Notes on Aharoni's rainbow cycle conjecture

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Abstract In 2017, Ron Aharoni made the following conjecture about rainbow cycles in edge-coloured graphs: If G is an n-vertex graph whose edges are coloured with n colours and each colour class has size at least r, then G contains a rainbow cycle of length at most $\lceil \frac{n}{r} \rceil$. One motivation for studying Aharoni's conjecture is that it is a strengthening of the Caccetta-Häggkvist conjecture on digraphs from 1978. In this article, we present a survey of Aharoni's conjecture, including many recent partial results and related conjectures. We also present two new results. Our first result (Theorem 21) provides sharp thresholds for rainbow cycles in edge-coloured graphs with 3 colours. Our second, and main, result (Theorem 35) is for the $r = 3$ case of Aharoni's conjecture. We prove that if G is an n-vertex graph whose edges are coloured with n colours and each colour class has size at least 3, then G contains a rainbow cycle of length at most $\frac{4n}{9} + 7$.

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1 Introduction

In 1978, Caccetta and Häggkvist made the following conjecture about directed cycles in digraphs.

Conjecture 1 (Caccetta-Häggkvist)**.** For all positive integers n, r, every simple[1] n-vertex digraph with minimum outdegree at least r contains a directed cycle of length at most $\lceil \frac{n}{r} \rceil$.

Despite considerable effort from numerous researchers, the Caccetta-Häggkvist conjecture remains open. A complete summary of the plethora of results related to the Caccetta-Häggkvist is beyond the scope of this survey. We refer the interested reader to Sullivan [23] for a brief synopsis. We instead focus on the following generalisation of Conjecture 1.

Conjecture 2 (Aharoni)**.** Let G be a simple[2] edge-coloured graph with n vertices and n colours, where each colour class has size at least r. Then G contains a rainbow[3] cycle of length at most $\lceil \frac{n}{r} \rceil$.

For completeness, we now give a proof that Conjecture 2 indeed implies Conjecture 1. In fact, as noted in [10], the following weakening of Aharoni's conjecture already implies the Caccetta-Häggkvist conjecture.

Conjecture 3 (DeVos et al. [10])**.** Let G be a simple edge-coloured graph with n vertices and n colours, where each colour class has size at least r. Then G contains a cycle C of length at most $\lceil \frac{n}{r} \rceil$ such that no two incident edges of C are the same colour.

Proof of Conjecture 1, assuming Conjecture 3. Let D be a simple digraph with n vertices and minimum outdegree at least r. Let G be the graph obtained from D by forgetting the orientations of all arcs. Colour $uv \in E(G)$ with colour u if $(u,v) \in E(D)$. Clearly, this colouring uses $|V(D)| = n$ colours. Moreover, since D has minimum outdegree at least r, each colour class has size at least r. Therefore, by Conjecture 3, G contains a properly edge-coloured cycle C of length at most $\lceil \frac{n}{r} \rceil$. The set of arcs in D corresponding to the edges of C must be a directed cycle; otherwise C is not properly edge-coloured. □

Despite the fact that Aharoni's conjecture implies the Caccetta-Häggkvist conjecture, a proof of Aharoni's conjecture may be easier to find than a proof of the Caccetta-Häggkvist conjecture. Although this might sound counterintuitive, the method of proving a stronger statement is very common in combinatorics. Moreover, generalisation often leads to new techniques and new questions which one would not even consider in the original setting. We will see that this is the case for Aharoni's conjecture in the next section.

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[1] A digraph is *simple* if for all vertices u and v, there is at most one arc from u to v.

[2] An edge-coloured graph G is *simple* if each colour class does not contain parallel edges.

[3] A subgraph of an edge-coloured graph is *rainbow* if no two of its edges are the same colour.

2 Related Results and Conjectures

In this section, we survey results and conjectures related to rainbow cycles. For a general survey on rainbow sets, we refer the reader to Aharoni and Briggs [2].

2.1 Larger Colour Classes

Much of the research on the Caccetta-Häggkvist conjecture has focused on the directed triangle case ($r = \lceil \frac{n}{3} \rceil$). A natural strategy is to increase the outdegree condition until one can prove the existence of a directed triangle. The best result in this direction is the following result of Hladký, Kráľ, and Norin [13], which uses the flag algebra machinery developed by Razborov [21].

