Product Structure of Graph Classes with Bounded Treewidth

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Abstract We show that many graphs with bounded treewidth can be described as subgraphs of the strong product of a graph with smaller treewidth and a bounded-size complete graph. To this end, define the *underlying treewidth* of a graph class $\mathscr{G}$ to be the minimum non-negative integer c such that, for some function f, for every graph $G \in \mathscr{G}$ there is a graph H with $\text{tw}(H) \leqslant c$ such that G is isomorphic to a subgraph of $H \boxtimes K_{f(\text{tw}(G))}$. We introduce disjointed coverings of graphs and show they determine the underlying treewidth of any graph class. Using this result, we prove that the class of planar graphs has underlying treewidth 3; the class of $K_{s,t}$-minor-free graphs has underlying treewidth s (for $t \geqslant \max\{s,3\}$); and the class of K_t-minor-free graphs has underlying treewidth $t-2$. In general, we prove that a monotone class has bounded underlying treewidth if and only if it excludes some fixed topological minor. We also study the underlying treewidth of graph classes defined by an excluded subgraph or excluded induced subgraph. We show that the class of graphs with no H subgraph has bounded underlying treewidth if and only if every component of H is a subdivided star, and that the class of graphs with no in-

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duced H subgraph has bounded underlying treewidth if and only if every component of H is a star.

1 Introduction

Graph product structure theory describes complicated graphs as subgraphs of strong products[1] of simpler building blocks. The building blocks typically have bounded treewidth, which is the standard measure of how similar a graph is to a tree. Examples of graphs classes that can be described this way include planar graphs [14, 29], graphs of bounded Euler genus [10, 14], graphs excluding a fixed minor [14], and various non-minor-closed classes [15, 21]. These results have been the key to solving several open problems regarding queue layouts [14], nonrepetitive colouring [13], p-centered colouring [11], adjacency labelling [12, 18], twin-width [1, 5], and comparable box dimension [16].

This paper shows that graph product structure theory can even be used to describe graphs of bounded treewidth in terms of simpler graphs. Here the building blocks are graphs of smaller treewidth and complete graphs of bounded size. For example, a classical theorem by the referee of [8] can be interpreted as saying that every graph G of treewidth k and maximum degree Δ is contained[2] in $T \boxtimes K_{O(k\Delta)}$ for some tree T.

This result motivates the following definition. The *underlying treewidth* of a graph class $\mathscr{G}$ is the minimum $c \in \mathbb{N}_0$ such that, for some function f, for every graph $G \in \mathscr{G}$ there is a graph H with $\mathrm{tw}(H) \leqslant c$ such that G is contained in $H \boxtimes K_{f(\mathrm{tw}(G))}$. If there is no such c, then $\mathscr{G}$ has *unbounded* underlying treewidth. We call f the *treewidth-binding function*. For example, the above-mentioned result in [8] says that any graph class with bounded degree has underlying treewidth at most 1 with treewidth-binding function $O(k)$.

This paper introduces disjointed coverings of graphs and shows that they are intimately related to underlying treewidth; see Section 3. Indeed, we show that disjointed coverings characterise the underlying treewidth of any graph class (Theorem 7). The remainder of the paper uses disjointed coverings to determine the underlying treewidth of several graph classes of interest, with a small treewidth-binding function as a secondary goal. In this extended abstract, most proofs are omitted; see [6] for all the details.

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[1] The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex-set $V(A) \times V(B)$, where distinct vertices $(v,x),(w,y) \in V(A) \times V(B)$ are adjacent if $v = w$ and $xy \in E(B)$, or $x = y$ and $vw \in E(A)$, or $vw \in E(A)$ and $xy \in E(B)$.

[2] A graph G is *contained* in a graph X if G is isomorphic to a subgraph of X.

2 Preliminaries

2.1 Basic Definitions

See [7] for graph-theoretic definitions not given here. We consider simple, finite, undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$. A *clique* in a graph is a set of pairwise adjacent vertices. Let $\omega(G)$ be the size of the largest clique in a graph G.

A *graph class* is a collection of graphs closed under isomorphism. A graph class is *monotone* if it is closed under taking subgraphs. A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A graph G is *H-minor-free* if H is not a minor of G. A graph class $\mathscr{G}$ is *minor-closed* if every minor of each graph in $\mathscr{G}$ is also in $\mathscr{G}$.

Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, \dots\}$. All logarithms in this paper are binary.

2.2 Tree-Decompositions

For a tree T, a *T-decomposition* of a graph G is a collection $\mathscr{W} = (W_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of T such that (i) for every edge $vw \in E(G)$, there exists a node $x \in V(T)$ with $v, w \in W_x$; and (ii) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in W_x\}$ induces a (connected) subtree of T. Each set W_x in $\mathscr{W}$ is called a *bag*. The *width* of $\mathscr{W}$ is $\max\{|W_x| : x \in V(T)\} - 1$. A *tree-decomposition* is a T-decomposition for any tree T. The *treewidth* $\operatorname{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G. Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [2, 20, 27] for surveys.

