

On Super Non-Abelian T-Duality of Symmetric and Semi-Symmetric Coset Sigma Models

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Abstract We review the algebraic approach to super non-Abelian T-Duality considered in [1], focusing on symmetric and semi-symmetric coset spaces on G/H . We discuss a potential impediment, appearing in these models when integrating out the gauge fields in favour of the dual variables. This process cannot be performed in general and we isolate the obstruction, highlighting three cases in which a solution can be found. After writing the T-dual action we provide solution for two specific models. The first based on the symmetric space $S^3 \simeq SO(4)/SO(3)$, well-known in the literature, the second on the semi-symmetric coset $OSp(1|2)/SO(1,1)$, a Green-Schwarz-like string sigma model satisfying the supergravity torsion constraints.¹

1 Introduction

We start by fixing notation on semi-symmetric cosets, as relevant formulae can be reduced to those for symmetric spaces. We consider two-dimensional sigma models involving a generic Lie (super)group G with associated Lie (super)algebra \mathfrak{g} and defined via smooth maps $g \in C^\infty(\Sigma, G)$ from a two-dimensional Lorentzian worldsheet Σ to G . The main building block of these models is the pull-back to Σ , via g , of the Maurer-Cartan form $j := g^{-1}dg \in \Omega^1(\Sigma, \mathfrak{g})$, invariant under the global left action G_L of the group on itself $g \rightarrow g_0^{-1}g$ and satisfying the Maurer-Cartan flatness condition $F_j := dj + \frac{1}{2}[j, j] = 0$, with d and $\Omega^p(\Sigma, \mathfrak{g})$ respectively denoting exterior derivative and \mathfrak{g} -valued p -forms on Σ . Semi-symmetric spaces are cosets on G/H where H , bosonic subgroup of the supergroup G with associated Lie algebra \mathfrak{h} , arises as the fixed point set of an automorphism of G of order four [2, 3]. At the Lie algebra level this implies the existence of an automorphism $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\sigma^4 = 1$. This allows to identify four subspaces $\mathfrak{g}_k \subset \mathfrak{g}$, $k = 0, 1, 2, 3$, charac-

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terised by $\sigma(\mathfrak{g}_k) = i^k \mathfrak{g}_k$ and $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{(k+l) \bmod 4}$. These are singled out by projectors $\mathfrak{g}_k = P_k(\mathfrak{g}) = \frac{1}{4}(1 + i^{3k}\sigma + i^{2k}\sigma^2 + i^k\sigma^3)(\mathfrak{g})$, leading to an orthogonal decomposition $\mathfrak{g} \simeq \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$. Noting that $\sigma^2 = (-1)^k$, the subspaces $\mathfrak{g}_0 \equiv \mathfrak{h}$, $\mathfrak{g}_2 \equiv \mathfrak{m}$ and $\mathfrak{g}_1 \equiv \mathfrak{p}$, $\mathfrak{g}_3 \equiv \mathfrak{q}$ are respectively recognised as purely bosonic and purely fermionic. σ is an involution on bosonic generators, hence semi-symmetric models are the superspace analogue of symmetric spaces, which are recovered by removing fermions. In turn, j decomposes as $j = A + p + m + q$, with the components transforming as $A \rightarrow h^{-1}Ah + h^{-1}dh$ and $\{p, m, q\} \rightarrow h^{-1}\{p, m, q\}h$ under the local right action H_R of H on G , $g \rightarrow gh$. A locally H_R -invariant and globally G_L -invariant action is then constructed out of p, m, q as $S_{SS} = \frac{1}{2} \int_{\Sigma} \langle m, \star m \rangle + \frac{1}{2} \int_{\Sigma} \langle p, q \rangle$. Wedge product is understood and \star is the Hodge star operator on Σ with respect to the world-sheet metric, satisfying $\star^2 = 1$ on $\Omega^1(\Sigma, \mathfrak{g})$. Brackets denote an Ad-invariant, non-degenerate graded-symmetric bilinear form on \mathfrak{g} compatible with the decomposition.