Theorem 4. *Every simple n-vertex digraph with minimum outdegree at least $0.3465n$ contains a directed triangle.*

Similarly, for Aharoni's conjecture, one can ask how large must the colour classes be to ensure a rainbow cycle of length at most $\lceil \frac{n}{r} \rceil$? The first non-trivial bound was proven by Hompe, Pelikánová, Pokorná, and Spirkl [17].

Theorem 5. *Let $r \geqslant 2$, $n \geqslant 1$, G be a simple edge-coloured graph with n vertices and n colours, where each colour class has size at least $301 r \log r$. Then G contains a rainbow cycle of length at most $\lceil \frac{n}{r} \rceil$.*

Theorem 5 was subsequently improved by Hompe and Spirkl [19], who removed the $\log r$ term.

Theorem 6. *Let $r, n \in \mathbb{N}$, G be a simple edge-coloured graph with n vertices and n colours, where each colour class has size at least $10^{11} r$. Then G contains a rainbow cycle of length at most $\frac{n}{r}$.*

2.2 Number of Colour Classes

One appealing aspect of Aharoni's conjecture is that we can ask what happens when the number of colour classes is different from the number of vertices. Note that this question does not even make sense in the digraph setting. To be precise, we now define a function $f(n,t,r)$ which will be useful to state many of the results that appear in this survey. The *rainbow girth* of an edge-coloured graph G, denoted $\mathrm{rg}(G)$, is the length of a shortest rainbow cycle in G. If G does not contain a rainbow cycle, then $\mathrm{rg}(G) = \infty$. Let

$$f(n,t,r) := \max\{\mathrm{rg}(G)\},$$

where the maximum is taken over all simple edge-coloured graphs G with n vertices and at least t colours, such that each colour class has size at least r.

We can rephrase Aharoni's conjecture via our function $f(n,t,r)$ as follows.

Conjecture 7 (Aharoni)**.** For all $n, r \geqslant 1$,

$$f(n,n,r) \leqslant \lceil \tfrac{n}{r} \rceil.$$

We believe that there is no reason to restrict attention to the case $t = n$, and that the following question is of independent interest.

Question 8. *Obtain good upper and lower bounds for $f(n,t,r)$ for all n, t, and r.*

An important special case of Question 8 is when $r = 1$, which was considered by Bollobás and Szemerédi [7].

Theorem 9. *For all $n \geqslant 4$ and $k \geqslant 2$,*

$$f(n,n+k,1) \leqslant \frac{2(n+k)}{3k}(\log k + \log\log k + 4).$$

In other words, Theorem 9 asserts that every n-vertex graph with at least $n+k$ edges contains a cycle of length at most $\frac{2(n+k)}{3k}(\log k + \log\log k + 4)$. This is a key tool used in many of the results presented in this survey.

Hompe and Spirkl [19] also obtained the following bounds when the number of colours is more than the number of vertices.

Theorem 10. *For all $n \geqslant 1$ and $k \geqslant 2$,*

$$f(n,n+k,10^9 k) \leqslant \max\{6, \frac{n(\log k)^2}{10k^{3/2}} + 14\log k\}.$$

When the number of colours is *less* than the number of vertices, DeVos et al. [10] obtained the following tight bounds for $r = 2$.

Theorem 11. *For all $n \geqslant 3$ and $t \leqslant n$,*

$$f(n,t,2) = \begin{cases} \infty & \text{if } t \leqslant n-2, \\ n-1 & \text{if } t = n-1, \\ \lceil \frac{n}{2} \rceil & \text{if } t = n. \end{cases}$$

2.3 Structured Colour Classes

In the proof of the Caccetta-Häggkvist conjecture (assuming Aharoni's conjecture) given in the Introduction, the colour classes of the derived edge-colouring are all stars. Therefore, it is natural to ask what happens when the colour classes are not stars. Note that if a colour class is not a star, then it must contain a matching of size 2, or it is a triangle.

In the extreme case when all colour classes contain a matching of size 2, Aharoni and Guo [5] proved that there is a much shorter rainbow cycle than the $\lceil \frac{n}{2} \rceil$ bound which follows from [10].