We use the following well-known facts about treewidth. Every minor H of a graph G satisfies $\operatorname{tw}(H) \leqslant \operatorname{tw}(G)$. In every tree-decomposition of a graph G, each clique of G appears in some bag. Thus $\operatorname{tw}(G) \geqslant \omega(G) - 1$ and $\operatorname{tw}(K_n) = n - 1$. If $\{v_1, \dots, v_k\}$ is a clique in a graph G_1 and $\{w_1, \dots, w_k\}$ is a clique in a graph G_2, and G is the graph obtained from the disjoint union of G_1 and G_2 by identifying v_i and w_i for each $i \in \{1, \dots, k\}$, then $\operatorname{tw}(G) = \max\{\operatorname{tw}(G_1), \operatorname{tw}(G_2)\}$. A greedy algorithm shows that $\chi(G) \leqslant \operatorname{tw}(G) + 1$.

2.3 Partitions

To describe our main results in Section 1, it is convenient to use the language of graph products. However, to prove our results, it is convenient to work with the equivalent notion of graph partitions, which we now introduce.

For graphs G and H, an *H-partition* of G is a partition $(V_x : x \in V(H))$ of $V(G)$ indexed by the nodes of H, such that for every edge vw of G, if $v \in V_x$ and $w \in V_y$, then $x = y$ or $xy \in E(H)$. We say that H is the *quotient* of such a partition. The *width* of an H-partition is $\max\{|V_x| : x \in V(H)\}$. For $c \in \mathbb{N}_0$, an H-partition where $\text{tw}(H) \leqslant c$ is called a *c-tree-partition*. The *c-tree-partition-width* of a graph G, denoted $\text{tpw}_c(G)$, is the minimum width of a c-tree-partition of G.

It follows from the definitions that a graph G has an H-partition of width at most ℓ if and only if G is contained in $H \boxtimes K_\ell$. Thus, $\text{tpw}_c(G)$ equals the minimum $\ell \in \mathbb{N}_0$ such that G is contained in $H \boxtimes K_\ell$ for some graph H with $\text{tw}(H) \leqslant c$. Hence, the underlying treewidth of a graph class $\mathscr{G}$ equals the minimum $c \in \mathbb{N}_0$ such that, for some function f, every graph $G \in \mathscr{G}$ has c-tree-partition-width at most $f(\text{tw}(G))$. We henceforth use this as our working definition of underlying treewidth.

Before continuing, we review work on the $c = 1$ case. A *tree-partition* is a T-partition for some tree T. The *tree-partition-width* of G, denoted by $\text{tpw}(G)$, is the minimum width of a tree-partition of G. Thus $\text{tpw}(G) = \text{tpw}_1(G)$, which equals the minimum $\ell \in \mathbb{N}_0$ for which G is contained in $T \boxtimes K_\ell$ for some tree T. Tree-partitions were independently introduced by Seese [28] and Halin [19], and have since been widely investigated [3, 4, 8, 9, 17, 30, 31].

Bounded tree-partition-width implies bounded treewidth, as noted by Seese [28]. This fact easily generalises for c-tree-partition-width:

$$\text{tw}(G) \leqslant (c+1)\,\text{tpw}_c(G) - 1.$$

Of course, $\text{tw}(T) = \text{tpw}(T) = 1$ for every tree T. But in general, $\text{tpw}(G)$ can be much larger than $\text{tw}(G)$. For example, fan graphs on n vertices have treewidth 2 and tree-partition-width $\Omega(\sqrt{n})$; see Lemma 8 below. On the other hand, the referee of [8] showed that if the maximum degree and treewidth are both bounded, then so is the tree-partition-width, which is one of the most useful results about tree-partitions.

Lemma 1 ([8])**.** *For $k, \Delta \in \mathbb{N}$, every graph of treewidth less than k and maximum degree at most Δ has tree-partition-width at most $24k\Delta$.*

This bound is best possible up to the multiplicative constant [31]. Note that bounded maximum degree is not necessary for bounded tree-partition-width (for example, stars). Ding and Oporowki [9] characterised graph classes with bounded tree-partition-width in terms of excluded topological minors. We give an alternative characterisation, which says that graph classes with bounded tree-partition-width are exactly those that have bounded treewidth and satisfy a further 'disjointedness' condition. Furthermore, this result naturally generalises for c-tree-partition-width and thus for underlying treewidth.

3 Disjointed Coverings

This section introduces disjointed coverings and shows that they can be used to characterise bounded c-tree-partition-width and underlying treewidth.

An *ℓ-covering* of a graph G is a set $\beta \subseteq 2^{V(G)}$ such that $|B| \leqslant \ell$ for every $B \in \beta$, and $\cup\{B\colon B \in \beta\} = V(G)$.[3] If $B_1 \cap B_2 = \varnothing$ for all distinct $B_1, B_2 \in \beta$, then β is an *ℓ-partition*. As illustrated in Figure 1, an ℓ-covering β of a graph G is *(c,d)-disjointed* if for every c-tuple $(B_1,\dots,B_c) \in \beta^c$ and every component X of $G-(B_1 \cup \dots \cup B_c)$ there exists $Q \subseteq V(X)$ with $|Q| \leqslant d$ such that for each component Y of $X-Q$, for some $i \in \{1,\dots,c\}$ we have $V(Y) \cap N_G(B'_i) = \varnothing$, where $B'_i := B_i \setminus (B_1 \cup \dots \cup B_{i-1})$. Note that we can take $Q = \varnothing$ if some $B'_i = \varnothing$, since $N_G(\varnothing) = \varnothing$.