T-duality can then be performed, assuming a topologically trivial world-sheet, by gauging a sub-(super)group $K_L \subseteq G_L$ of the isometries with gauge fields $\omega \in \Omega^1(\Sigma, \mathfrak{k}_L)$ and introducing a term enforcing their flatness $F_\omega := d\omega + \frac{1}{2}[\omega, \omega] = 0$ by means of Lagrange multipliers $\Lambda \in C^\infty(\Sigma, \mathfrak{k}_L)$, which transform as $\Lambda \rightarrow h^{-1}\Lambda h$ under local H_R . Integrating out the multipliers, ω is set to be pure gauge and the initial model can be recovered, while integrating out ω a T-dual model is obtained in which the multipliers play the role of dual coordinates. The gauged action, with Lagrange multipliers, reads $S_{SS}^\omega = \frac{1}{2} \int_{\Sigma} \langle m_\omega, \star m_\omega \rangle + \frac{1}{2} \int_{\Sigma} \langle p_\omega, q_\omega \rangle + \frac{1}{2} \int_{\Sigma} \langle D(j_\omega), j_\omega \rangle + \int_{\Sigma} \langle \tilde{\Lambda}, F_{j_\omega} \rangle$. Where $\tilde{\Lambda} := g^{-1}\Lambda g + g^{-1}D(g)$, while $j_\omega := j + g^{-1}\omega g$ and $A_\omega, p_\omega, m_\omega, q_\omega$ are its projections on the subspaces. $D : \mathfrak{g} \rightarrow \mathfrak{g}$ defines a deformation of the initial model introduced in [4, 5, 6], which will be set to zero in our examples. We shall dualise the whole isometry groups and also choose gauge $g = \mathbb{1}$. The EOM for ω reads $\star m_\omega - \frac{1}{2}p_\omega + \frac{1}{2}q_\omega + \nabla_{j_\omega} \tilde{\Lambda} - D(j_\omega) = 0$ and its projections on $\mathfrak{p}, \mathfrak{m}, \mathfrak{q}$ can be solved for $p_\omega, m_\omega, q_\omega$ [1]. The projection on \mathfrak{h} reads

$$[\tilde{\Lambda}_q, p_\omega] + D_{\tilde{\Lambda}_m}(m_\omega) + [\tilde{\Lambda}_p, q_\omega] = \nabla_{A_\omega} \tilde{\Lambda}_h \quad \text{with} \quad \begin{cases} D_{\tilde{\Lambda}_m} := D + ad_{\tilde{\Lambda}_m} \\ \nabla_{j_\omega} := d + ad_{j_\omega} \end{cases} \quad (1)$$

and cannot be generally solved for A_ω due to the lack of linear terms. This issue is a result of local H_R -invariance and forces a case-by-case study.

2 Solving the EOM - two simple examples

Exploiting the solution for $p_\omega, m_\omega, q_\omega$, equation (1) can be rearranged as

$$W(A_\omega) + Z(\star A_\omega) = \zeta \quad \text{with} \quad W, Z : \mathfrak{h} \rightarrow \mathfrak{h}, \quad A_\omega, \zeta \in \Omega^1(\Sigma, \mathfrak{h}) \quad \text{and} \quad \begin{cases} W := ad_{\tilde{\Lambda}_h} + N + (D_{\tilde{\Lambda}_m} - M^\dagger) \circ \sum_{k=0}^{\infty} S^{2k+1} \circ (D_{\tilde{\Lambda}_m} + M) \\ Z := (D_{\tilde{\Lambda}_m} - M^\dagger) \circ \sum_{k=0}^{\infty} S^{2k} \circ (D_{\tilde{\Lambda}_m} + M) \\ \zeta := d\tilde{\Lambda}_h + \xi + (D_{\tilde{\Lambda}_m} - M^\dagger) \circ (\star + S) \circ \sum_{k=0}^{\infty} S^{2k} \circ (d\tilde{\Lambda}_m + \chi) \end{cases} \quad (2)$$

Where we defined $S := ad_{\tilde{\lambda}_p} + L$, with $L := ad_{\tilde{\lambda}_p} \circ \mathcal{O}_1 + ad_{\tilde{\lambda}_q} \circ \mathcal{O}_2$, and