Theorem 12. *There exists an absolute constant C such that if G is a simple edge-coloured graph with n vertices, n colours, and each colour class is a matching of size 2, then G contains a rainbow cycle of length at most $C\log n$.*

Kevin Hendrey (private communication) proved that an $\mathcal{O}(\log n)$ bound also holds in the case when all colour classes are a triangle. The same proof also appears in a recent paper of Aharoni, Berger, Chudnovsky, Guo, and Zerbib [1].

Theorem 13. *There exists an absolute constant C such that if G is a simple edge-coloured graph with n vertices, n colours, and each colour class is a triangle, then G contains a rainbow cycle of length at most $C\log n$.*

Some mixed cases were investigated by Guo [11].

Theorem 14. *There exists an absolute constant C such that if G is a simple edge-coloured graph with n vertices, n colours, and each colour class is a matching of size 2 or a triangle, then G contains a rainbow cycle of length at most $C\log n$.*

Theorem 15. *For any constants $0 \leqslant \alpha < 1$ and $0 \leqslant \beta \leqslant \alpha$ with $\beta < (1-\alpha)/3$, there exists a constant $C(\alpha,\beta)$ such that if G is an n-vertex simple edge-coloured graph containing at least $(\alpha-\beta)n$ colour classes consisting of a single edge and at least $(1-\alpha-\beta)n$ colour classes consisting of a triangle, then G contains a rainbow cycle of length at most $C(\alpha,\beta)\log n$.*

2.4 Rainbow Triangles

The rainbow triangle case (when $r = \lceil n/3 \rceil$) of Aharoni's conjecture states that $f(n,n,\lceil n/3 \rceil) \leqslant 3$. This is of course still open since the directed triangle case of the Caccetta-Häggkvist conjecture is still open. However, there have been partial results which increase the number of colours or sizes of the colour classes. Two such results were obtained by Aharoni, DeVos, and Holzman [4].

Theorem 16. *For all $n \geqslant 1$,*

$$f(n, 9n/8, n/3) \leqslant 3.$$

Theorem 17. *For all $n \geqslant 1$,*

$$f(n, n, 2n/5) \leqslant 3.$$

Both of these results have been subsequently improved by Hompe, Qu, and Spirkl [18].

Theorem 18. *For all $n \geqslant 1$,*

$$f(n, 1.1077n, n/3) \leqslant 3.$$

Theorem 19. *For all $n \geqslant 1$,*

$$f(n, n, 0.3988n) \leqslant 3.$$

Aharoni, DeVos, González Hermosillo de la Maza, Montejano, and Šámal [3] proved the following theorem showing when a simple edge-coloured graph with 3 colours contains a rainbow triangle. Their theorem actually implies Mantel's theorem.

Theorem 20. *Every n-vertex simple edge-coloured graph with 3 colours and each colour class of size at least $\frac{26-2\sqrt{7}}{81} \cdot n^2$ contains a rainbow triangle.*

They also prove that the constant $\frac{26-2\sqrt{7}}{81}$ in Theorem 20 is best possible. However, their extremal example has rainbow 2-cycles. Our first new result is the following sharp thresholds for $f(n, 3, r)$.

Theorem 21. *For all $n \geqslant 100$,*

$$f(n,3,r) = \begin{cases} \infty & \text{if } r \leqslant \lfloor \binom{n}{2}/3 \rfloor, \\ 2 & \text{if } \lfloor \binom{n}{2}/3 \rfloor < r \leqslant \binom{n}{2}, \\ 1 & \text{if } r > \binom{n}{2}. \end{cases}$$

Proof. First suppose $r > \lfloor \binom{n}{2}/3 \rfloor$. Let G be a simple edge-coloured graph with n vertices, 3 colours and each colour class of size at least r. Since K_n has only $\binom{n}{2}$ edges, G must contain a loop or a rainbow 2-cycle. Moreover, if $r > \binom{n}{2}$, then G must contain a loop. The n-vertex edge-coloured graph with a red, blue, and green edge between every pair of vertices proves equality.

