebraic geometry, nilpotent elements in a commutative ring are essential in forming infinitesimal deformations. In supergeometry, nilpotent elements are part of the foundation. They represent coordinates on 'superspace' in their own right rather than deformations of coordinate rings.

There are a number of approaches to the construction of supermanifolds, many of which are detailed in [1]. In this article we study supermanifolds from two approaches: firstly as local, supercommutative ringed spaces (LSRS) developed by Berezin [2], Kostant [3] and Leites [4]; secondly, in line with the approach by DeWitt in [5].[1] Among our main results in this article is Theorem 5.14 relating the two approaches in the category of superschemes.

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[1] Supermanifolds in this approach could also be attributed to Rogers, e.g., [6], and so might be referred to as 'Rogers-DeWitt' supermanifolds. We will refer to them as 'DeWitt supermanifolds' however since the approach by Rogers is substantially more general, see e.g., [7].

1.1 Article summary and outline

We begin in §2 with an introduction to supermanifolds in the LSRS approach and present their classification to first order. Subsequently we give a general characterization of submanifolds in supermanifolds. In §3 we illustrate this classification for affine superspace, projective superspace and super Riemann surfaces.

A deep viewpoint propounded by Grothendieck [22] is to consider the functor of points represented by a space to be the more natural object to study, rather than the space itself. A number of authors have adapted this viewpoint to supermanifold theory, yielding significant results. For instance, Vaintrob in [28] used this viewpoint to study supersymplectic structures; Stolz and Teichner et. al. in [29, 30] relied heavily on this viewpoint in deriving links between supergeometry and topological invariants. For a more introductory account of the functor-of-points viewpoint to supermanifold theory, see Manin in [26], [27, Ch. 4] and Deligne and Morgan in [8].

In this article we will also consider the functor-of-points viewpoint to supermanifolds. In §4 we show how to recover this aforementioned classification of supermanifolds to first order through the functorial viewpoint. This is the content in Theorem 4.5 and Theorem 4.6. The example of a quadratic form in dimension $(1|1)$ is then given to illustrate the distinction between L-points for $L = 0, 1, 2$.

One of our motivations to study the functor of points of a supermanifold is to arrive at a framework to unify other approaches in supermanifold theory in [1]. We propose that such a unification amounts to understanding these approaches as Grothendieck topologies on the category of superschemes. This in line with a similar program outlined by Molotkov in [20] and furthered by Sachse [21]. In Definition 4.4 we establish the notion of 'L-points of a supermanifold'. In §5 we present a brief introduction to the DeWitt approach to supermanifold theory, following [5]. Subsequently, we revisit the locus of L-points of a superscheme. Our main result is Theorem 5.14 where it is shown how this locus can be given the structure of a DeWitt supermanifold. This leads to Question 5.15 and 5.16 on understanding supermanifold topologies as Grothendieck topologies.

1.2 Further remarks

Through a unification proposed as above we can hope to arrive at a deeper understanding into superspace constructions and relations among them. As an illustration of a typical problem, consider that of relating constructions in super Teichmüller theory with supermoduli theory. Penner and Zeitlin in [23] have shown, through DeWitt's approach, that super Teichmüller space is supersymplectic; Sachse in [21] give models of super Teichmüller space from both the DeWitt and LSRS approaches; Rothstein in [24] gives a classification of supersymplectic structures on supermanifolds as LSRS; Donagi and Witten in [25] have shown, using the LSRS approach, that the supermoduli space of curves admits a natural gauge pairing between certain cohomology classes and deformation parameters. In light of these results, some natural questions are:

- does Penner and Zeitlin's supersymplectic structure translate to a supersymplectic structure on super Teichmüller space as LSRS?
- can we apply Rothstein's classification to study Penner and Zeitlin's supersymplectic structure?
- can Penner and Zeitlin's supersymplectic form be expressed in terms of Donagi and Witten's deformation gauge pairing?

Progress on the above questions will invariably involve a unification of different approaches to supermanifold theory. As such, it would lead to a wealth of insight into the nature of supermanifolds more broadly.

Note: throughout this article, by 'rational points' it is meant $\mathbb{K}$-rational points for a field $\mathbb{K}$.

2 Supermanifolds

Supermanifolds are, in analogy with complex analytic spaces, a geometric object whose functions under multiplication form a *supercommutative ring*.

2.1 Supercommutative rings

Fix a field $\mathbb{K}$. A $\mathbb{K}$-algebra A is said to be a *superalgebra* if it is equipped with a morphism $A \stackrel{p}{\to} \mathbb{Z}_2$. The *even subalgebra* is then the preimage $A_{\bar{0}} := p^{-1}\bar{0}$ and the *fermionic part* is the preimage $A_{\bar{1}} := p^{-1}\bar{1}$. See that $A_{\bar{0}} \subset A$ is indeed subalgebra while $A_{\bar{1}}$ is an $A_{\bar{0}}$-module. This gives a $\mathbb{Z}_2$-decomposition of A over $A_{\bar{0}}$, $(A, p) \cong A_{\bar{0}} \oplus A_{\bar{1}}$.

The $\mathbb{K}$-superalgebra (A, p) is said to be a *supercommutative ring* if, for homogeneous $a, b \in A$, that

$$a \cdot b = (-1)^{p(a)p(b)} b \cdot a. \tag{1}$$

From (1) we can immediately deduce the existence of nilpotent elements: if $a \in A_{\bar{1}}$ then $a \cdot a = -a \cdot a$ giving $a^2 = 0$.

Remark 2.1 *While a supercommutative ring A is not* strictly *commutative, it is weakly commutative in the sense that* (1) *gives rise to an equivalence between the categories of left- and right A-modules.* [2] □

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[2] c.f., the commutativity isomorphism in [8, p. 62].

2.1.1 First properties

Recall that any superalgebra (A, p) admits a *canonical* $\mathbb{Z}_2$-decomposition $(A, p) \cong A_{\bar{0}} \oplus A_{\bar{1}}$ with $A_{\bar{1}}$ the 'fermionic part'. It lends itself to the construction of the left- and right- *fermionic ideals*: $J_{left} := A_{\bar{1}} \cdot A$ and $J_{right} := A \cdot A_{\bar{1}}$. By Remark 2.1, if A is supercommutative then $J_{left} \cong J_{right}$ and we will just refer to $J_{left} = J_{right} = J \subset A$ as the fermionic ideal. This leads now to two characterizing structures derived from A:

I) the *reduced ring* $R := A/J$;
II) the *fermionic module* $F := J/J^2$.

Note that R here is commutative and F is an R-module. Despite the prefix 'reduced', R may be non-reduced as a commutative ring.

Example 2.2 Over a field $\mathbb{K}$ let V be a $\mathbb{K}$-vector space and $\wedge^\bullet_{\mathbb{K}} V$ its exterior algebra. Then $\wedge^\bullet_{\mathbb{K}} V$ is a supercommutative ring with: I) $\mathbb{K}$ as its reduced ring; II) V as its fermionic module. □

The prototypical example above of a supercommutative ring motivates the following generalization of affineness from commutative algebra.

Definition 2.3 *An* affine supercommutative ring over $\mathbb{K}$ *is a supercommutative ring over* $\mathbb{K}$ *for which:*

i) its reduced ring R is affine;
ii) it is isomorphic to $\wedge^\bullet_R V/I$ for V a free R-module and $I \subset \wedge^\bullet_R V$ an ideal. □

Accordingly, the exterior algebra from Example 2.2 is an example of an affine, supercommutative ring.

2.2 Local supercommutative ringed spaces

2.2.1 Local supercommutative rings

Recall that a ring is said to be *local* if it contains a unique, maximal ideal $\mathfrak{m} \subset R$. Similarly, a supercommutative algebra A is local if it contains a unique, maximal ideal $\mathfrak{m} \subset A$.

Lemma 2.4 Let A be a supercommutative ring and $J \subset A$ the fermionic ideal. Then A is local if and only if A/J is local. □

Proof Let A be a local, supercommutative ring and $J \subset A$ the fermionic ideal. Recall that the reduced ring $R = A/J$ is commutative. Let $A \stackrel{p}{\to} A/J = R$ be the quotient morphism. Suppose $\mathfrak{m} \subset A$ is maximal and let $I \subset R$ be an ideal. Since p is an epimorphism (onto), $p(p^{-1}I) = I$. Moreover, p preserves inclusions giving $I = p(p^{-1}I) \subset p(\mathfrak{m})$. As this holds for any ideal $I \subset R$, it follows that $p(\mathfrak{m}) \subset R$ is maximal.

Conversely, let $m \subset R$ be a maximal ideal. Set $\mathfrak{m} = p^{-1}m$ and let $J' \subset A$ be an ideal. Since p is a morphism, $p(J') \subset R$ is an ideal and so contained in m. Thus $J' \subset p^{-1}(p(J')) \subset p^{-1}m = \mathfrak{m}$, i.e., that $J' \subset \mathfrak{m}$, so $\mathfrak{m} \subset A$ must be maximal. □

Example 2.5 Continuing on from Example 2.2, the maximal ideal in a field $\mathbb{K}$ is the zero ideal $\{0\}$. By Lemma 2.4, the maximal ideal in the supercommutative ring $\wedge^\bullet_{\mathbb{K}}V$ is then the preimage of $\{0\}$ under the quotient $\wedge^\bullet_{\mathbb{K}}V \xrightarrow{p} \mathbb{K}$. Equivalently, it is the kernel $J = \ker\{\wedge^\bullet_{\mathbb{K}}V \xrightarrow{p} \mathbb{K}\}$. Since $\mathbb{K}$ is the reduced ring of $\wedge^\bullet_{\mathbb{K}}V$, the maximal ideal $J \subset \wedge^\bullet_{\mathbb{K}}V$ is precisely the fermionic ideal. □

The above example leads to a more general result pertaining to maximal ideals.

Lemma 2.6 Let A be a supercommutative ring and suppose its reduced ring is also reduced as a commutative ring. Then the fermionic ideal $J \subset A$ is maximal.

2.2.2 Manifolds

Let X be a topological space. A *presheaf* of rings O on X assigns to each open set $U \subset X$ a ring $O(U)$. Elements of O are referred to as functions. To inclusions of open sets $U \subset V$ the presheaf O comes with a ring morphism, referred to as restriction $res_{V,U} : O(V) \to O(U)$. If $U = V$, $res_{U,V}$ is the identity. The presheaf O is a *sheaf* if it satisfies the *locality* and *glueing* axioms. Locality requires, for functions $s, t \in O(V)$ and $U \subset V$, if $res_{U,V}\, s = res_{U,V}\, t$, then $s = t$. Glueing requires, for functions $s_U \in O(U), s_V \in O(V)$, if $res_{U,U\cap V}\, s_U = res_{V,V\cap U}\, s_V$, then there exists a function $s \in O(U \cup V)$ such that $s_U = res_{U\cup V,U}\, s$ and $s_V = res_{U\cup V,V}\, s$. The *stalk* of the sheaf O on X at a point $P \in X$ is the pullback of O along the inclusion of the point $\iota_P : \{P\} \hookrightarrow X$ and denoted O_P. The pullback morphism of sheaves on X, $O \to \iota_{P*}O_P$ given by $f \mapsto f(P)$ is referred to as *evaluation at P*. Note that for different points $P, P' \in X$ the evaluation is valued in the rings O_P resp., $O_{P'}$. As such, it doesn't make sense to generally ask whether functions can separate points. This question does make sense if the rings O_P are local for each $P \in X$ however, since then one can instead ask whether functions vanish modulo the unique, maximal ideal.