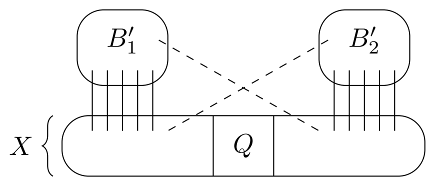

Fig. 1 A disjointed partition with $c = 2$, where non-edges are dashed.

Given positive integers ℓ and t and an ℓ-covering β of a graph G, we define $\beta[t] := \{\bigcup \mathscr{B}\colon \mathscr{B} \subseteq \beta, |\mathscr{B}| \leqslant t\}$. So $\beta[t]$ is a $t\ell$-covering of G. For a function $f\colon \mathbb{N} \to \mathbb{R}^+$ we say that β is *(c,f)-disjointed* if $\beta[t]$ is $(c,f(t))$-disjointed for every $t \in \mathbb{N}$.

While (c,d)-disjointed coverings are conceptually simpler than (c,f)-disjointed coverings, we show they are roughly equivalent (Theorem 3). Moreover, (c,f)-disjointed coverings are essential for the main proof (Lemma 6) and give better bounds on the c-tree-partition-width, leading to smaller treewidth-binding functions when determining the underlying treewidth of several graph classes of interest (for K_t-minor-free graphs for example).

Note that we often consider the singleton partition $\beta := \{\{v\}\colon v \in V(G)\}$ of a graph G, which is (c,f)-disjointed if and only if, for every $t \in \mathbb{N}$, every t-partition of G is $(c,f(t))$-disjointed.

This section characterises c-tree-partition-width in terms of (c,d)-disjointed coverings (or partitions) and (c,f)-disjointed coverings (or partitions). The following observation deals with the $c = 0$ case.

Observation 2. *The following are equivalent for any graph G and $d \in \mathbb{N}$:*

(a) *G has a $(0,d)$-disjointed covering*;
(b) *every covering of G is $(0,d)$-disjointed*;
(c) *each component of G has at most d vertices*;
(d) *G has 0-tree-partition-width at most d.*

Observation 2 implies that a graph class $\mathscr{G}$ has underlying treewidth 0 if and only if there is a function f such that every component of every graph $G \in \mathscr{G}$ has at most $f(\operatorname{tw}(G))$ vertices.

[3] Our definition of ℓ-covering differs from the standard usage where it refers to a covering in which each element of the ground set is covered ℓ times.

We prove the following characterisation of bounded c-tree-partition-width (which is new even in the $c=1$ case).

Theorem 3. *For fixed $c \in \mathbb{N}_0$, the following are equivalent for a graph class $\mathscr{G}$ with bounded treewidth:*

(a) *$\mathscr{G}$ has bounded c-tree-partition-width*;
(b) *for some $d,\ell \in \mathbb{N}$, every graph in $\mathscr{G}$ has a (c,d)-disjointed ℓ-partition*;
(c) *for some $d,\ell \in \mathbb{N}$, every graph in $\mathscr{G}$ has a (c,d)-disjointed ℓ-covering*;
(d) *for some $\ell \in \mathbb{N}$ and function f, every graph in $\mathscr{G}$ has a (c,f)-disjointed ℓ-partition*;
(e) *for some $\ell \in \mathbb{N}$ and function f, every graph in $\mathscr{G}$ has a (c,f)-disjointed ℓ-covering.*

Proof. Observation 2 handles the $c=0$ case. Now assume that $c \geqslant 1$. Lemma 5 below says that (a) implies (b). Since every ℓ-partition is an ℓ-covering, (b) implies (c), and (d) implies (e). Lemma 4 below says that (c) implies (d). Finally, Lemma 6 below says that (e) implies (a). ☐

By definition, every (c,f)-disjointed ℓ-covering is $(c,f(1))$-disjointed. The next lemma gives a qualitative converse to this.

Lemma 4. *Let $\ell,c,d \in \mathbb{N}$, and let β be a (c,d)-disjointed ℓ-covering of a graph G. Then β is (c,f)-disjointed, where $f(t) := dt^c$ for each $t \in \mathbb{N}$.*

Proof. Fix $t \in \mathbb{N}$. Let $B_1,\dots,B_c \in \beta[t]$. Let X be a component of $G-(B_1\cup\dots\cup B_c)$. For each $i \in \{1,\dots,c\}$, let $\mathscr{B}_i$ be a set of at most t elements of β whose union is B_i. Let $\mathscr{F} := \mathscr{B}_1 \times \dots \times \mathscr{B}_c$, and for each $y=(A_1,\dots,A_c) \in \mathscr{F}$, define Q_y as follows. Let X_y the component of $G-(A_1\cup\dots\cup A_c)$ containing X. Since β is (c,d)-disjointed, there exists $Q_y \subseteq V(X_y)$ of size at most d such that for every component Y of $X_y - Q_y$ there is some $i \in \{1,\dots,c\}$ such that $V(Y)\cap N_G(A_i \setminus (A_1\cup\dots\cup A_{i-1}))$ is empty. Now let $Q := \bigcup_{y\in\mathscr{F}} Q_y$, and note that $|Q| \leqslant d|\mathscr{F}| \leqslant dt^c$.