We now show that $f(n, 3, r) = \infty$ if $n \geqslant 100$ and $r \leqslant \lfloor \binom{n}{2}/3 \rfloor$. Partition the vertices of K_n as $X \cup Y \cup Z$ where $|X| = |Y| = \lceil \frac{2n}{5} \rceil$. Colour all edges in $K_n[X]$ or $K_n[Y]$ or $K_n[Z]$ red, all edges between X and Y blue, and all edges between Z and $X \cup Y$ green. Observe that this is an edge-colouring of K_n with no rainbow triangle. The colour classes do not quite have the same size, which we now fix. Since $n \geqslant 100$, we have $|X||Y| \leqslant \binom{n}{2}/3$ and $|Z|(|X| + |Y|) \leqslant \binom{n}{2}/3$. Thus, we may recolour some of the edges in $K_n[X]$ blue, and some of the edges in $K_n[Y]$ green, so that the number of red, blue, and green edges differ by at most 1. This new edge-colouring of K_n still does not contain a rainbow triangle. This of course implies that there are no rainbow cycles since there are only 3 colours. □

2.5 *Fixed Values of* r

The Caccetta-Häggkvist conjecture is known to hold for small values of r. The $r = 1$ case is trivial, the $r = 2$ case was proven by Caccetta and Häggkvist [8]; the $r = 3$ case was proven by Hamidoune [12]; and the $r \in \{4,5\}$ cases were proven by Hoàng and Reed [14]. For Aharoni's conjecture, the $r = 2$ case was proven by DeVos et al. [10].

Theorem 22. *For all* $n \geqslant 1$,
$$f(n,n,2) \leqslant \lceil \tfrac{n}{2} \rceil.$$

In Section 3, we prove the following result for $r = 3$.

Theorem 23. *For all* $n \geqslant 1$,
$$f(n,n,3) \leqslant \tfrac{4n}{9} + 7.$$

After a previous version of this paper was made publicly available [9], Hompe and Huynh [16] extended our methods to prove the following theorem for all fixed values of r.

Theorem 24. *For all* $r \geqslant 1$, *there exists a constant* c_r *such that*
$$f(n,n,r) \leqslant \tfrac{n}{r} + c_r,$$
for all $n \geqslant 1$.

2.6 *Non-uniform Versions*

For the Caccetta-Häggkvist conjecture, it is natural to ask if there is a version which takes into account *all* the outdegrees rather than just the minimum outdegree. Seymour (see [15]) proposed the following generalization. Given a digraph D with no sink[4], define
$$\psi(D) := \sum_{v \in V(D)} \frac{1}{\deg^+(v)}.$$

Conjecture 25. Every simple digraph D with no sink contains a directed cycle of length at most $\lceil \psi(D) \rceil$.

Note that in the case that all outdegrees are r, then $\psi(D) = \frac{n}{r}$, so Conjecture 25 implies the Caccetta-Häggkvist conjecture. Unfortunately, Conjecture 25 was disproved by Hompe [15]. However, Aharoni et al. [1] proved that "half" of Conjecture 25 holds.

Theorem 26. *Every simple digraph* D *with no sink contains a directed cycle of length at most* $2\psi(D)$.

[4] A *sink* is a vertex with outdegree zero.

Theorem 26 has a natural generalization in the rainbow setting. Given an edge-coloured graph G, define

$$\psi(G) = \sum_{A} \frac{1}{|A|},$$

where the sum is taken over all colour classes A of G.

Conjecture 27. Every simple edge-coloured graph G with the same number of colours as vertices contains a rainbow cycle of length at most $2\psi(G)$.

In the case that all colour classes have size at most 2, Aharoni et al. [1] proved the following strengthening of Conjecture 27.

Theorem 28. *Let G be a simple edge-coloured graph with the same number of colours as vertices and such that each colour class has size at most 2. Then G contains a rainbow cycle of length at most $\lceil \psi(G) \rceil$.*

Note that Theorem 28 is a strengthening of Theorem 22 since it allows colour classes of size 1.

2.7 Matroids

We can generalise Aharoni's conjecture to any setting in which the notion of 'cycle' makes sense. One natural candidate is that of a *matroid*. For the reader unfamiliar with matroids, we introduce all the necessary definitions now. For a more thorough introduction to matroids, we refer the reader to Oxley [20].