Definition 2.7 *A pair (X, O_X), for X a topological space and O_X a sheaf of rings is said to be a* locally ringed space *if, for each point $p \in X$, the stalk $O_{X,p}$ is a local ring. In this case O_X is referred to as the* structure sheaf. □

Example 2.8 The pair $(\{pt\}, \mathbb{K})$ for $\{pt\}$ the one-point, topological space and $\mathbb{K}$ a field defines a locally ringed space. More generally, for any commutative ring R, its spectrum Spec $R = (R^{primes}, \widetilde{R})$, for R^{primes} the set of prime ideals in R equipped with the Zariski topology and $\widetilde{R}$ the sheafification of the presheaf assigning R to the open sets in R^{primes} defines a locally ringed space. See e.g., [9, Ch. II] or [10, §3] for a detailed introduction bypassing the language of sheaves. □

Example 2.9 Let $X = \mathbb{R}^m$. A smooth structure on X is a identified with the sheaf of smooth, $\mathbb{R}$-valued functions C^∞. To each open set $U \subset X$, $C^\infty(U) = C^\infty(U, \mathbb{R})$ is the ring of smooth functions $U \xrightarrow{f} \mathbb{R}$. At each point $P \in X$ the stalk C^∞_P is a local ring with maximal ideal $\mathfrak{m}_P \subset C^\infty_P$ comprising those (germs of) functions vanishing at P, i.e., $\mathfrak{m}_P = \{f \in C^\infty \mid f(P) = 0\}$. Smooth, m-dimensional Euclidean space is then the locally ringed space $(\mathbb{R}^m, C^\infty)$. □

Example 2.10 Replacing $\mathbb{R}^m$ with $\mathbb{C}^m$ and the sheaf C^∞ in Example 2.9 with the sheaf of complex-valued, analytic functions O defines the m-dimensional, analytic Euclidean space $(\mathbb{C}^m, O)$. □

Definition 2.11 *A morphism of locally ringed spaces $\phi : (X, O_X) \to (Y, O_Y)$ consists of a pair $\phi = (f, h^\sharp)$ where $f : X \to Y$ is a continuous map of topological spaces and $h^\sharp : f^* O_Y \to O_X$ is a morphism of sheaves of local rings on X.* □

Remark 2.12 *Adjunction states* $\mathrm{Hom}_{\mathsf{Sh}(X)}(f^* O_Y, O_X) \cong \mathrm{Hom}_{\mathsf{Sh}(Y)}(O_Y, f_* O_X)$. *As such, W.L.O.G., $h^\sharp$ in Definition 2.11 can be understood equivalently as a morphism of sheaves of local rings $O_Y \to f_* O_X$ on Y.* □

With Definition 2.7 and 2.11, locally ringed spaces form a category. It makes sense then to ask whether two such spaces are isomorphic. Accordingly, through Example 2.9 and 2.10 we can define smooth and analytic manifolds, circumventing the more familiar definition involving charts and atlases.[3]

Definition 2.13 *A smooth resp., complex analytic, m-dimensional manifold is a locally ringed space (X, O_X), locally isomorphic to $(\mathbb{R}^m, C^\infty)$ resp., $(\mathbb{C}^m, O)$.* □

2.2.3 Supermanifolds

Definition 2.13 captures the familiar intuition that the real and analytic Euclidean spaces in Example 2.9 and 2.10 are model spaces on which manifolds are built. The notion of a supermanifold can be arrived at in parallel.

Definition 2.14 *A* supercommutative space $\mathfrak{X}$ *is a pair $\mathfrak{X} = (X, O_{\mathfrak{X}})$ where X is a topological space and $O_{\mathfrak{X}}$ is a sheaf of local, supercommutative rings.* □

Following Definition 2.11, a *morphism* of supermanifolds $\mathfrak{X} = (X, O_{\mathfrak{X}})$ and $\mathfrak{Y} = (Y, O_{\mathfrak{Y}})$ is defined by a pair $(f, h^\sharp) : \mathfrak{X} \to \mathfrak{Y}$ where $f : X \to Y$ is continuous and $h^\sharp : f^* O_{\mathfrak{Y}} \to O_{\mathfrak{X}}$ is a morphism of sheaves of local, supercommutative rings on X (or equivalently on Y by Remark 2.12). The analogue of smooth and analytic Euclidean space in the present context is obtained by reference to the prototypical, supercommutative ring from Example 2.2.

Example 2.15 Fix an n-dimensional $\mathbb{R}$- or $\mathbb{C}$-vector space V and take as the structure sheaf $\wedge^\bullet_{C^\infty} V$ resp., $\wedge^\bullet_O V$. Note that these are indeed local, supercommutative rings by Lemma 2.6. The pair $(\mathbb{R}^m, \wedge^\bullet_{C^\infty} V)$ resp., $(\mathbb{C}^m, \wedge^\bullet_O V)$ defines $m|n$-dimensional real, resp., analytic Euclidean superspace. They are denoted $\mathbb{R}^{m|n}$ and $\mathbb{C}^{m|n}$ respectively.□

[3] see e.g., [10, §2.2].

Remark 2.16 *Throughout this article we will denote by* $\mathbb{K}$ *a field, which we intend to mean either* $\mathbb{R}$ *or* $\mathbb{C}$. *Then* $\mathbb{K}^{m|n}$ *is taken to mean either* $\mathbb{R}^{m|n}$ *or* $\mathbb{C}^{m|n}$ *from Example 2.15.* □

Following Definition 2.13 we arrive at the notion of a supermanifold.[4]

Definition 2.17 *An* $m|n$*-dimensional real resp., complex supermanifold* $\mathfrak{X}$ *is a supercommutative space* $(X, \mathcal{O}_{\mathfrak{X}})$ *which is locally isomorphic to the Euclidean superspace* $\mathbb{R}^{m|n}$ *resp.,* $\mathbb{C}^{m|n}$. □

2.3 First properties of supermanifolds

We denote by $\mathfrak{X}$ a supermanifold and $\mathcal{O}_{\mathfrak{X}}$ its structure sheaf. As $\mathcal{O}_{\mathfrak{X}}$ is a sheaf of supercommutative rings it will imply $\mathfrak{X}$ can be classified to first order through analogous properties I) and II) for supercommutative rings in §2.1.1. Denote by $|\mathfrak{X}|$ the underlying topological space so that, as a supercommutative space, $\mathfrak{X} = (|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}})$. Over each open set $U \subset |\mathfrak{X}|$, $\mathcal{O}_{\mathfrak{X}}(U)$ is a supercommutative ring and so decomposes into a (commutative) subring and a module over it, $\mathcal{O}_{\mathfrak{X}}(U) \cong A_{\bar{0}} \oplus A_{\bar{1}}$. Projecting onto $A_{\bar{0}}$ for each U defines then a presheaf of commutative rings $U \overset{\mathcal{O}_{\mathfrak{X},\bar{0}}}{\mapsto} A_{\bar{0}}$ and a presheaf of $\mathcal{O}_{\mathfrak{X},\bar{0}}$-modules $U \overset{\mathcal{O}_{\mathfrak{X},\bar{1}}}{\mapsto} A_{\bar{1}}$. Since $\mathcal{O}_{\mathfrak{X}}$ is a sheaf we have:

Lemma 2.18 The presheaves $U \overset{\mathcal{O}_{\mathfrak{X},\bar{0}}}{\mapsto} A_{\bar{0}}$ and $U \overset{\mathcal{O}_{\mathfrak{X},\bar{1}}}{\mapsto} A_{\bar{1}}$ define a sheaf of commutative rings $\mathcal{O}_{\mathfrak{X},\bar{0}}$ and a sheaf of $\mathcal{O}_{\mathfrak{X},\bar{0}}$-modules $\mathcal{O}_{\mathfrak{X},\bar{1}}$ on $|\mathfrak{X}|$ respectively.

As a consequence of Lemma 2.18 we obtain an isomorphism of *sheaves on* $|\mathfrak{X}|$, $\mathcal{O}_{\mathfrak{X}} \overset{\cong}{\to} \mathcal{O}_{\mathfrak{X},\bar{0}} \oplus \mathcal{O}_{\mathfrak{X},\bar{1}}$. The sheaf of fermionic ideals is $\mathcal{J}_{\mathfrak{X}} := \mathcal{O}_{\mathfrak{X}} \cdot \mathcal{O}_{\mathfrak{X},\bar{1}}$ and, modulo $\mathcal{J}_{\mathfrak{X}}$ we obtain a commutative sheaf of rings $C = \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_{\mathfrak{X}}$. By Lemma 2.4, C will be a sheaf of local, commutative rings on $|\mathfrak{X}|$. Hence by Definition 2.13, the pair $M = (|\mathfrak{X}|, C)$ will define a differentiable manifold. This is the analogue of property I) from §2.1.1 for supermanifolds.

Definition 2.19 *The differentiable manifold* $M = (|\mathfrak{X}|, C)$ *associated to any supermanifold* $\mathfrak{X}$ *is referred to as the* body *of* $\mathfrak{X}$. □

For each $U \subset |\mathfrak{X}|$ we have the fermonic ideal $\mathcal{J}_{\mathfrak{X}}(U) \subset \mathcal{O}_{\mathfrak{X}}(U)$ giving, over $C(U)$, the fermionic modules $\mathcal{J}_{\mathfrak{X}}(U)/\mathcal{J}_{\mathfrak{X}}(U)^2$. For $\mathfrak{X}$ a supermanifold, it is locally isomorphic to $\mathbb{K}^{m|n}$. Therefore, there exists a covering $(U_i)_i$ such that $\mathcal{J}_{\mathfrak{X}}(U_i)/\mathcal{J}_{\mathfrak{X}}(U_i)^2$ is a free $C(U_i)$-module for all i. Hence $\mathcal{J}/\mathcal{J}^2_{\mathfrak{X}}$ is a locally free sheaf of C-modules on $|\mathfrak{X}|$ and so will be the sheaf of sections of a vector bundle $\mathbf{V}_{\mathfrak{X}} \to M$ over the body of $\mathfrak{X}$. This is the analogue of property II) in §2.1.1 for supermanifolds.

Definition 2.20 *The vector bundle* $\mathbf{V}_{\mathfrak{X}} \to M$ *is referred to as the* soul *of the supermanifold* $\mathfrak{X}$. □

[4] This viewpoint was developed by the authors [2, 4, 3] and is sometimes referred to as the 'BLK-approach' to supermanifold theory.

The body and soul of $\mathfrak{X}$ admit the following 'geometric relation' to $\mathfrak{X}$.