$$\begin{cases} M := ad_{\tilde{\lambda}_p} \circ \mathcal{O}_3 + ad_{\tilde{\lambda}_q} \circ \mathcal{O}_4 \\ N := ad_{\tilde{\lambda}_q} \circ \mathcal{O}_3 + ad_{\tilde{\lambda}_p} \circ \mathcal{O}_4 \\ \xi := \mathcal{O}_3^\dagger(d\tilde{\Lambda}_q) + \mathcal{O}_4^\dagger(d\tilde{\Lambda}_p) \\ \chi := \mathcal{O}_1^\dagger(d\tilde{\Lambda}_q) + \mathcal{O}_2^\dagger(d\tilde{\Lambda}_p) \end{cases} \quad \begin{cases} \mathcal{O}_1 := R_{11} \circ ad_{\Lambda_q} + R_{12} \circ ad_{\Lambda_p} \\ \mathcal{O}_2 := R_{21} \circ ad_{\Lambda_q} + R_{22} \circ ad_{\Lambda_p} \\ \mathcal{O}_3 := R_{12} \circ ad_{\Lambda_q} + R_{11} \circ ad_{\Lambda_p} \\ \mathcal{O}_4 := R_{22} \circ ad_{\Lambda_q} + R_{21} \circ ad_{\Lambda_p} \end{cases} \quad (3)$$

We refer to [1] for the expressions of R_{ij} and clarify that any \mathcal{O}^\dagger is defined, with respect to the inner product $\langle \mathcal{O}^\dagger(X), Y \rangle = \langle X, \mathcal{O}(Y) \rangle$ for any two 1-forms X, Y , using $\langle R_{12}(X), Y \rangle = -\langle X, R_{12}(Y) \rangle$, $\langle R_{21}(X), Y \rangle = -\langle X, R_{21}(Y) \rangle$, $\langle R_{11}(X), Y \rangle = -\langle X, R_{22}(Y) \rangle$. As anticipated, all the above formulae reduce to those for symmetric spaces, in which $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{m}$, by simply setting to zero any element in \mathfrak{p} and \mathfrak{q} . In symmetric supercosets, \mathfrak{h} and \mathfrak{m} contain both bosonic and fermionic generators.

The possibility of solving (2) for A_ω , depends on the invertibility of W and Z and in turn on the structure of the algebra. This potential obstruction has been mentioned in [7, 5, 6] and problems in the presence of fermions have been discussed in [8]. We now highlight three situations in which the equation can be solved: in the first two cases W and $1 \pm ZW^{-1}$ or Z and $1 \pm WZ^{-1}$ are invertible, while in the third one neither W nor Z is invertible, but their sum and difference are

$$A_\omega = \frac{1}{2}(\zeta + \star\zeta)B_+ + \frac{1}{2}(\zeta - \star\zeta)B_- \quad \text{with} \quad \begin{cases} B_\pm := W^{-1}[(1 \pm ZW^{-1})^{-1}] \\ B_\pm := \pm Z^{-1}[(1 \pm WZ^{-1})^{-1}] \\ B_\pm := (W \pm Z)^{-1} \end{cases} \quad (4)$$

Writing $A_\omega = \star\alpha + \beta$ with $\alpha := \frac{1}{2}(\zeta_+ B_+ - \zeta_- B_-)$, $\beta := \frac{1}{2}(\zeta_+ B_+ + \zeta_- B_-)$ and $\zeta_\pm := d\tilde{\Lambda}_p + \xi \pm (D_{\tilde{\Lambda}_m} - M^\dagger) \circ \sum_{k=0}^\infty (\pm S)^k \circ (d\tilde{\Lambda}_m + \chi)$ defined as $(1 \pm \star)\zeta = (1 \pm \star)\zeta_\pm$, one obtains the full T-dual action $\tilde{S}_{SS} = \int_\Sigma \tilde{g} + \tilde{B}$ from the hybrid one in [1]

$$\tilde{g} := \frac{1}{2} \langle \lambda_-, \frac{1}{1-S} \star \lambda_+ \rangle - \langle \nabla_\beta \tilde{\Lambda}_p + \mathcal{O}_3^\dagger(\nabla_\beta \tilde{\Lambda}_q) + \mathcal{O}_4^\dagger(\nabla_\beta \tilde{\Lambda}_p), \star \alpha \rangle \quad (5)$$

$$\begin{aligned} \tilde{B} := & \frac{1}{2} \langle \lambda_-, \frac{1}{1-S} \lambda_+ \rangle + \langle \tilde{\Lambda}_p, F_\beta - \frac{1}{2}[\alpha, \alpha] \rangle + \frac{1}{2} \langle \alpha, N(\alpha) \rangle + \\ & + \frac{1}{2} \langle \nabla_\beta \tilde{\Lambda}_p, R_{21}(\nabla_\beta \tilde{\Lambda}_p) + R_{22}(\nabla_\beta \tilde{\Lambda}_q) \rangle + \frac{1}{2} \langle \nabla_\beta \tilde{\Lambda}_q, R_{11}(\nabla_\beta \tilde{\Lambda}_p) + R_{12}(\nabla_\beta \tilde{\Lambda}_q) \rangle \end{aligned}$$

with $\lambda_\pm := \nabla_{\beta \pm \alpha} \tilde{\Lambda}_m + \mathcal{O}_1^\dagger(\nabla_{\beta \pm \alpha} \tilde{\Lambda}_q) + \mathcal{O}_2^\dagger(\nabla_{\beta \pm \alpha} \tilde{\Lambda}_p) - D(\beta \pm \alpha)$.