Suppose for contradiction that there exists a component Y of $X-Q$ such that for all $i \in \{1,\dots,c\}$, there is a vertex $b_i \in N_G(Y)\cap B'_i$, where $B'_i := B_i \setminus (B_1\cup\dots\cup B_{i-1})$. Let $y=(A_1,\dots,A_c) \in \mathscr{F}$ be such that $(b_1,\dots,b_c) \in A_1\times\dots\times A_c$, and consider that component Y' of $X_y - Q_y$ containing Y. By the definition of Q_y, there is some $i \in \{1,\dots,c\}$ such that Y' contains no neighbour of a vertex in $A_i \setminus (A_1 \cup \dots \cup A_{i-1})$. In particular, all neighbours of vertices of Y are either vertices of Y' or neighbours of vertices of Y', so b_i is not a neighbour of any vertex of Y, a contradiction. ☐

Having a (c,d)-disjointed partition is necessary for bounded c-tree-partition-width.

Lemma 5. *For all $c,\ell \in \mathbb{N}_0$, every graph G with c-tree-partition-width ℓ has a $(c,c\ell)$-disjointed ℓ-partition.*

Proof. By assumption, G has an H-partition $\beta = (V_h : h \in V(H))$ where H is a graph of treewidth at most c and $|V_h| \leqslant \ell$ for all h. We first show that the singleton partition of H is (c,c)-disjointed. Let $v_1, \dots, v_c \in V(H)$ and let X be a component of $H - \{v_1, \dots, v_c\}$. Let $(W_x : x \in V(T))$ be a tree-decomposition of H where $|W_x| \leqslant c+1$ for all $x \in V(T)$. We may assume that $W_x \neq W_y$ whenever $x \neq y$. For each $i \in \{1, \dots, c\}$, let T_i be the subtree of T induced by $\{x \in V(T) : v_i \in W_x\}$.

First suppose that $V(T_i) \cap V(T_j) = \varnothing$ for some $i, j \in \{1, \dots, c\}$. Let $z \in V(T_i)$ be the closest node (in T) to T_j. Let $Q := W_z \cap X$. Note that $Q \subseteq W_z \setminus \{v_i\}$ so $|Q| \leqslant c$. Any path from v_i to v_j in H passes through W_z, so each component of $X - Q$ is disjoint from $N_H(v_i)$ or $N_H(v_j)$.

Now assume that $V(T_i) \cap V(T_j) \neq \varnothing$ for all $i, j \in \{1, \dots, c\}$. Let T_X be the subgraph of T induced by $\{x \in V(T) : V(X) \cap W_x \neq \varnothing\}$. Since X is connected, T_X is a subtree of T. Suppose that $V(T_i) \cap V(T_X) = \varnothing$ for some i. Since every neighbour of v_i in H is in $\bigcup(W_x : x \in V(T_i))$, it follows that $N_H(v_i) \cap V(X) = \varnothing$ and so we may take $Q := \varnothing$ in this case. Now assume that $V(T_i) \cap V(T_X) \neq \varnothing$ for all $i \in \{1, \dots, c\}$. By the Helly property, $\tilde{T} := T_1 \cap \dots \cap T_c \cap T_X$ is a non-empty subtree of T. For each $x \in V(\tilde{T})$, we have $|W_x| \leqslant c+1$ and so $W_x = \{v_1, \dots, v_c, u\}$ for some $u \in V(X)$. First assume $|V(\tilde{T})| \geqslant 2$. Then there are adjacent $x, y \in V(\tilde{T})$ with $W_x = \{v_1, \dots, v_c, u\}$ and $W_y = \{v_1, \dots, v_c, v\}$ for $u, v \in V(X)$. Since $W_x \neq W_y$, we have $u \neq v$ and thus there is no (u,v)-path in $H - \{v_1, \dots, v_c\}$, contradicting the connectedness of X. Hence $\tilde{T}$ consists of a single vertex z; thus $W_z = \{v_1, \dots, v_c, u\}$ for some $u \in V(X)$. Let $Q := \{u\}$ and consider a component Y of $X - Q$. Let T_Y be the subtree of T induced by $\{y \in V(T) : V(Y) \cap W_y \neq \varnothing\}$. Since T_Y is connected and does not contain z, it is disjoint from some T_i. As above, $N_H(v_i) \cap V(Y) = \varnothing$, as required.

We have shown that the singleton partition of H is (c,c)-disjointed. Now focus on G. By assumption, β is an ℓ-partition of G. Let $V_{v_1}, \dots, V_{v_c}$ be parts in β, and let X be a component of $G - (V_{v_1} \cup \dots \cup V_{v_c})$. Then $X \subseteq \bigcup\{V_h : h \in X'\}$ where X' is a component of $H - \{v_1, \dots, v_c\}$. Since H is (c,c)-disjointed, there exists $Q' \subseteq V(X')$ of size at most c such that each component $X' - Q'$ is disjoint from some $N_H(v_i)$. Let $Q := \bigcup\{V_h : h \in Q'\}$, which has size at most $c\ell$. Each component of $X - Q$ is disjoint from some $N_G(V_{v_i})$. □

The next lemma lies at the heart of the paper.

Lemma 6. *Let $k, c, \ell \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{R}^+$. For any graph G, if* $\operatorname{tw}(G) < k$ *and G has a (c,f)-disjointed ℓ-covering, then G has c-tree-partition-width at most* $\max\{12\ell k, 2c\ell f(12k)\}$.