Proposition 2.21 The body of a supermanifold $\mathfrak{X}$ is naturally embedded in $\mathfrak{X}$; the soul of $\mathfrak{X}$ is the conormal bundle to the embedding of its body. □

Proof Recall that the body of $\mathfrak{X}$ is the locally ringed space $M = (|\mathfrak{X}|, C)$ where $C = \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_{\mathfrak{X}}$. An embedding is a morphism of supermanifolds $(f, h^\sharp)$ where f is an embedding of the underlying, topological space and $h^\sharp$ is an epimorphism on structure sheaves. Denote by $h^\sharp$ the quotient $\mathcal{O}_{\mathfrak{X}} \stackrel{h^\sharp}{\to} \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_{\mathfrak{X}} = C$. Then we obtain a morphism $\iota = (\mathbf{1}, h^\sharp) : M \to \mathfrak{X}$, where $\mathbf{1} : |\mathfrak{X}| \to |\mathfrak{X}|$ is the identity map. The morphism ι is an embedding of supermanifolds.[5] The ideal sheaf of the body embedding $\iota : M \hookrightarrow \mathfrak{X}$ is $\ker h^\sharp = \mathcal{J}_{\mathfrak{X}}$. The conormal bundle to ι is a vector bundle over the body M whose sheaf of sections is given by $\mathcal{J}_{\mathfrak{X}}/\mathcal{J}^2_{\mathfrak{X}}$. This is precisely the sheaf of fermonic modules in $\mathfrak{X}$. Hence the soul of $\mathfrak{X}$ *is* the conormal bundle to the body embedding ι. □

Lemma 2.22 Let $\mathfrak{X}$ be a supermanifold and $\mathcal{J}_{\mathfrak{X}} \subset \mathcal{O}_{\mathfrak{X}}$ the fermionic ideal. Then the soul $\mathbf{V}_{\mathfrak{X}}$ of $\mathfrak{X}$ is given by $\mathbf{V}_{\mathfrak{X}} = \mathrm{Spec}\, \Gamma(M, \mathrm{Sym}^\bullet(\mathcal{J}_{\mathfrak{X}}/\mathcal{J}^2_{\mathfrak{X}}))$. □

Proof This follows from the characterization of vector bundles as locally ringed spaces from algebraic geometry (see e.g., [11, Tag 01M1]) along with the identification of the sheaf of sections of $\mathbf{V}_{\mathfrak{X}}$ in the proof of Proposition 2.21. □

2.4 First properties of supermanifold embeddings

The following result characterizes the body and soul of submanifolds[6] of a supermanifold $\mathfrak{X}$ in terms of the body and soul of $\mathfrak{X}$.[7]

Theorem 2.23 Let $\mathfrak{X}$ be a supermanifold, $\mathfrak{Y} \subset \mathfrak{X}$ a submanifold and suppose that $\mathcal{I}_{\mathfrak{Y}} \cap \mathcal{J}_{\mathfrak{X}} = (0)$, for $\mathcal{I}_{\mathfrak{Y}}$ the ideal sheaf of $\mathfrak{Y}$ and $\mathcal{J}_{\mathfrak{X}} \subset \mathcal{O}_{\mathfrak{X}}$ the fermionic ideal. Then:

(i) the body N of $\mathfrak{Y}$ is a submanifold of the body M of $\mathfrak{X}$;
(ii) the ideal sheaf of $N \subset M$ is given by the image of $\mathcal{I}_{\mathfrak{Y}}$ in $\mathcal{O}_M$;
(iii) $N \cong \mathfrak{Y} \cap M$;
(iv) $\mathbf{V}_{\mathfrak{Y}} \cong \mathbf{V}_{\mathfrak{X}}|_{|\mathfrak{Y}|}$

where $\mathbf{V}_{\mathfrak{Y}}$ resp., $\mathbf{V}_{\mathfrak{X}}$ denotes the soul of $\mathfrak{Y}$ resp., $\mathfrak{X}$. □

Proof *(i)* Let $(i, h^\sharp) : \mathfrak{Y} \subset \mathfrak{X}$ denote the embedding of supermanifolds where $i : |\mathfrak{Y}| \subset |\mathfrak{X}|$ the embedding of topological spaces and $h^\sharp : \mathcal{O}_{\mathfrak{X}} \to i_*\mathcal{O}_{\mathfrak{Y}}$ an epimorphism. The body of $\mathfrak{Y}$ resp., $\mathfrak{X}$ is the locally ringed space $N = (|\mathfrak{Y}|, \mathcal{O}_{\mathfrak{Y}}/\mathcal{J}_{\mathfrak{Y}})$

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[5] Scheme-theoretically, ι is a *closed immersion*.

[6] By a submanifold of a supermanifold, it is meant a subspace which is itself a supermanifold.

[7] Theorem 2.23 is adapted from work in [12]. We refer here for more details on the nature of supermanifold embeddings.

resp., $M = (|\mathfrak{X}|, \mathcal{O}_\mathfrak{X}/\mathcal{J}_\mathfrak{X})$. Since $h^\sharp$ preserves the fermionic ideal it gives a commuting diagram,

$$\begin{array}{ccccc} \mathcal{O}_\mathfrak{X} & \xrightarrow{h^\sharp} & i_*\mathcal{O}_\mathfrak{Y} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}_M & \xrightarrow{\bar{h}^\sharp} & \mathcal{O}_N & \longrightarrow & 0 \end{array}$$

Hence we have an embedding $(i, \bar{h}^\sharp) : N \subset M$.

(ii) Let $\mathcal{I}_\mathfrak{Y}$ denote the ideal sheaf of $\mathfrak{Y} \subset \mathfrak{X}$. There exist short exact sequences,

$$0 \to \mathcal{I}_\mathfrak{Y} \cap \mathcal{J}_\mathfrak{X} \to \mathcal{I}_\mathfrak{Y} \to \mathcal{O}_M \to 0 \quad \text{and} \quad 0 \to \mathcal{I}_\mathfrak{Y} \cap \mathcal{J}_\mathfrak{X} \to \mathcal{J}_\mathfrak{X} \to \mathcal{O}_\mathfrak{Y} \to 0 \tag{2}$$

where $\mathcal{I}_\mathfrak{Y} \to \mathcal{O}_M$ is the composition $\mathcal{I}_\mathfrak{Y} \to \mathcal{O}_\mathfrak{X} \to \mathcal{O}_M$ and $\mathcal{J}_\mathfrak{X} \to \mathcal{O}_\mathfrak{Y}$ is the composition $\mathcal{J}_\mathfrak{X} \to \mathcal{O}_\mathfrak{X} \to \mathcal{O}_\mathfrak{Y}$. Assuming $\mathcal{I}_\mathfrak{Y} \cap \mathcal{J}_\mathfrak{X} = (0)$, the ideal $\mathcal{I}_\mathfrak{Y}$ can be identified with its image in $\mathcal{O}_M$ and $\mathcal{J}_\mathfrak{X}$ with its image in $\mathcal{O}_\mathfrak{Y}$. The support of $\mathcal{O}_M/\mathcal{I}_\mathfrak{Y}$ is $|\mathfrak{X}| \setminus (|\mathfrak{X}| \setminus |\mathfrak{Y}|) = |\mathfrak{Y}|$. Set $N' := (|\mathfrak{Y}|, \mathcal{O}_M/\mathcal{I}_\mathfrak{Y})$. We need to show $N' \cong N$, for N the body of $\mathfrak{Y}$. To show this, firstly note $\mathcal{I}_\mathfrak{Y} = \ker\{\mathcal{O}_\mathfrak{X} \stackrel{h^\sharp}{\to} i_*\mathcal{O}_\mathfrak{Y}\}$. Therefore $\mathcal{J}_\mathfrak{Y} = \mathcal{J}_\mathfrak{X}/\mathcal{I}_\mathfrak{Y}\mathcal{J}_\mathfrak{X}$. Since the morphism $\mathcal{J}_\mathfrak{X} \to \mathcal{O}_\mathfrak{Y}$ from (2) is injective we have the commuting diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_\mathfrak{Y}\mathcal{J}_\mathfrak{X} & \longrightarrow & \mathcal{J}_\mathfrak{X} & \longrightarrow & \mathcal{J}_\mathfrak{Y} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{O}_\mathfrak{Y} & = & \mathcal{O}_\mathfrak{Y} & & \end{array}$$

Hence the fermionic ideal for $\mathfrak{Y}$ satisfies $\mathcal{J}_\mathfrak{Y} = \mathrm{im}\{\mathcal{J}_\mathfrak{X} \to \mathcal{O}_\mathfrak{Y}\}$. Therefore,

$$\frac{\mathcal{O}_\mathfrak{Y}}{\mathcal{J}_\mathfrak{Y}} = \frac{\mathcal{O}_\mathfrak{Y}}{\mathcal{J}_\mathfrak{X}} \cong \frac{(\mathcal{O}_\mathfrak{X}/\mathcal{I}_\mathfrak{Y})}{\mathcal{J}_\mathfrak{X}} \cong \frac{(\mathcal{O}_\mathfrak{X}/\mathcal{J}_\mathfrak{X})}{\mathcal{I}_\mathfrak{Y}} = \frac{\mathcal{O}_M}{\mathcal{I}_\mathfrak{Y}}. \tag{3}$$

Denoting by φ the isomorphism in (3) we have an isomorphism of locally ringed spaces $(\mathbf{1}, \varphi) : N \cong N'$.

(iii) Note that $\mathfrak{Y} \cap M \cong \mathfrak{Y} \times_\mathfrak{X} M$, i.e., that the intersection can be identified with the fiber product of embeddings $\mathfrak{Y} \subset \mathfrak{X}$ and $M \subset \mathfrak{X}$. This leads to $i_*\mathcal{O}_{\mathfrak{Y}\cap M} = i_*\mathcal{O}_\mathfrak{Y} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_M$. With $\mathcal{O}_M = \mathcal{O}_\mathfrak{X}/\mathcal{J}_\mathfrak{X}$ we have,

$$i_*\mathcal{O}_{\mathfrak{Y}\cap M} = i_*\mathcal{O}_\mathfrak{Y} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_M = i_*\mathcal{O}_\mathfrak{Y} \otimes_{\mathcal{O}_\mathfrak{X}} (\mathcal{O}_\mathfrak{X}/\mathcal{J}_\mathfrak{X}) \cong \frac{i_*\mathcal{O}_\mathfrak{Y}}{\mathcal{J}_\mathfrak{X} \cdot i_*\mathcal{O}_\mathfrak{Y}} \cong \frac{i_*\mathcal{O}_\mathfrak{Y}}{i_*\mathcal{J}_\mathfrak{Y}} = i_*\mathcal{O}_N$$

for $N = (|\mathfrak{Y}|, \mathcal{O}_\mathfrak{Y}/\mathcal{J}_\mathfrak{Y})$ the body of $\mathfrak{Y}$. In identifying $|\mathfrak{Y}| = |\mathfrak{Y}| \cap |\mathfrak{X}|$ see that, as a locally ringed space, $\mathfrak{Y} \cap M = (|\mathfrak{Y}|, \mathcal{O}_{\mathfrak{Y}\cap M})$. Now since i is an embedding, the isomorphism $i_*\mathcal{O}_{\mathfrak{Y}\cap M} \cong i_*\mathcal{O}_N$ implies $\mathcal{O}_{\mathfrak{Y}\cap M} \cong \mathcal{O}_N$ as sheaves on $|\mathfrak{Y}|$. Hence we obtain the desired isomorphism $N \cong \mathfrak{Y} \cap M$.