We shall now solve (2) in two examples. The first one involves dualisation of the $SO(4)$ isometry of the coset space $SO(4)/SO(3) \simeq S^3$. This model has been studied in the literature [9, 7] and it is thus interesting to understand how the procedure goes through from the purely algebraic point of view. Given the $SO(4)$ algebra

$$[R_{IJ}, R_{KL}] = -\frac{i}{2}(\delta_{IK}R_{JL} - \delta_{JK}R_{IL} - \delta_{IL}R_{JK} + \delta_{JL}R_{IK}) \quad (6)$$

one can separate the $\mathfrak{h} = SO(3) = \{H_i := -\frac{1}{2}\varepsilon_i{}^{jk}R_{jk}\}$ subalgebra from the rest $\mathfrak{m} = \{M_i := R_{i4}\}$ by using indices $I, J = \{1, 2, 3, 4\}$ and $i, j = \{1, 2, 3\}$. Indices are raised

and lowered using the Euclidean metric δ_{IJ} . Generators in \mathfrak{h} and \mathfrak{m} satisfy

$$[H_i, H_j] = \frac{i}{2} \varepsilon_{ij}{}^k H_k \quad [M_i, H_j] = \frac{i}{2} \varepsilon_{ij}{}^k M_k \quad [M_i, M_j] = \frac{i}{2} \varepsilon_{ij}{}^k H_k \quad (7)$$

Writing the multipliers, i.e. the dual coordinate, as $\Lambda_{\mathfrak{h}} := \tilde{y}^i H_i$ and $\Lambda_{\mathfrak{m}} := \tilde{x}^i M_i$, recalling that $A_{\omega} := A_{\omega}^i H_i$ and using the above commutators, one finds

$$\begin{aligned} W(A_{\omega}) &= v^i \varepsilon_{ij}{}^k A_{\omega}^j H_k \quad \text{with} \quad v^i := \left[\frac{i}{2} \tilde{y}^i + \frac{i(\tilde{x} \cdot \tilde{y})}{2(4 - \tilde{y}^2)} \tilde{x}^i \right] \\ Z(\star A_{\omega}) &= \frac{1}{4(\tilde{y}^2 - 4)} \{ [(\tilde{y}^2 - 4)\tilde{x}^2 - (\tilde{y} \cdot \tilde{x})^2] \delta_j^k + \\ &\quad + [(\tilde{y} \cdot \tilde{x})\tilde{y}_j - (\tilde{y}^2 - 4)\tilde{x}_j] \tilde{x}^k + [(\tilde{y} \cdot \tilde{x})\tilde{x}_j - \tilde{x}^2 \tilde{y}_j] \tilde{y}^k \} \star A_{\omega}^j H_k \end{aligned} \quad (8)$$

It is clear that v_k and \tilde{x}_k respectively lie in the kernels of W_j^k and Z_j^k , but it is also not hard to find that $W \pm Z$ can be inverted, so that $B_{\pm} := (W \pm Z)^{-1}$ reads

$$\begin{aligned} (B_{\pm})_k{}^l &= a_1^{\pm} \delta_k^l + \tilde{x}_k (a_2^{\pm} \tilde{x}^l + a_3^{\pm} \tilde{y}^l) + \tilde{y}_k (a_4^{\pm} \tilde{x}^l + a_5^{\pm} \tilde{y}^l) + \varepsilon_{ak}{}^l (a_6^{\pm} \tilde{x}^a + a_7^{\pm} \tilde{y}^a) + \\ &\quad + \tilde{x}^a \tilde{y}^b \varepsilon_{ab}{}^l (a_8^{\pm} \tilde{x}_k + a_9^{\pm} \tilde{y}_k) + \tilde{x}^a \tilde{y}^b \varepsilon_{abk} (a_{10}^{\pm} \tilde{x}^l + a_{11}^{\pm} \tilde{y}^l) + a_{12}^{\pm} \tilde{x}^a \tilde{y}^b \varepsilon_{abk} \tilde{x}^c \tilde{y}^d \varepsilon_{cd}{}^l \end{aligned} \quad (9)$$

with $a_1^{\pm}, \dots, a_{12}^{\pm}$ complicated functions of $\tilde{x}^2, \tilde{y}^2, (\tilde{x} \cdot \tilde{y})$ which we omit for brevity.