Lemmas 4 and 6 imply the following result:

Corollary 1. *Let $k, c, d, \ell \in \mathbb{N}$. For any graph G, if* $\operatorname{tw}(G) < k$ *and G has a (c,d)-disjointed ℓ-covering, then G has c-tree-partition-width at most* $2cd\ell(12k)^c$.

Observe that the singleton partition of any graph with maximum degree Δ is $(1,\Delta)$-disjointed. So Corollary 1 with $c = \ell = 1$ and $d = \Delta$ implies Lemma 1 (even with the same constant 24). Indeed, the proof of Lemma 6 in the case of graphs with bounded degree is equivalent to the proof of Lemma 1.

To conclude this section, Lemmas 5 and 6 imply the following characterisation of underlying treewidth.

Theorem 7. *The underlying treewidth of a graph class $\mathscr{G}$ is equal to the minimum $c \in \mathbb{N}_0$ such that, for some function $g\colon \mathbb{N} \to \mathbb{N}$, every graph $G \in \mathscr{G}$ has a $(c, g(\operatorname{tw}(G)))$-disjointed $g(\operatorname{tw}(G))$-partition.*

4 Lower Bounds

We now define two graphs that provide lower bounds on the underlying treewidth of various graph classes. For a graph G and $\ell \in \mathbb{N}$, let ℓG be the union of ℓ vertex-disjoint copies of G. Let $\widehat{G}$ be the graph obtained from G by adding one dominant vertex. Observe that $\operatorname{tw}(\widehat{G}) = \operatorname{tw}(G)+1$ and $\operatorname{tw}(\ell G) = \operatorname{tw}(G)$ for any $\ell \in \mathbb{N}$, implying $\operatorname{tw}(\widehat{\ell G}) = \operatorname{tw}(G)+1$. For $c, \ell \in \mathbb{N}$, define graphs $G_{c,\ell}$ and $C_{c,\ell}$ recursively as follows. First, $G_{1,\ell} \coloneqq P_{\ell+1}$ is the path on $\ell+1$ vertices, and $C_{1,\ell} \coloneqq K_{1,\ell}$ is the star with ℓ leaves. Further, for $c \geqslant 2$, let $G_{c,\ell} \coloneqq \widehat{\ell G_{c-1,\ell}}$ and $C_{c,\ell} \coloneqq \widehat{\ell C_{c-1,\ell}}$.

The next lemma collects together some useful and well-known properties of $G_{c,\ell}$ and $C_{c,\ell}$.

Lemma 8. *For all $c, \ell \in \mathbb{N}$,*

(i) $\operatorname{tw}(G_{c,\ell}) = \operatorname{tw}(C_{c,\ell}) = c$;
(ii) *for any ℓ-partition of $G \in \{G_{c,\ell}, C_{c,\ell}\}$, there is a $(c+1)$-clique in G whose vertices are in distinct parts*;
(iii) *$G_{c,\ell}$ and $C_{c,\ell}$ both have $(c-1)$-tree-partition-width greater than ℓ*;
(iv) *$G_{2,\ell}$ is outerplanar, and $G_{3,\ell}$ is planar*;
(v) *$G_{c,\ell}$ is $K_{c,\max\{c,3\}}$-minor-free*;
(vi) *$C_{c,\ell}$ does not contain P_4 as an induced subgraph*;
(vii) *$C_{c,\ell}$ does not contain P_n as a subgraph for $n \geqslant 2^{c+1}$*.

Proof. Since $\operatorname{tw}(\widehat{\ell G}) = \operatorname{tw}(G)+1$ for any graph G and $\ell \in \mathbb{N}$, part (i) follows by induction.

We establish (ii) by induction on c. In the case $c = 1$, every ℓ-partition of $P_{\ell+1}$ or $K_{1,\ell}$ contains an edge whose endpoints are in different parts, and we are done. Now assume the claim for $c-1$ ($c \geqslant 2$) and let $G \in \{G_{c-1,\ell}, C_{c-1,\ell}\}$. Consider an ℓ-partition of $\widehat{\ell G}$. At most $\ell-1$ copies of G contain a vertex in the same part as the dominant vertex v. Thus, some copy G_0 of G contains no vertices in the same part as v. By induction, G_0 contains a c-clique K whose vertices are in distinct parts. Since v is dominant, $K \cup \{v\}$ satisfies the induction hypothesis.

Let $G \in \{G_{c,\ell}, C_{c,\ell}\}$. Consider an H-partition of G of width at most ℓ. By (ii), G contains a $(c+1)$-clique whose vertices are in distinct parts. So $\omega(H) \geqslant c+1$, implying $\operatorname{tw}(H) \geqslant c$. This establishes (iii).

Observe that $G_{2,\ell}$ is outerplanar (called a *fan* graph). The disjoint union of outerplanar graphs is outerplanar and the graph obtained from any outerplanar graph by adding a dominant vertex is planar; thus $G_{3,\ell}$ is planar.