(iv) Recall that $\mathcal{I}_{\mathfrak{Y}} = \ker\{\mathcal{O}_{\mathfrak{X}} \stackrel{h^\sharp}{\to} i_*\mathcal{O}_{\mathfrak{Y}}\}$ is the ideal sheaf defining the embedding $\mathfrak{Y} \subset \mathfrak{X}$. Accordingly we have a relation between fermonic ideals, $\mathcal{J}_{\mathfrak{Y}} = \mathcal{J}_{\mathfrak{X}}/\mathcal{I}_{\mathfrak{Y}}\mathcal{J}_{\mathfrak{X}}$. This leads to the following morphism of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_{\mathfrak{Y}}\mathcal{J}_{\mathfrak{X}}^2 & \longrightarrow & \mathcal{J}_{\mathfrak{X}}^2 & \longrightarrow & i_*\mathcal{J}_{\mathfrak{Y}}^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_{\mathfrak{Y}}\mathcal{J}_{\mathfrak{X}} & \longrightarrow & \mathcal{J}_{\mathfrak{X}} & \longrightarrow & i_*\mathcal{J}_{\mathfrak{Y}} & \longrightarrow & 0 \end{array}$$

giving the right-exact sequence of $\mathcal{O}_M$-modules,

$$\frac{\mathcal{I}_{\mathfrak{Y}}\mathcal{J}_{\mathfrak{X}}}{\mathcal{I}_{\mathfrak{Y}}\mathcal{J}_{\mathfrak{X}}^2} \longrightarrow \frac{\mathcal{J}_{\mathfrak{X}}}{\mathcal{J}_{\mathfrak{X}}^2} \longrightarrow i_*\frac{\mathcal{J}_{\mathfrak{Y}}}{\mathcal{J}_{\mathfrak{Y}}^2} \longrightarrow 0 \tag{4}$$

With $\mathcal{O}_M = \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_{\mathfrak{X}}$ we have,

$$\frac{\mathcal{I}_{\mathfrak{Y}}\mathcal{J}_{\mathfrak{X}}}{\mathcal{I}_{\mathfrak{Y}}\mathcal{J}_{\mathfrak{X}}^2} \cong (\mathcal{I}_{\mathfrak{Y}}\mathcal{J}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_M \cong \mathcal{I}_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{J}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_M \cong \mathcal{I}_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \frac{\mathcal{J}_{\mathfrak{X}}}{\mathcal{J}_{\mathfrak{X}}^2} \cong \mathcal{I}_N \otimes_{\mathcal{O}_M} \frac{\mathcal{J}_{\mathfrak{X}}}{\mathcal{J}_{\mathfrak{X}}^2}$$

where $\mathcal{I}_N = \mathrm{im}\{\mathcal{I}_{\mathfrak{Y}} \to \mathcal{O}_M\}$ is the ideal sheaf of $N \subset M$ by part *(ii)*. In pulling back the sequence (4) to N now, the morphism $i^*(\mathcal{I}_{\mathfrak{Y}}\mathcal{J}_{\mathfrak{X}}/\mathcal{I}_{\mathfrak{Y}}\mathcal{J}_{\mathfrak{X}}^2) \to i^*(\mathcal{J}_{\mathfrak{X}}/\mathcal{J}_{\mathfrak{X}}^2)$ necessarily vanishes. Hence that $i^*(\mathcal{J}_{\mathfrak{X}}/\mathcal{J}_{\mathfrak{X}}^2) \cong \mathcal{J}_{\mathfrak{Y}}/\mathcal{J}_{\mathfrak{Y}}^2$. As these are (respectively) the sheaves of sections of souls $\mathbf{V}_{\mathfrak{Y}}$ and $\mathbf{V}_{\mathfrak{X}}$, it follows that $\mathbf{V}_{\mathfrak{Y}} \cong \mathbf{V}_{\mathfrak{X}}|_N$. □

3 Examples

3.1 Affine algebraic superspaces

Let $A = \mathbb{K}[x^1, \ldots, x^m|\theta_1, \ldots, \theta_n]$ be the free algebra on $(m+n)$-generators subject to the relations $x^i x^j = x^j x^i, x^i\theta_a = \theta_a x^i$ and $\theta_a\theta_b = -\theta_b\theta_a$. Then A defines a supercommutative ring. The reduced ring of A is the polynomial ring $\mathbb{K}[x^1, \ldots, x^n]$. If we denote by V a free $\mathbb{K}[x^1, \ldots, x^m]$-module of rank n, then $A \cong \wedge^\bullet_{\mathbb{K}[x^1,\ldots,x^m]} V$. Hence by Definition 2.3, A defines an *affine*, supercommutative ring. The spectrum construction from algebraic geometry generalizes to the supergeometry setting and establishes a duality between affine algebraic superspaces and supercommutative rings.[8] The $m|n$-dimensional, affine algebraic superspace over $\mathbb{K}$, denoted $\mathbb{A}^{m|n}_{\mathbb{K}}$, is then the spectrum of A, i.e., $\mathbb{A}^{m|n}_{\mathbb{K}} = \mathrm{Spec}\,\mathbb{K}[x^1, \ldots, x^m|\theta_1, \ldots, \theta_n]$.

(i) the body of $\mathbb{A}^{m|n}_{\mathbb{K}}$ is $\mathrm{Spec}\,\mathbb{K}[x^1, \ldots, x^m] = \mathbb{A}^m_{\mathbb{K}}$;

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[8] See e.g., [13].

(ii) the soul of $\mathbb{A}^{m|n}_{\mathbb{K}}$ is the free $\mathbb{K}[x^1, \ldots, x^m]$-module V of rank n such that $\mathcal{O}(\mathbb{A}^{m|n}_{\mathbb{K}}) \cong \wedge^{\bullet}_{\mathbb{K}[x^1,\ldots,x^m]} V$.

3.1.1 Affine subvarieties

As in algebraic geometry, affine subvarieties are subsets $S \subset \mathbb{A}^{m|n}_{\mathbb{K}}$ defined by a loci of polynomials. With $\mathbb{A}^{m|n}_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[x^1, \ldots, x^m|\theta_1, \ldots, \theta_n]$, the coordinate ring $\mathbb{K}[S]$ of S is given by $\mathbb{K}[S] = \mathbb{K}[x^1, \ldots, x^m|\theta_1, \ldots, \theta_n]/I$ for I an ideal. Note by Definition 2.3, since $\mathbb{K}[x^1, \ldots, x^m|\theta_1, \ldots, \theta_n]$ is affine, then so is the coordinate ring $\mathbb{K}[S]$. Suppose $I = (f_i)_i$ for $f_i \in \mathbb{K}[x^1, \ldots, x^m|\theta_1, \ldots, \theta_n]$. With the characterization of the body and soul of submanifolds in Theorem 2.23 we have:

(i) the body of S is given by the intersection of S with the body of $\mathbb{A}^{m|n}_{\mathbb{K}}$. This is precisely the locus,

$$f_i(x^1, \ldots, x^m|\theta_1, \ldots, \theta_n) = 0 \qquad \text{and} \qquad \theta_1 = \cdots = \theta_n = 0 \tag{5}$$

for all i.

(ii) The soul of S is given by restricting the soul of $\mathbb{A}^{m|n}_{\mathbb{K}}$ to the body of S. With $\mathcal{O}(\mathbb{A}^{m|n}_{\mathbb{K}}) \cong \wedge^{\bullet}_{\mathbb{K}[x^1,\ldots,x^m]} V$, the soul of S is then $V|_{|S|}$.

3.1.2 Superschemes

With affine superspaces over a field $\mathbb{K}$ now established, we can define superschemes following Definition 2.17 and in line with Leites in [13].

Definition 3.1 *An* algebraic superscheme *over a field $\mathbb{K}$ is supercommutative space which is locally isomorphic to an affine superspace over $\mathbb{K}$.* □

3.2 Projective superspaces

The multiplicative group $\mathbb{G}_m$ acts on the affine superspace $\mathbb{A}^{m|n}_{\mathbb{K}}$ by scalar multiplication. On generators $x|\theta$ of the coordinate ring of $\mathbb{A}^{m|n}_{\mathbb{K}}$ and $\lambda \in \mathbb{G}_m$ we have the scaling $x|\theta \mapsto \lambda x|\lambda\theta$. The Euclidean superspace $\mathbb{K}^{m|n}$ was formed as a local, supercommutative ringed space in Example 2.15. Projective superspace over $\mathbb{K}$ is then formed as the quotient $\mathbb{P}^{m-1|n}_{\mathbb{K}} := (\mathbb{K}^{m|n} \setminus \{(0|0)\})/\mathbb{G}_m$.

(i) the body of $\mathbb{P}^{m-1|n}_{\mathbb{K}}$ is $\mathbb{P}^{m-1}_{\mathbb{K}}$;

(ii) the soul of $\mathbb{P}^{m-1|n}_{\mathbb{K}}$ is the n-fold sum of the tautological line bundle over $\mathbb{P}^{m-1}_{\mathbb{K}}$. Its sheaf of sections is $\oplus^n \mathcal{O}_{\mathbb{P}^m_{\mathbb{K}}}(-1)$.

An atlas on projective superspace $\mathbb{P}^{m-1|n}_{\mathbb{K}}$ can be described in a manner analogous to the case of $\mathbb{P}^{m-1}_{\mathbb{K}}$. An open set in this atlas can be described as follows. Denoting by $A = \mathbb{K}[X^1, \ldots, X^m|\Theta_1, \ldots, \Theta_n]$ the homogeneous coordinate ring, let U_i be the localization of A at X^i. Set $x^j = \frac{X^j}{X^i}$ and $\theta_a = \frac{\Theta_a}{x^i}$. Then $\mathcal{O}(U_i) = \mathbb{K}[x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^m|\theta_1, \ldots, \theta_n] \cong A_{X^i}$ is the coordinate ring of the open set $U_i \subset \mathbb{P}^{m-1|n}_{\mathbb{K}}$. From here it ought to be clear why the projective superspace is $m-1|n$-dimensional.

3.2.1 Projective superspace varieties

Homogeneous ideals in the coordinate ring $\mathbb{K}[X^1, \ldots, X^m|\Theta_1, \ldots, \Theta_n]$ with respect to the $\mathbb{G}_m$-action define subvarieties of projective superspace. As an illustration over $\mathbb{K} = \mathbb{C}$, the superspace quadric $Q \subset \mathbb{P}^{2|2}_{\mathbb{C}}$ is given by the ideal $((X^1)^2 + (X^2)^2 + (X^3)^2 + \Theta_2\Theta_2) \subset \mathbb{C}[X^1, X^2, X^3|\Theta_1, \Theta_2]$. In the open set U_1 we have the affine subvariety $(1 + (x^2)^2 + (x^3)^2 + \theta_1\theta_2) \subset \mathbb{C}[x^2, x^3|\theta_1, \theta_2]$ where $x^i = X^i/x^1$ and $\theta_a = \Theta_a/x^1$.