The second example has to do with super T-dualisation of the $OSp(1|2)$ isometry of the semi-symmetric space $OSp(1|2)/SO(1, 1)$. The interest in such a coset, which has also been considered in the context of holography [10], is due to its structure, which is that of a 2d Green-Schwarz string sigma model satisfying the torsion constraints of supergravity. For this reason, dualising such a model would not only represent a natural next step to the super T-dualisation of the Principal Chiral Model on $OSp(1|2)$, for which in [1] it was argued that T-duality breaks the supergravity constraint, but also a concrete example of super T-duality on supercosets, that could be compared to [4, 5, 6]. Given the $OSp(1|2)$ algebra in light-cone notation

$$\begin{aligned} \{Q_{\pm}, Q_{\pm}\} &= L_{\pm\pm} & \{Q_+, Q_-\} &= L_{+-} & [L_{\pm\pm}, Q_{\mp}] &= \mp i Q_{\pm} \\ [L_{+-}, L_{\pm\pm}] &= \pm i L_{\pm\pm} & [L_{++}, L_{--}] &= -2i L_{+-} & [L_{+-}, Q_{\pm}] &= \pm \frac{i}{2} Q_{\pm} \end{aligned} \quad (10)$$

the four subspaces are $\mathfrak{h} = \{L_{+-}\}$ $\mathfrak{p} = \{Q_+\}$ $\mathfrak{m} = \{L_{++}, L_{--}\}$ $\mathfrak{q} = \{Q_-\}$. Then, writing the Lagrange multipliers, i.e. the dual coordinates, as $\Lambda_{\mathfrak{h}} := \tilde{y} L_{+-}$, $\Lambda_{\mathfrak{p}} := \tilde{\theta}^+ Q_+$, $\Lambda_{\mathfrak{m}} := \tilde{x}^{++} L_{++} + \tilde{x}^{--} L_{--}$, $\Lambda_{\mathfrak{q}} := \tilde{\theta}^- Q_-$ and recalling that $A_{\omega} := A_{\omega}^{+-} L_{+-}$, one can exploit the above commutators to compute $W(A_{\omega}) = 0$ and

$$Z(\star A_{\omega}) = \frac{4\tilde{x}^{++}\tilde{x}^{--}}{1 + \tilde{y}^2} \left[1 + \frac{4i\tilde{\theta}^+\tilde{\theta}^-}{(1 - i\tilde{y})[4\tilde{x}^{++}\tilde{x}^{--} + (1 + i\tilde{y})^2]} \right] \star A_{\omega}^{+-} L_{+-} \quad (11)$$

Equation (2) can thus be immediately solved as in (4) with $B_{\pm} := \pm Z^{-1}$ and

$$Z^{-1} = \frac{1 + \tilde{y}^2}{4\tilde{x}^{++}\tilde{x}^{--}} \left[1 - \frac{4i\tilde{\theta}^+\tilde{\theta}^-}{(1 - i\tilde{y})[4\tilde{x}^{++}\tilde{x}^{--} + (1 + i\tilde{y})^2]} \right] \quad (12)$$

3 Conclusions and Outlook

We focused on a delicate step in the T-dualisation procedure of symmetric and semi-symmetric cosets, namely solving the EOM for the gauge fields ω in the \mathfrak{h} subspace. We highlighted how such EOM is not necessarily solvable due to the lack of a linear term in A_ω : this may heavily affect T-duality, as the removal of A_ω is necessary to write down the full dual action. Rewriting the EOM as in (2), three cases can be recognised in which the solution is of the form (4). This has been used to write the full T-dual action and construct solutions for two explicit examples.

An important step toward a better comprehension of T-dualisation would be represented by a deeper understanding of the EOM (2) and of the constraints its resolution imposes on the underlying algebra. Additionally, given the physical relevance of the coset $OSp(1|2)/SO(1,1)$ and the simple result (12) for its EOM, it would certainly be very interesting to carry out the dualisation of such model in full details, so as to have a concrete example of super non-Abelian T-duality on supercosets, which could hopefully serve as a base model in view of more complicated ones.

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