We next show that $G_{c,\ell}$ is $K_{c,\max\{c,3\}}$-minor-free. $G_{1,\ell}$ is a path and so has no $K_{1,3}$-minor. $G_{2,\ell}$ is outerplanar and so has no $K_{2,3}$-minor. Let $c \geqslant 3$ and assume the result holds for smaller c. Suppose that $G_{c,\ell}$ contains a $K_{c,c}$-minor. Since $K_{c,c}$ is 2-connected, some copy of $G_{c-1,\ell}$ in $G_{c,\ell}$ contains a $K_{c-1,c}$-minor. This contradiction establishes (v).

We show that $C_{c,\ell}$ contains no induced P_4 by induction on c. First, $C_{1,\ell} = K_{1,\ell}$ does not contain P_4. Next, suppose that $C_{c-1,\ell}$ does not contain an induced P_4. P_4 does not have a dominant vertex and so any induced P_4 in $C_{c,\ell}$ must lie entirely within one copy of $C_{c-1,\ell}$. In particular, $C_{c,\ell}$ does not contain an induced P_4. This proves (vi).

Finally, Nešetřil and Ossona de Mendez [26] proved (vii). □

The underlying treewidth of the class of graphs of treewidth at most k is obviously at most k. Lemma 8 (i) and (iii) imply the following.

Corollary 2. *The underlying treewidth of the class of graphs of treewidth at most k equals k.*

Corollary 3. *The classes $\{G_{c,\ell}\colon c,\ell \in \mathbb{N}\}$ and $\{C_{c,\ell}\colon c,\ell \in \mathbb{N}\}$ both have unbounded underlying treewidth.*

Proof. Suppose that $\{G_{c,\ell}\colon c,\ell \in \mathbb{N}\}$ has underlying treewidth b. Thus, for some function f, for all $c,\ell \in \mathbb{N}$, we have $\operatorname{tpw}_b(G_{c,\ell}) \leqslant f(\operatorname{tw}(G_{c,\ell})) = f(c)$. In particular, with $c := b+1$ and $\ell := f(c)$, we have $\operatorname{tpw}_{c-1}(G_{c,\ell}) \leqslant \ell$, which contradicts Lemma 8 (iii). The proof for $\{C_{c,\ell}\colon c,\ell \in \mathbb{N}\}$ is analogous. □

5 Excluding a Minor

This section uses disjointed partitions to determine the underlying treewidth of several minor-closed classes of interest.

The next definition enables K_t-minor-free graphs and $K_{s,t}$-minor-free graphs to be handled simultaneously. For $s,t \in \mathbb{N}$, let $\mathscr{K}_{s,t}$ be the class of graphs G for which there is a partition $\{A,B\}$ of $V(G)$ such that $|A| = s$ and $|B| = t$; $vw \in E(G)$ for all $v \in A$ and $w \in B$; and $G[B]$ is connected. Obviously, every graph in $\mathscr{K}_{s,t}$ contains $K_{s,t}$. Similarly, we obtain K_t as a minor of any $G \in \mathscr{K}_{t-2,t}$ by contracting a matching between A and B of size $t-2$ whose end-vertices are distinct from the end-vertices of some edge of $G[B]$.

The next lemma is proved by a well-known technique [23, 24].

Lemma 9. *Let G be a graph with no minor in $\mathscr{K}_{s,t}$. Assume $\{A,B\}$ is a partition of $V(G)$ such that $G[B]$ is connected and every vertex in B has at least s neighbours in A. Then $|B| \leqslant \delta|A|$ for some $\delta = \delta(s,t)$.*

The following well-known Erdős–Pósa type result is useful for showing disjointedness.

Lemma 10. *Let $\mathscr{H}$ be a set of connected subgraphs of a graph G. Then, for every non-negative integer ℓ, either there are $\ell+1$ vertex-disjoint graphs in $\mathscr{H}$ or there is a set $Q \subseteq V(G)$ of size at most $\ell(\mathrm{tw}(G)+1)$ such that $G-Q$ contains no graph of $\mathscr{H}$.*

Lemma 11. *For fixed $s,t \in \mathbb{N}$, every graph G with no minor in $\mathscr{K}_{s,t}$ and of treewidth k has s-tree-partition-width $O(k^2)$.*

Proof. By Lemma 6 it suffices to show that the singleton partition of G is (s,f)-disjointed, where $f(n) := \delta sn(k+1)$ and $\delta := \delta(s,t)$ from Lemma 9. Let $S_1,\dots,S_s$ be subsets of $V(G)$ of size at most n, let $S := S_1 \cup \dots \cup S_s$, and for each $i \in \{1,\dots,s\}$ let $S_i' := S_i \setminus (S_1 \cup \dots \cup S_{i-1})$. Let X be a connected component of $G-S$. Let $\mathscr{H}$ be the set of connected subgraphs H of X such that $H \cap N(S_i') \neq \varnothing$ for all $i \in \{1,\dots,s\}$. Say $\mathscr{R}$ is a maximum-sized set of pairwise disjoint subgraphs in $\mathscr{H}$. We may assume that $\bigcup\{V(R)\colon R \in \mathscr{R}\} = V(X)$. Let X' be the graph obtained from $G[S \cup V(X)]$ by contracting each subgraph $R \in \mathscr{R}$ into a vertex v_R. So $V(X') = S \cup \{v_R\colon R \in \mathscr{R}\}$. Since X is connected, $\{v_R\colon R \in \mathscr{R}\}$ induces a connected subgraph of X'. By construction, in X', each vertex v_R has at least s neighbours in S. By Lemma 9, $|\mathscr{R}| \leqslant \delta|S|$. By Lemma 10, there is a set $Q \subseteq V(X)$ of size at most $\delta|S|(k+1) \leqslant f(n)$ such that $X-Q$ contains no graph in $\mathscr{H}$. Thus each component Y of $X-Q$ satisfies $V(Y) \cap N_G(S_i') = \varnothing$ for some $i \in \{1,\dots,s\}$. Hence, the singleton partition of G is (s,f)-disjointed. □

We now determine the underlying treewidth of K_t- and $K_{s,t}$-minor-free graphs.