(i) by (5), the body of Q is the projective subvariety defined by the ideal $((X^1)^2 + (X^2)^2 + (X^3)^2) \subset \mathbb{C}[X^1, X^2, X^3]$, which describes the degree two embedding $\mathbb{P}^1_{\mathbb{C}} \subset \mathbb{P}^2_{\mathbb{C}}$. Hence the body of Q is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$;

(ii) by Theorem 2.23, the soul of Q is the restriction of the soul of $\mathbb{P}^{2|2}_{\mathbb{C}}$ to the degree two embedding $\mathbb{P}^1_{\mathbb{C}} \subset \mathbb{P}^2_{\mathbb{C}}$. Recall from §3.2 that the soul of $\mathbb{P}^{2|2}_{\mathbb{C}}$ is $\oplus^2\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-1)$. Hence, the soul of Q is $\oplus^2\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2)$.

For more details and theory on projective superspaces and subvarieties, including their weighted analogues, see [14].

3.3 Super Riemann surfaces

Definition 3.2 *A* super Riemann surface *is a pair* $(\mathcal{S}, \mathcal{D})$ *where* $\mathcal{S}$ *is a* $(1|1)$*-dimensional, complex supermanifold and* $\mathcal{D} \subset T_{\mathcal{S}}$ *is a totally non-integrable distribution, i.e., defines a short exact sequence,*

$$0 \longrightarrow \mathcal{D} \longrightarrow T_{\mathcal{S}} \longrightarrow \mathcal{D} \otimes \mathcal{D} \longrightarrow 0$$

where $T_{\mathcal{S}} = Der\ \mathcal{O}_{\mathcal{S}}$ *is the tangent sheaf.* □

In coordinates $x|\theta$ on the complex Euclidean superspace $\mathbb{C}^{1|1}$, the vector field $\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ generates a totally non-integrable distribution $D \subset T_{\mathbb{C}^{1|1}}$. As a result $(\mathbb{C}^{1|1}, D)$ defines a (non-compact) super Riemann surface. It is in fact the local model for any super Riemann surface. In more detail, let $U, V \subset \mathcal{S}$ denote open sets, $x|\theta$ resp., $y|\eta$ coordinates on U resp., V and $\rho_{UV} = \rho^+_{UV}|\rho^-_{UV}$ transition functions on $U \cap V$,

$$y = \rho^+(x|\theta) = f(x) \qquad\qquad \eta = \rho^-(x|\theta) = g(x)\theta. \tag{6}$$

The system $(U, V, U \cap V, \{\rho_{UV}, \rho_{VU}\})$ form (part of) an atlas for a super Riemann surface $(\mathcal{S}, \mathcal{D})$ iff ρ_{UV} preserves the generators of $\mathcal{D}|_U$ and $\mathcal{D}|_V$. With $\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial y}$ the generators of $\mathcal{D}|_U$ and $\mathcal{D}|_V$ respectively, that ρ_{UV} preserves $\mathcal{D}$ means $(\rho_{UV})_*(\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}) \propto \frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial y}$. Enforcing this necessitates $g(x) = \sqrt{f'(x)}$ in (6). We then have as a consequence:

(i) the body $|\mathcal{S}|$ of a super Riemann surface $(\mathcal{S}, \mathcal{D})$ is a Riemann surface;
(ii) the soul of a super Riemann surface is a spin structure $(T^*_{|\mathcal{S}|})^{1/2}$ on the body.

Evidently,

Theorem 3.3 The set of totally non-integrable distributions $\mathcal{D}$ on compact, $(1|1)$-dimensional supermanifold $\mathcal{S}$ is in bijective correspondence with spin structures on the body $|\mathcal{S}|$ of $\mathcal{S}$.

Remark 3.4 *Authors such as Rabin and Crane in [15] and Witten in [16] take a broader definition of a super Riemann surface. For these authors, the functions in* (6) *are themselves valued in an auxiliary exterior algebra. The author in [17] views these describing infinitesimal deformations however, which is in line with the treatment of super Riemann surfaces and their moduli from [18]. Through the viewpoint in this article, we would understand the super Riemann surfaces from [15, 16] as 'DeWitt structures on a specified locus of points' of super Riemann surfaces from [17, 18].* □

4 Points of a supermanifold

4.1 *L*-points

We have so far introduced supermanifolds as local, supercommutative ringed spaces. That is, a supermanifold $\mathfrak{X}$ consists principally of a topological space $|\mathfrak{X}|$ and a sheaf of local, supercommutative rings $O_{\mathfrak{X}}$ on $|\mathfrak{X}|$. It is arguably more natural however to adopt the functor of points approach and view a supermanifold $\mathfrak{X}$ through the representable functor it defines, $\underline{\mathfrak{X}} = \mathrm{Hom}_{\mathsf{SM}}(-, \mathfrak{X})$ for SM the category of supermanifolds. This allows for defining points of a supermanifold in a very general setting. Over a field $\mathbb{K}$ and for an integer L, consider the supercommutative ring $\wedge^\bullet_{\mathbb{K}}\mathbb{K}^L$. By §3.1, its spectrum is the affine superspace $\mathbb{A}^{0|L}_{\mathbb{K}}$.

Definition 4.1 *An L-point of a supermanifold* $\mathfrak{X}$ *is a morphism* $\mathbb{A}^{0|L}_{\mathbb{K}} \to \mathfrak{X}$*. If* $L = 0$*, we refer to the L-point as* reduced*. If* $L = 1$ *the L-point is referred to as a* superpoint. □

In analogy with schemes we have:[9]

[9] c.f., [11, Lemma 01J6]

Lemma 4.2 An L-point $\mathbb{A}^{0|L}_{\mathbb{K}} \stackrel{\phi}{\to} \mathfrak{X}$ is equivalent to a homomorphism of (local) supercommutative rings $\mathcal{O}_{\mathfrak{X},P} \to \mathbb{K}[\xi_1, \ldots, \xi_L]$, where P is the image of ϕ and $\mathbb{A}^{0|L}_{\mathbb{K}} = \mathrm{Spec}\ \mathbb{K}[\xi_1, \ldots, \xi_L]$. □

Proof With $\mathbb{A}^{0|L}_{\mathbb{K}} = \mathrm{Spec}\ \mathbb{K}[\xi_1, \ldots, \xi_L]$, as a locally ringed space we can write $\mathbb{A}^{0|L}_{\mathbb{K}} = (\{pt\}, \mathbb{K}[\xi_1, \ldots, \xi_L])$ as in Example 2.8. With $\mathbb{A}^{0|L}_{\mathbb{K}} \stackrel{\phi}{\to} \mathfrak{X}$ an L-point of $\mathfrak{X}$ it is a morphism of loca, supercommutative ringed spaces. Accordingly, we can write $\phi = (i, h^\sharp)$ where $i : \{pt\} \to |\mathfrak{X}|$ and $h^\sharp : i^*\mathcal{O}_{\mathfrak{X}} \to \mathbb{K}[\xi_1, \ldots, \xi_L]$. Writing $P = i(\{pt\}) \in |\mathfrak{X}|$ note that $i^*\mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_{\mathfrak{X},P}$. As such ϕ is completely determined by the homomorphism of supercommutative rings $\mathcal{O}_{\mathfrak{X},P} \stackrel{h^\sharp}{\to} \mathbb{K}[\xi_1, \ldots, \xi_L]$. □

To expand further on Lemma 4.2: with the L-point $\mathbb{A}^{0|L}_{\mathbb{K}} \stackrel{\phi}{\to} \mathfrak{X}$ equivalent to a homomorphism of local, supercommutative rings $\mathcal{O}_{\mathfrak{X},P} \stackrel{h^\sharp}{\to} \mathbb{K}[\xi_1, \ldots, \xi_L]$, recall from Example 2.5 that the maximal ideal in $\mathbb{K}[\xi_1, \ldots, \xi_L]$ is the fermionic ideal $J = \ker\{\mathbb{K}[\xi_1, \ldots, \xi_L] \to \mathbb{K}\}$. As such $h^\sharp$ sends the unique maximal ideal $\mathfrak{m}_P \subset \mathcal{O}_{\mathfrak{X},P}$ to J. Hence that it induces a morphism of short exact sequences,

$$\begin{CD} 0 @>>> \mathfrak{m}_P @>>> \mathcal{O}_{\mathfrak{X},P} @>>> \mathbb{K} @>>> 0 \\ @. @VVV @VV{h^\sharp}V @VVV @. \\ 0 @>>> J @>>> \mathbb{K}[\xi_1, \ldots, \xi_L] @>>> \mathbb{K} @>>> 0 \end{CD} \tag{7}$$

A function $f \in \mathcal{O}_{\mathfrak{X}}$ *vanishes at the L-point ϕ* if its germ $[f]_P$ vanishes modulo the maximal ideal $\mathfrak{m}_P$. Equivalently by (7), if $h^\sharp([f]_P)$ vanishes modulo the fermionic ideal J.

Remark 4.3 *For convenience we will often conflate the point P with the image of the L-point $\phi = (i, h^\sharp)$ on $\{pt\}$* □

Recall our notation: for any supermanifold $\mathfrak{X}$, $\underline{\mathfrak{X}} := \mathrm{Hom}_{\mathsf{SM}}(-, \mathfrak{X})$.

Definition 4.4 *The* locus of L-points of $\mathfrak{X}$ *is the image $\underline{\mathfrak{X}}(\mathbb{A}^{0|L}_{\mathbb{K}})$ in* Set. *For brevity, it is denoted by $\underline{\mathfrak{X}}(L)$* □

4.2 First properties

Theorem 4.5 The locus of reduced points of a supermanifold is isomorphic to the locus of rational points of its body. □

Proof Recall that a reduced point of $\mathfrak{X}$ is an L-point with $L = 0$. The locus of reduced points is then $\underline{\mathfrak{X}}(0)$. By Lemma 4.2 and the diagram in (7), any reduced point P is determined by a morphism $\mathcal{O}_{\mathfrak{X},P} \to \mathbb{K}$. Since $\mathbb{K}$ is commutative we have an isomorphism of sets[10] $\mathrm{Hom}_{\mathsf{SCR}}(O_{\mathfrak{X},P},\mathbb{K}) \cong \mathrm{Hom}_{\mathsf{CR}}(O_{M,P},\mathbb{K})$ where M is the body of $\mathfrak{X}$.[11] Therefore $\underline{\mathfrak{X}}(0) \cong \underline{M}(\mathrm{Spec}\,\mathbb{K})$. As $\underline{M}(\mathrm{Spec}\,\mathbb{K})$ is precisely the locus of rational points of M, the theorem follows. □

Theorem 4.6 The locus of superpoints in a supermanifold is isomorphic to the locus of rational points of its soul. □

Proof We need to show there exists an isomorphism $\underline{\mathfrak{X}}(1) \cong \underline{\mathbf{V}}_{\mathfrak{X}}(\mathrm{Spec}\,\mathbb{K})$ where $\mathbf{V}_{\mathfrak{X}}$ is the soul of $\mathfrak{X}$, characterized in Lemma 2.22. We begin by constructing a map $\underline{\mathfrak{X}}(1) \to \underline{\mathbf{V}}_{\mathfrak{X}}(\mathrm{Spec}\,\mathbb{K})$. Let $\mathbb{A}^{0|1}_{\mathbb{K}} \stackrel{\phi}{\to} \mathfrak{X}$ be a superpoint with image P in $|\mathfrak{X}|$. By Lemma 4.2 and the diagram in (7), ϕ is determined by the morphism $h^\sharp$ on local supercommutative rings,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}_P & \longrightarrow & O_{\mathfrak{X},P} & \longrightarrow & \mathbb{K} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle h^\sharp} & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{K} & \stackrel{\xi}{\longrightarrow} & \mathbb{K}[\xi] & \longrightarrow & \mathbb{K} & \longrightarrow & 0 \end{array} \tag{8}$$

where we have identified $\ker\{\mathbb{K}[\xi] \to \mathbb{K}\} \cong \mathbb{K}$ as $\mathbb{K}$-modules and $\mathbb{K} \stackrel{\xi}{\to} \mathbb{K}[\xi]$ denotes multiplication by ξ. For any $[f]_P \in O_{\mathfrak{X},P}$ we can write $h^\sharp([f]_P) = a([f]_P) + \xi b([f]_P)$. Since $h^\sharp$ is a morphism of supercommutative rings, $a : O_{\mathfrak{X},p} \to \mathbb{K}$ will be an even homomorphism and $b : O_{\mathfrak{X},P} \to \mathbb{K}[\xi]$ will be an odd derivation over a, i.e., that

$$b([ff']_P) = a([f]_P)b([f']_P) + b([f]_P)a([f']_P). \tag{9}$$

By Theorem 4.5, a defines a rational point in the body of $\mathfrak{X}$. Now with $\mathcal{J}_{\mathfrak{X}} \subset O_{\mathfrak{X}}$ the fermionic ideal and $\mathcal{J}_{\mathfrak{X},P} \subset O_{\mathfrak{X},P}$ the stalk at P, $h^\sharp$ induces, analogously to (8), the morphism of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}_{\mathfrak{X},P} & \longrightarrow & O_{\mathfrak{X},P} & \longrightarrow & C_{M,P} & \longrightarrow & 0 \\ & & {\scriptstyle h^\sharp}\downarrow & \stackrel{b}{\swarrow} & \downarrow{\scriptstyle h^\sharp} & \stackrel{a}{\searrow} & \downarrow & & \\ 0 & \longrightarrow & \mathbb{K} & \longrightarrow & \mathbb{K}[\xi] & \longrightarrow & \mathbb{K} & \longrightarrow & 0 \end{array} \tag{10}$$

where C_M is the structure sheaf of the body of $\mathfrak{X}$. Since $a : O_{\mathfrak{X},P} \to \mathbb{K}$ is even and $\mathbb{K}$ contains no non-trivial, nilpotent elements, it follows that $a(j) = 0, \forall j \in \mathcal{J}_{\mathfrak{X},P}$. Then by (9) see that for any element $j'' = jj' \in \mathcal{J}^2_{\mathfrak{X},P}$,

[10] Here SCR refers to the category of supercommutative rings while CR is that of commutative rings

[11] To see that we will indeed have an isomorphism of sets note that conversely, given a rational point of M, this is a morphism $\mathrm{Spec}\,\mathbb{K} \stackrel{\psi}{\to} M$. It defines on local rings, $C_{M,P} \stackrel{g^\sharp}{\to} \mathbb{K}$. Using that $C_{M,P} = O_{\mathfrak{X},P}/\mathcal{J}_{\mathfrak{X},P}$, the composition $h^\sharp : O_{\mathfrak{X},P} \to C_{M,P} \stackrel{g^\sharp}{\to} \mathbb{K}$ defines a reduced point of $\mathfrak{X}$.

$$b(j'') = b(jj') = b(j)a(j') + a(j)b(j') = 0.$$

Therefore $h^\sharp(j'') = 0$ so it will induce a commuting diagram,

$$\begin{array}{ccc} \mathcal{J}_{\mathfrak{X},P} & \xrightarrow{h^\sharp} & \mathbb{K} \\ \downarrow & \nearrow_{b} & \\ \mathcal{J}_{\mathfrak{X},P}/\mathcal{J}^2_{\mathfrak{X},P} & & \end{array} \tag{11}$$

Recall now the universal property for the symmetric algebra which says: for any commutative algebra A over a commutative R, an R-module V and any R-linear morphism $V \stackrel{b}{\to} A$, there exists a unique, algebra morphism $\widetilde{b} : \mathrm{Sym}^\bullet_R V \to A$ commuting the following diagram,

$$\begin{array}{ccc} V & \longrightarrow & \mathrm{Sym}^\bullet_R V \\ & {}_{b}\searrow \quad \swarrow_{\widetilde{b}} & \\ & A & \end{array}$$

where $V \subset \mathrm{Sym}^\bullet_R V$ is the inclusion of V as the degree one component of its symmetric algebra. Applying this universal property to (11) yields a unique morphism of algebras $\widetilde{b} : \mathrm{Sym}^\bullet_{\mathcal{O}_{\mathfrak{X},P}}(\mathcal{J}_{\mathfrak{X},P}/\mathcal{J}^2_{\mathfrak{X},P}) \to \mathbb{K}$. With $\mathbf{V}_{\mathfrak{X}} = \mathrm{Spec}\,\mathrm{Sym}^\bullet(\mathcal{J}_{\mathfrak{X},P}/\mathcal{J}^2_{\mathfrak{X},P})$ from Lemma 2.22 see that $\widetilde{b}$ now defines a rational point $\mathrm{Spec}\,\mathbb{K} \to \mathbf{V}_{\mathfrak{X}}$. We have therefore constructed a map $\underline{\mathfrak{X}}(1) \to \underline{\mathbf{V}}_{\mathfrak{X}}(\mathrm{Spec}\,\mathbb{K})$. A inverse map $\underline{\mathbf{V}}_{\mathfrak{X}}(\mathrm{Spec}\,\mathbb{K}) \to \underline{\mathfrak{X}}(1)$ can be formed by suitably reversing the construction given above. We omit the details here. □

4.3 A (1|1)-quadratic form

To illustrate the distinction between L-points for $L = 0, 1, 2$ we present the (1|1)-dimensional, quadratic form as follows. Consider a form Q given by

$$\langle (x|\theta), (y|\eta)\rangle = xy + \theta\eta. \tag{12}$$

Here Q defines a function $\mathbb{A}^{2|2}_{\mathbb{K}} \cong \mathbb{A}^{1|1}_{\mathbb{K}} \times \mathbb{A}^{1|1}_{\mathbb{K}} \to \mathbb{A}^{1|1}_{\mathbb{K}}$. As a pairing, we can write

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

so that $\langle (x|\theta), (y|\eta)\rangle = (x|\theta)^T Q (y|\eta)$.

On the locus of superpoints, set $x = z^{1,0}$, $y = z^{2,0}$, $\theta = {\vartheta_1}^1\xi$ and $\eta = {\vartheta_2}^1\xi$. Then Q defines on $\underline{\mathbb{A}}^{2|2}_{\mathbb{K}}(1)$,

$$h^\sharp(\langle(x|\theta),(y|\eta)\rangle) = z^{1,0}z^{2,0} + ({\vartheta_1}^1\xi)({\vartheta_2}^1\xi) = z^{1,0}z^{2,0}.$$

As a pairing then on $\underline{\mathbb{A}}^{2|2}_{\mathbb{K}}(1) \cong \mathbb{K}^{1+1} = \mathbb{K}^2 \cong \mathbb{K}\times\mathbb{K}$, Q from (13) is represented by the 1×1-matrix, $\underline{Q}(1) = (1)$.[12]

We obtain a more interesting expression on the locus of 0|2-points. Here, write

$$\begin{aligned} x &\overset{h^\sharp}{\longmapsto} z^{1,0} + z^{1,12}\xi_1\xi_2;\\ y &\overset{h^\sharp}{\longmapsto} z^{2,0} + z^{2,12}\xi_1\xi_2;\\ \theta &\overset{h^\sharp}{\longmapsto} {\vartheta_1}^1\xi_1 + {\vartheta_1}^2\xi_2;\\ \eta &\overset{h^\sharp}{\longmapsto} {\vartheta_2}^1\xi_1 + {\vartheta_2}^2\xi_2. \end{aligned}$$

Then,

$$h^\sharp(\langle(x|\theta),(y|\eta)\rangle) = z^{1,0}z^{2,0} + \big(z^{1,0}z^{2,12} + z^{2,0}z^{1,12} + {\vartheta_1}^1{\vartheta_2}^2 - {\vartheta_1}^2{\vartheta_2}^1\big)\xi_1\xi_2$$

As a pairing on $\underline{\mathbb{A}}^{2|2}_{\mathbb{K}}(2) \cong \mathbb{K}^{2^{2-1}(2+2)} = \mathbb{K}^8 \cong \mathbb{K}^4\times\mathbb{K}^4$, Q from (13) is represented by the 4×4-matrix,

$$\underline{Q}(2) = \begin{pmatrix} 1 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1\\ & & -1 & 0 \end{pmatrix}$$

where the blank spaces in the above matrix are zeroes. Note that $\underline{Q}(2)$ not symmetric, in contrast to (13). This is because, due to anti-commutativity of the odd variables, the product (12) itself is not symmetric, i.e., that $\langle(x|\theta),(y|\eta)\rangle \neq \langle(y|\eta),(x|\theta)\rangle$. Indeed, we have the relations $\langle(x|\theta),(y|\eta)\rangle + \langle(y|\eta),(x|\theta)\rangle = 2xy$ and $\langle(x|\theta),(y|\eta)\rangle - \langle(y|\eta),(x|\theta)\rangle = 2\theta\eta$. In a sense, these relations are not 'seen' by the reduced and superpoints.

5 The DeWitt topology

In §2 we presented the local, supercommutative ringed space (LSRS) approach to supermanifold theory (c.f., Definition 2.17). In §4 we recovered its first properties through its functor of points and arrived general class of points in Definition 4.4. In the present section we detail the approach to supermanifold theory instigated by DeWitt in [5], aptly termed the 'DeWitt approach'. Our main result here is Theorem 5.14. In analogy with the complex topology in algebraic geometry, we construct in

[12] Note, this coincides with $\underline{Q}(0)$.

Theorem 5.14 a topology on the locus of L-points of a superscheme, referred to as the *DeWitt topology*.