Theorem 12. *For fixed $t \in \mathbb{N}$ with $t \geqslant 2$, the underlying treewidth of the class of K_t-minor-free graphs equals $t-2$. In particular, every K_t-minor-free graph of treewidth k has $(t-2)$-tree-partition-width $O(k^2)$.*

Proof. Since K_t is a minor of every graph in $\mathscr{K}_{t-2,t}$, Lemma 11 implies that every K_t-minor-free graph of treewidth k has $(t-2)$-tree-partition-width $O(k^2)$. Thus the underlying treewidth of the class of K_t-minor-free graphs is at most $t-2$. Suppose for contradiction that equality does not hold. That is, for some function f, every K_t-minor-free graph G has $(t-3)$-tree-partition-width at most $f(\mathrm{tw}(G))$. Let $\ell := f(t-2)$. The graph $G_{t-2,\ell}$ in Lemma 8 has treewidth $t-2$ and is thus K_t-minor-free. However, by Lemma 8, $\mathrm{tpw}_{t-3}(G_{t-2,\ell}) > \ell = f(t-2) = f(\mathrm{tw}(G_{t-2,\ell}))$, which is the required contradiction. □

A similar proof shows:

Theorem 13. *For fixed $s,t \in \mathbb{N}$ with $t \geqslant \max\{s,3\}$, the underlying treewidth of the class of $K_{s,t}$-minor-free graphs equals s. For $s,t \in \mathbb{N}$ with $s \leqslant t \leqslant 2$, the underlying treewidth of the class of $K_{s,t}$-minor-free graphs equals $s-1$.*

The class of planar graphs is minor-closed. Since planar graphs are K_5- and $K_{3,3}$-minor-free, Theorem 12 or Theorem 13 imply the next result (where the lower bound holds since the graph $G_{3,\ell}$ in Lemma 8 is planar).

Theorem 14. *The underlying treewidth of the class of planar graphs equals* 3*. In particular, every planar graph of treewidth k has* 3*-tree-partition-width $O(k^2)$.*

The class of graphs embeddable on a given surface (that is, a closed compact 2-manifold) is minor-closed. The *Euler genus* of a surface with h handles and c cross-caps is $2h+c$. The *Euler genus* of a graph G is the minimum $g \in \mathbb{N}_0$ such that there is an embedding of G in a surface of Euler genus g; see [25] for more about graph embeddings in surfaces.

It follows from Euler's formula that every graph with Euler genus at most g is $K_{3,2g+3}$-minor-free. Thus Lemma 8 and Theorem 13 imply the following.

Theorem 15. *The underlying treewidth of the class of graphs embeddable on any fixed surface Σ equals* 3*. In particular, every graph embeddable in Σ and of treewidth k has* 3*-tree-partition-width $O(k^2)$.*

6 Excluding a Topological Minor

A graph $\tilde{G}$ is a *subdivision* of a graph G if $\tilde{G}$ can be obtained from G by replacing each edge vw by a path P_{vw} with endpoints v and w (internally disjoint from the rest of $\tilde{G}$). If each P_{vw} has t internal vertices, then $\tilde{G}$ is the *t-subdivision* of G. If each P_{vw} has at most t internal vertices, then $\tilde{G}$ is a *$(\leqslant t)$-subdivision* of G. A graph H is a *topological minor* of G if a subgraph of G is isomorphic to a subdivision of H. A graph G is *H-topological-minor-free* if H is not a topological minor of G. Using disjointed partitions, we prove:

Theorem 16. *For every fixed multigraph X with p vertices, every X-topological minor-free graph G of treewidth k has p-tree-partition-width $O(k^2)$.*

Theorem 16 implies the upper bound in the next result, which we show is tight.

Theorem 17. *The underlying treewidth of the class of K_t-topological-minor-free graphs equals $t-2$ if $t \in \{2,3,4\}$ and equals t if $t \geqslant 5$.*

Determining the underlying treewidth of the class of $K_{s,t}$-topological-minor-free graphs is an interesting open problem (for $s \geqslant 4$).

We show that c-tree-partition-width is well-behaved (in a certain sense) under subdivisions. The next theorem follows.

Theorem 18. *A monotone graph class $\mathcal{G}$ has bounded underlying treewidth if and only if $\mathcal{G}$ excludes some fixed topological minor.*

7 Excluding a Subgraph or Induced Subgraph

For a graph H, a graph G is *H-free* if G contains no subgraph isomorphic to H. For a finite set of graphs $\mathscr{H}$, we say that G is *$\mathscr{H}$-free* if G is H-free for all $H \in \mathscr{H}$. Let $\mathcal{G}_H$

be the class of H-free graphs and let $\mathcal{G}_{\mathcal{H}}$ be the class of $\mathcal{H}$-free graphs. The next result characterise when $\mathcal{G}_{\mathcal{H}}$ has bounded underlying treewidth, and determines the exact underlying treewidth for several natural classes. The proof is based on disjointed coverings.