5.1 The DeWitt approach

5.1.1 DeWitt superspaces

Fix a field $\mathbb{K}$, an integer L and set $B_L := \wedge^\bullet_{\mathbb{K}}\mathbb{K}^L$. It is referred to as the *auxiliary algebra on L generators*. In the vector space topology, $B_L \cong \mathbb{K}^{2^{L-1}(1+1)}$. Let $B_L^{m,n}$ now denote the product,

$$B_L^{m,n} := \prod^m B_{L,\bar{0}} \times \prod^n B_{L,\bar{1}} \tag{14}$$

where $B_{L,\bar{0}}$ resp., $B_{L,\bar{1}}$ is the subring resp., $B_{L,\bar{0}}$ module of even and odd elements of B_L. Evidently $B_L^{1,1} \cong B_L$. In the vector space topology $B_L^{m,n} \cong \mathbb{K}^{2^{L-1}(m+n)}$. Since B_L is an exterior algebra it comes equipped with a projection $\sigma : B_L^{m,n} \to \mathbb{K}^m$ induced by $B_L \to \mathbb{K}$. We refer to $B_L^{m,n} \stackrel{\sigma}{\to} \mathbb{K}^m$ as the *body projection*.

Definition 5.1 *The topology on $B_L^{m,n}$ comprised of open sets of the form $\sigma^{-1}U$ for $U \subset \mathbb{K}^m$ open is referred to as the* DeWitt topology on $B_L^{m,n}$. □

Relating the vector space and DeWitt topologies on $B_L^{m,n}$, we have for any open set $U \subset \mathbb{K}^m$ that $\sigma^{-1}U \cong U \times \mathbb{K}^{2^{L-1}(m+n)-m}$.

Definition 5.2 *The spaces $B_L^{m,n}$ equipped with the DeWitt topology from* (14) *are referred to as* (m,n)-dimensional, DeWitt superspaces over B_L. □

Recall from the LSRS approach (Definition 2.17), a supermanifold is built on model superspaces $(\mathbb{R}^m, \wedge^\bullet_{C^\infty}V)$ and $(\mathbb{C}^m, \wedge^\bullet_{\mathcal{O}}V)$ from Example 2.15. In the DeWitt approach, a *DeWitt supermanifold* is built by patching together DeWitt superspaces over a fixed, auxiliary algebra B_L with respect to a class of smooth functions.

5.1.2 Supersmooth functions

Let $z^\alpha|\vartheta_a$ denote coordinates on $B_L^{m,n}$ where $\alpha = 1, \ldots, m$ and $a = 1, \ldots, n$. Here z^α is even, so valued in $B_{L,\bar{0}}$ while ϑ_a is odd, so valued in $B_{L,\bar{1}}$. With $1, \xi_1, \ldots, \xi_L$ denoting generators for B_L we write,

$$z^\alpha = z^{\alpha,0} + z^{\alpha,ij}\xi_i\xi_j + \cdots \qquad \text{and} \qquad \vartheta_a = \vartheta_a^{\ i}\xi_i + \vartheta_a^{\ ijk}\xi_i\xi_j\xi_k + \cdots \tag{15}$$

If $P \in B_L^{m,n}$ denotes a point so that $z^\alpha(P) \in B_{L,\bar{0}}$ and $\vartheta_a(P) \in B_{L,\bar{1}}$, then $z^{\alpha,0}(P)$ are coordinates for the body $\sigma(P) \in \mathbb{K}^m$, i.e., $z^{\alpha,0}(P) = z^{\alpha,0}(\sigma(P))$.

Remark 5.3 *More generally, in the vector space topology we have the isomorphism* $\varphi : B_L^{m,n} \cong \mathbb{K}^{2^{L-1}(m+n)}$. *On coordinates, this establishes a unique representation of* $z^\alpha|\vartheta_a$ *in* (15) *with the tuple* $z^\alpha \mapsto (z^{\alpha,0}, z^{\alpha,ij}, \ldots)$ *and* $\vartheta_a \mapsto (\vartheta_a^{\ i}, \vartheta_a^{\ ijk}, \ldots)$. *In the case* $m = 1, n = 1$, *for* $P \in B_L^{1,1}$ *a point,* $\varphi(P) \in \mathbb{K}^{2^{L-1}(1+1)}$ *is a point in a classical, Euclidean space. Coordinates of* $\varphi(P)$ *are* $(z^0(P), z^{ij}(P), \ldots, \vartheta^i(P), \vartheta^{ijk}(P), \ldots)$. □

Definition 5.4 *The ring of smooth functions on the DeWitt superspace* $B_L^{m,n}$ *is given by the tensor product,*

$$C^\infty(B_L^{m,n}) := C^\infty(\sigma(B_L^{m,n})) \otimes B_L$$

where $\sigma(B_L^{m,n}) = \mathbb{K}^m$ *is the body projection and* $C^\infty(\sigma(B_L^{m,n})) = C^\infty(\mathbb{K}^m)$ *are smooth functions on* $\mathbb{K}^m$. □

With the above notion of smooth functions on DeWitt superspaces, the topological isomorphism $B_L^{m,n} \cong \mathbb{K}^{2^{L-1}(m+n)}$ does *not* induce an isomorphism of smooth structures. Indeed, the ring of smooth functions on $\mathbb{K}^{2^{L-1}(m+n)}$ is much larger than that of $B_L^{m,n}$.[13] The above notion of smoothness captures the property that any smooth function F on $B_L^{m,n}$ can be written in coordinates $z|\vartheta$,

$$F(z|\vartheta) = F^0(z) + F^a(z)\vartheta_a + F^{aa'}(z)\vartheta_a\vartheta_{a'} + \cdots \tag{16}$$

The $\mathcal{Z}$-expansion introduced by Bruzzo et. al. in [1] is the morphism of rings $C^\infty(\mathbb{K}^m) \xrightarrow{\mathcal{Z}} C^\infty(B_L^{m,0})$ given by

$$\mathcal{Z}(f)(z) = f(z^0) + (z - z^0)f'(z^0) + \frac{1}{2!}(z - z^0)^2 f''(z^0) + \cdots$$

In [1, §2] it is shown:

Lemma 5.5 The $\mathcal{Z}$-expansion $C^\infty(\mathbb{K}^m) \xrightarrow{\mathcal{Z}} C^\infty(B_L^{m,0})$ defines a monomorphism of rings.

Comparing smooth functions on DeWitt superspaces from (16) with the image of $\mathcal{Z}$ leads to the notion of 'supersmoothness'.

Definition 5.6 *A smooth function* $F \in C^\infty(B_L^{m,n})$ *is said to be* supersmooth *if the coefficient functions* F^I *in* (16) *are in the image of* $\mathcal{Z}$. □

Example 5.7 We give here an example of a function which is, and which is not supersmooth. Let $z|\vartheta$ be coordinates on $B_2^{1,1}$ so that $z = z^0 + z^{12}\xi_1\xi_2$ and $\vartheta = \vartheta^1\xi_1 + \vartheta^2\xi_2$. Now let $F \in C^\infty(B_2^{1,1})$ be given by,

$$F(z|\vartheta) = z$$

[13] E.g., the function $\exp(\vartheta^1)$, which is a smooth function in $C^\infty(\mathbb{K}^{2^{L-1}(m+n)})$, will not come from a smooth function on $B_L^{m,n}$.

Then F is supersmooth since,

$$F(z|\vartheta) = z = z^0 + z^{12}\xi_1\xi_2 = \mathcal{Z}(f)(z)$$

where $f \in C^\infty(\sigma(B_2^{1,1})) = C^\infty(\mathbb{K})$ is given by $f(z^0) = z^0$. Indeed, any algebraic function $F(z|\vartheta) = G(z)$ will be supersmooth and so any $F(z|\vartheta) = G_0(z) + G_1(z)\vartheta + \cdots$ where $G_i(z)$ are algebraic functions of z. The following simple function

$$F(z|\vartheta) = z + \xi_1\xi_2$$

is not supersmooth however. □

5.1.3 DeWitt supermanifolds

Definition 5.8 *An* (m, n)-dimensional, DeWitt supermanifold over B_L *is a locally ringed space which admits an open covering by open sets* $(\mathcal{U}_i)_{i\in I}$ *for* $\mathcal{U}_i \subset B_L^{m,n}$ *open in the DeWitt topology and such that over intersections the transition function*

$$\varphi_j^{-1}(\mathcal{U}_i \cap \mathcal{U}_j) \xrightarrow{\varphi_j^{-1}} \mathcal{U}_i \cap \mathcal{U}_j \xrightarrow{\varphi_i} \varphi_i(\mathcal{U}_i \cap \mathcal{U}_j)$$

is supersmooth for all i, j *for which* $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$*.* □

Remark 5.9 *Recalling that* $B_L^{m,n} \cong \mathbb{K}^{2^{L-1}(m+n)}$*, it follows from Definition 5.8 that underlying any* (m, n)*-dimensional, DeWitt supermanifold over* B_L *is a* $2^{L-1}(m+n)$*-dimensional, differentiable manifold over* $\mathbb{K}$*. In parallel with Definition 5.8 one can construct a wider class of supermanifold where the open sets are open in the vector space topology on* $B_L^{m,n}$*, i.e., open under the isomorphism* $B_L^{m,n} \cong \mathbb{K}^{2^{L-1}(m+n)}$*. The contrast to the DeWitt case was studied by Rabin and Crane in [7] and it was shown by Bruzzo et. al. in [1, Ch. 5] how to distinguish different topologies with an analogue of de Rham cohomology.* □

5.2 Superschemes and DeWitt supermanifolds

5.2.1 The atlas for DeWitt supermanifolds

As for manifolds, an atlas for an (m, n)-dimensional, DeWitt supermanifold over B_L consists of a collection of disjoint, DeWitt open sets $(\mathcal{U}_i)_{i\in I}$, $\mathcal{U}_i \subset B_L^{m,n}$, distinguished DeWitt open subsets $\mathcal{U}_{ij} \subset \mathcal{U}_i$ and supersmooth diffeomorphisms $\mathcal{U}_{ij} \xrightarrow{\cong} \mathcal{U}_{ji}$ which is the identity if $i = j$ and for all i, j, k fits into a commuting diagram,

$$\begin{array}{ccc} \mathcal{U}_{ij} & \xrightarrow{\cong} & \mathcal{U}_{jk} \\ & \searrow^{\cong} \quad \nearrow_{\cong} & \\ & \mathcal{U}_{ik} & \end{array}$$

The construction $\frac{\bigsqcup_{i\in I}\mathcal{U}_i}{\sim}$, where the equivalence relation is defined though the transition functions $(\mathcal{U}_{ij} \cong \mathcal{U}_{ji})_{ij}$, defines a DeWitt supermanifold up to isomorphism. In the DeWitt topology, recall that each open set is of the form $\mathcal{U}_i = \sigma^{-1}U_i$ for $U_i \subset \mathbb{K}^m$. As such the glueing data above describes a classical, m-dimensional submanifold of the DeWitt supermanifold, referred to as the *body*. From [5, pp. 55-6]:

Theorem 5.10 Underlying any (m,n)-dimensional, DeWitt supermanifold over B_L is a rank-$(2^{L-1}(m+n)-m)$ vector bundle over the body.

Remark 5.11 *It is suggestive to compare Theorem 5.10 with the first properties of supermanifolds as LSRS, described in §2.3 and §4.2.* □

5.2.2 *L*-points of affine superspace

To prepare our argument for superschemes more generally, we establish the case of affine superspaces in what follows. Over a field $\mathbb{K}$, recall the affine superspace $\mathbb{A}^{m|n}_{\mathbb{K}}$ from §3.1. By Theorem 4.5, the rational points $(L=0)$ of $\mathbb{A}^{m|n}_{\mathbb{K}}$ recover the Euclidean space $\mathbb{K}^m$. By Theorem 4.6, the superpoints $(L=1)$ recover a rank n, trivial vector bundle over $\mathbb{K}^m$. Hence that $\underline{\mathbb{A}}^{m|n}_{\mathbb{K}}(0) \cong \mathbb{K}^m$ and $\underline{\mathbb{A}}^{m|n}_{\mathbb{K}}(1) \cong \mathbb{K}^{m+n}$. More generally we have for the L-points:

Lemma 5.12

$$\underline{\mathbb{A}}^{m|n}_{\mathbb{K}}(L) \cong B^{m,n}_L.$$

Proof Firstly recall $B^{m,n}_L \cong \mathbb{K}^{2^{L-1}(m+n)}$. The statement in the lemma now relates the functor of points $\underline{\mathbb{A}}^{m|n}_{\mathbb{K}}$ with that of the affine space $\underline{\mathbb{A}}^{2^{L-1}(m+n)}_{\mathbb{K}}$, where $\mathbb{K}^{2^{L-1}(m+n)} \cong \underline{\mathbb{A}}^{2^{L-1}(m+n)}_{\mathbb{K}}(\mathrm{Spec}\ \mathbb{K})$. That is, we need to show there exists a natural isomorphism between the sets,[14]

$$\mathrm{Hom}_{\mathsf{LSRS}}(\mathrm{Spec}\ \mathbb{K}[\xi_1,\ldots,\xi_L], \mathbb{A}^{m|n}_{\mathbb{K}}) \cong \mathrm{Hom}_{\mathsf{LSRS}}(\mathrm{Spec}\ \mathbb{K}, \mathbb{A}^{2^{L-1}(m+n)}_{\mathbb{K}}) \tag{17}$$

where $\mathbb{A}^{0|L}_{\mathbb{K}} = \mathrm{Spec}\ \mathbb{K}[\xi_1,\ldots,\xi_L]$. Write, $\mathbb{A}^{m|n}_{\mathbb{K}} = \mathrm{Spec}\ \mathbb{K}[x^1,\ldots,x^m|\theta_1,\ldots,\theta_n]$ and $\mathbb{A}^{2^{L-1}(m+n)}_{\mathbb{K}} = \mathrm{Spec}\ \mathbb{K}[z^I,\vartheta^J]$ where I and even resp., odd J are multi-indices on L-generators. Then $x^\alpha|\theta_a$ resp., z^I,ϑ^J are coordinates on $\mathbb{A}^{m|n}_{\mathbb{K}}$ resp., $\mathbb{A}^{2^{L-1}(m+n)}_{\mathbb{K}}$. The statement in (17) is now equivalent to the following as $\mathbb{K}$-algebras,

[14] here LSRS denotes the category of local, supercommutative ringed spaces

$$\mathrm{Hom}_{\mathbb{K}\text{-ALG}}(\mathbb{K}[x^\alpha|\theta_a], \mathbb{K}[\xi_1, \ldots, \xi_L]) \cong \mathrm{Hom}_{\mathbb{K}\text{-ALG}}(\mathbb{K}[z^{\alpha I}, \vartheta_a^{\ J}], \mathbb{K}). \tag{18}$$

The above isomorphism is a consequence of the familiar tensor-hom adjunction adapted to supercommutative rings. Thus we obtain the isomorphism (17) and so the lemma. □

To elaborate further on the proof of Lemma 5.12, with $\mathbb{A}^{m|n}_{\mathbb{K}} = \mathrm{Spec}\ \mathbb{K}[x^\alpha|\theta_a]$, $\alpha = 1, \ldots, m$ and $a = 1, \ldots, n$; the coordinate ring of $\mathbb{A}^{m|n}_{\mathbb{K}}$ is $\mathbb{K}[x^\alpha|\theta_a]$. An L-point $\phi \in \underline{\mathbb{A}}^{m|n}_{\mathbb{K}}(\mathbb{K}^{0|L})$ is a morphism $\phi = (f, h^\sharp) : \mathbb{K}^{0|L} \to \mathbb{A}^{m|n}_{\mathbb{K}}$ where $f : \{pt\} \to \mathbb{A}^m$ is a rational point of $\mathbb{A}^{m|n}_{\mathbb{K}}$ (c.f., Theorem 4.5) and $h^\sharp : \mathbb{K}[x^i|\theta_a] \to \mathbb{K}[\xi_1, \ldots, \xi_L]$ is an morphism of supercommutative rings. By Lemma 4.2, ϕ is completely determined by $h^\sharp$ which, on generators, is

$$h^\sharp(x^\alpha) = z^{\alpha,0} + z^{\alpha,ij}\xi_i\xi_j + \cdots \quad \text{and} \quad h^\sharp(\theta_a) = \vartheta_a^{\ i}\xi_i + \vartheta_a^{\ ijk}\xi_i\xi_j\xi_k + \cdots \tag{19}$$

The coefficients of $h^\sharp$ generate the algebra $\mathbb{K}[z^{\alpha,I}, \vartheta_a^{\ J}]$ which is the coordinate ring for $\mathbb{A}^{2^{L-1}(m+n)}_{\mathbb{K}}$. The assignment $\xi_j \mapsto 1, \forall j$ mediates the isomorphism (18). As coordinates, let $P = f(\{pt\})$ be the image of f. Lemma 5.12 says: $h^\sharp(x^\alpha(P)) = h^\sharp(x^\alpha)(P)$ and $h^\sharp(\theta_a(P)) = h^\sharp(\theta_a)(P)$ and by (19) we have a correspondence,

$$h^\sharp(x^\alpha(P)) = (z^{\alpha,0}(P), z^{\alpha,ij}(P), \ldots); \quad h^\sharp(\theta_a(P)) = (\vartheta_a^{\ i}(P), \vartheta_a^{\ ijk}(P), \ldots). \tag{20}$$

This shows explicitly how the locus of L-points of $\mathbb{A}^{m|n}_{\mathbb{K}}$ are precisely tuples (rational points) in the Euclidean space $\mathbb{K}^{2^{L-1}(m+n)}$, or equivalently, the superspace $B^{m,n}_L$ by Remark 5.3. Note, we have only considered $B^{m,n}_L$ here with its vector space topology.

Proposition 5.13 $\underline{\mathbb{A}}^{m|n}_{\mathbb{K}}(L)$ can be given the structure of a DeWitt superspace. □

Proof Recall the body map $B^{m,n}_L \stackrel{\sigma_L}{\to} \mathbb{K}^m$ defining the DeWitt topology. Then σ_L defines the pullback morphism on functions $\sigma^\sharp_L : \mathbb{K}[x^1, \ldots, x^m] \to C^\infty(B^{m,n}_L)$. We can extend this to a morphism $\widetilde{\sigma}^\sharp_L : \mathbb{K}[x^1, \ldots, x^m|\theta_1, \ldots, \theta_n] \to C^\infty(B^{m,n}_L)$ through the assignment $\theta_i \mapsto \vartheta_i$. Setting $h^\sharp = \sigma^\sharp_L$ in (20), the body map σ_L leads thereby to an identification $\underline{\mathbb{A}}^{m|n}_{\mathbb{K}}(L) \cong \sigma_L^{-1}\underline{\mathbb{A}}^{m|n}_{\mathbb{K}}(0)$. This establishes the proposition. □

5.2.3 *L*-points of superschemes

Theorem 5.14 For any $m|n$-dimensional superscheme $\mathfrak{X}$ over $\mathbb{K}$, its locus of L-points $\underline{\mathfrak{X}}(L)$ admits the structure of an (m, n)-dimensional, DeWitt supermanifold over B_L. □

Proof From [19] we can write $\mathfrak{X} = \frac{\bigsqcup_i U_i}{\sim}$ where $U_i \subset \mathbb{K}^{m|n}$ are a collection of disjoint, open sets which cover $\mathfrak{X}$ and the equivalence relation $\sim$ is defined by isomorphisms $\rho_{ij} : U_{ij} \cong U_{ji}$ between distinguished subsets $U_{ij} \subset U_i$ and $U_{ji} \subset U_j$. In the case where $\mathfrak{X}$ is a superscheme over $\mathbb{K}$, the open sets are isomorphic to affine superspaces $U_i \cong \mathbb{A}^{m|n}_{\mathbb{K}}$. By Lemma 5.12, the L-points of each open U_i is isomorphic to $B^{m,n}_L$, so we have isomorphisms $\underline{U}_i(L) \cong \mathcal{U}_i \cong B^{m,n}_L$ and therefore a collection of distinguished subsets $\underline{U}_{ij}(L) = \mathcal{U}_{ij} \subset \mathcal{U}_i \cong B^{m,n}_L$. By Proposition 5.13 these open sets are open in the DeWitt topology on $B^{m,n}_L$. Now recall from Lemma 5.5 that the $\mathcal{Z}$-expansion is a monomorphism. Accordingly, it will preserve isomorphisms. Applying the $\mathcal{Z}$-expansion then to the transition functions $\rho_{ij} : U_{ij} \cong U_{ji}$ leads to the supersmooth isomorphisms $\mathcal{Z}(\rho_{ij}) : \mathcal{U}_{ij} \cong \mathcal{U}_{ji}$ between DeWitt superspaces. Hence from the covering data (U_i, U_{ij}, ρ_{ij}) defining $\mathfrak{X}$, its L-points give covering data $(\mathcal{U}_i, \mathcal{U}_{ij}, \mathcal{Z}(\rho_{ij}))$ which can be compiled as in §5.2.1 to form an atlas for a DeWitt supermanifold. The locus of L-points the superscheme $\mathfrak{X}$ therefore admits the structure of a DeWitt supermanifold. □

From [10, §4], the *complex topology* on a scheme M over $\mathbb{C}$ is a topology on the rational points $\underline{M}(\mathrm{Spec}\ \mathbb{C})$ of M. It inherits this topology from complex Euclidean space in a manner similar to our proof of Theorem 5.14. Now in Theorem 5.14 above we have shown, for any superscheme $\mathfrak{X}$, that its locus of L-points $\underline{\mathfrak{X}}(L)$ can be given the structure of a DeWitt supermanifold for any L. In analogy with the complex topology then, we can understand Theorem 5.14 as describing a topology which by Definition 5.1 we can refer to as: *the DeWitt topology on the superscheme* $\mathfrak{X}$.

Question 5.15 Can one interpret the DeWitt topology from Theorem 5.14 as a Grothendieck topology on the category of superschemes SSCH? □

Recall from the proof of Theorem 5.14 that the DeWitt topology on $\mathfrak{X}$ is a consequence of Proposition 5.13. Note by Lemma 5.12 however that it is feasible to form more general topologies on $\mathfrak{X}$ than the DeWitt topology.

Question 5.16 Can one interpret the other topologies on supermanifolds from [1] as Grothendieck topologies on SSCH? □

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