Let P_n be the n-vertex path. A *spider* is a subdivision of a star and a *spider-forest* is a subdivision of a star-forest. For $s,t \in \mathbb{N}$ with $s \geqslant 2$, the *(s,t)-spider*, denoted $S_{s,t}$, is the $(t-1)$-subdivision of $K_{1,s}$. If v is the centre of $S_{s,t}$, then each component of $S_{s,t} - v$ is called a *leg*.

Theorem 19. *For all $\ell,n,s,t \in \mathbb{N}$ where $n,s \geqslant 3$ and $\ell \geqslant 2$, and for every finite set $\mathcal{H}$ of graphs,*

(i) *$\mathcal{G}_{\mathcal{H}}$ has bounded underlying treewidth if and only if $\mathcal{H}$ contains a spider-forest*;
(ii) *the underlying treewidth of $\mathcal{G}_{P_n}$ equals $\lfloor \log n \rfloor - 1$*;
(iii) *the underlying treewidth of $\mathcal{G}_{\ell P_n}$ equals $\lfloor \log n \rfloor$*;
(iv) *the underlying treewidth of $\mathcal{G}_{S_{s,t}}$ equals $\lfloor \log t \rfloor + 1$*;
(v) *the underlying treewidth of $\mathcal{G}_{\ell S_{s,t}}$ equals $\lfloor \log t \rfloor + 2$.*

For a graph H, let $\mathcal{I}_H$ be the class of graphs with no induced subgraph isomorphic to H. We characterise the graphs H such that $\mathcal{I}_H$ has bounded underlying treewidth, and determines the precise underlying treewidth for each such H.

Theorem 20. *For any graph H,*

(i) *$\mathcal{I}_H$ has bounded underlying treewidth if and only if H is a star-forest*;
(ii) *if H is a star-forest, then $\mathcal{I}_H$ has underlying treewidth at most* 2;
(iii) *$\mathcal{I}_H$ has underlying treewidth at most* 1 *if and only if H is a star or each component of H is a path on at most three vertices*;
(iv) *$\mathcal{I}_H$ has underlying treewidth* 0 *if and only if H is a path on at most three vertices, or $E(H) = \varnothing$.*

8 Graph Drawings

A graph is *k-planar* if it has a drawing in the plane with at most k crossings on each edge, where we assume that no three edges cross at the same point. Of course, the class of 0-planar graphs is the class of planar graphs, which has underlying treewidth 3 (Theorem 14). However, 1-planar graphs behave very differently. It is well-known that every graph has a 1-planar subdivision: take an arbitrary drawing of G and for each edge e add a subdivision vertex between consecutive crossings on e. Since the class of 1-planar graphs is monotone, Theorem 18 implies that the class of 1-planar graphs has unbounded underlying treewidth.

By restricting the type of drawing, we obtain positive results. A *circular drawing* of a graph G positions each vertex on a circle in the plane, and draws each edge as an arc across the circle, such that no two edges cross more than once. A graph is *outer k-planar* if it has a circular drawing such that each edge is involved in at most k

crossings. The outer 0-planar graphs are precisely the outerplanar graphs, which have treewidth 2. We show below that for each $k \in \mathbb{N}$, the class of outer k-planar graphs has underlying treewidth 2. In fact, we prove a slightly more general result. A graph is *weakly outer k-planar* if it has a circular drawing such that whenever two edges e and f cross, e or f crosses at most k edges. Clearly every outer k-planar graph is weakly outer k-planar.

Theorem 21. *Each weakly outer k-planar graph has 2-tree-partition-width $O(k^3)$.*

Theorem 21 implies the next result, where the lower bound holds since $G_{2,\ell}$ from Lemma 8 is outerplanar.

Theorem 22. *For every fixed $k \in \mathbb{N}$, the underlying treewidth of the class of weakly outer k-planar graphs equals 2, with constant treewidth-binding function.*

9 Universal Graphs

A graph U is *universal* for a graph class $\mathscr{G}$ if $U \in \mathscr{G}$ and U contains every graph in $\mathscr{G}$. This definition is only interesting when considering infinite graphs. For each $k \in \mathbb{N}$ there is a universal graph $\mathscr{T}_k$ for the class of countable graphs of treewidth k. Huynh, Mohar, Šámal, Thomassen, and Wood [22] gave an explicit construction for $\mathscr{T}_k$, and showed how product structure theorems for finite graphs lead to universal graphs. Their results imply that for any hereditary class $\mathscr{G}$ of countable graphs, if the class of finite graphs in $\mathscr{G}$ has underlying treewidth c with treewidth-binding function f, then every graph in $\mathscr{G}$ of treewidth at most k is contained in $\mathscr{T}_c \boxtimes K_{f(k)}$. This result is applicable to all minor-closed classes, monotone classes, and hereditary classes. For example, every countable K_t-minor free graph of treewidth k is contained in $\mathscr{T}_{t-2} \boxtimes K_{O(k^2)}$.

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