A *matroid* is a pair $M = (E, \mathscr{C})$ where E is a finite set, called the *ground set* of M, and $\mathscr{C}$ is a collection of subsets of E, called *circuits*, satisfying

1. $\emptyset \notin \mathscr{C}$,
2. if C' is a proper subset of $C \in \mathscr{C}$, then $C' \notin \mathscr{C}$,
3. if C_1 and C_2 are distinct members of $\mathscr{C}$ and $e \in C_1 \cap C_2$, then there exists $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

We now give some examples of matroids. Let G be a graph. We will consider two different matroids with ground set $E(G)$. The circuits of the first matroid are the (edges of) cycles of G. This is the *cycle matroid* of G, denoted $M(G)$. A matroid is *graphic* if it is isomorphic to the cycle matroid of some graph. A *cocycle* of G is an inclusion-wise minimal edge-cut of G. The collection of cocycles of G is also a matroid, called the *cocycle matroid* of G, and is denoted $M(G)^*$. A matroid is *cographic* if it is isomorphic to the cocycle matroid of some graph.

Let $\mathbb{F}$ be a field. An *$\mathbb{F}$-matrix* is a matrix with entries in $\mathbb{F}$. Let A be an $\mathbb{F}$-matrix whose columns are labelled by a finite set E. The *column matroid* of A, denoted $M[A]$, is the matroid with ground set E whose circuits correspond to the minimal (under inclusion) linearly dependent columns of A. A matroid is *representable over $\mathbb{F}$* if it is isomorphic to $M[A]$ for some $\mathbb{F}$-matrix A. A matroid is *binary* if it is representable over the two-element field, and it is *regular* if it is representable over every field.

In order to formulate Aharoni's conjecture for matroids, we need to define simple matroids and how to express the number of vertices of a graph as a matroid parameter. We do this now. A matroid is *simple* it it does not contain any circuits of size 1 or 2. A set $I \subseteq E$ is *independent* if it does not contain a circuit. The *rank* of $X \subseteq E$ is the size of a largest independent set contained in X, and is denoted $r_M(X)$. The *rank* of M is $r(M) := r_M(E)$. Notice that the number of vertices of a connected graph G is $r(M(G)) - 1$. Thus, Aharoni's conjecture can be phrased in the language of matroids as follows.

Conjecture 29 (Aharoni)**.** Let M be a simple rank-$(n-1)$ graphic matroid and c be a colouring of $E(M)$ with n colours, where each colour class has size at least r. Then M contains a rainbow circuit of size at most $\lceil \frac{n}{r} \rceil$.

One way to generalise Conjecture 1 is to replace 'graphic matroid' with some larger superclass of matroids. In [10], it was shown that one cannot replace 'graphic matroid' by 'binary matroid' in Conjecture 29.

Theorem 30. *For all $n \geqslant 6$, there exists a simple rank-$(n-1)$ binary matroid M on $2n$ elements, and a colouring of $E(M)$ where each colour class has size 2, such that all rainbow circuits of M have size strictly greater than $\lceil \frac{n}{2} \rceil$.*

The main result of [10] can be phrased in matroid language as follows.

Theorem 31. *Let M be a simple rank-$(n-1)$ graphic matroid and c be a colouring of $E(M)$ with n colours, where each colour class has size at least 2. Then M contains a rainbow circuit of size at most $\lceil \frac{n}{2} \rceil$.*

In [10], it is also proved that the matroid analogue of Theorem 22 holds for cographic matroids.

Theorem 32. *Let N be a simple rank-$(n-1)$ cographic matroid and c be a colouring of $E(N)$ with n colours, where each colour class has size at least 2. Then N contains a rainbow circuit of size at most $\lceil \frac{n}{2} \rceil$.*

Regular matroids are a well-studied superclass of graphic matroids, and are 'essentially' graphic or cographic via Seymour's regular matroid decomposition theorem [22]. Therefore, by combining Theorem 31, Theorem 32, and Seymour's regular matroid decomposition theorem, it may be possible to prove the following conjecture.

Conjecture 33 (DeVos et al. [10])**.** Let M be a simple rank-$(n-1)$ regular matroid and c be a colouring of $E(M)$ with n colours, where each colour class has size at least 2. Then M contains a rainbow circuit of size at most $\lceil \frac{n}{2} \rceil$.

Of course, we can also consider Question 8 for various classes of matroids. Bérczi and Schwarcz [6] obtained one such result.

Theorem 34. *Let M be an n-element rank-t binary matroid whose ground set is coloured with t colours. Then M either contains a rainbow circuit or a monochromatic cocircuit.*[5]

[5] A *monochromatic cocircuit* is a circuit in the dual matroid whose elements are all the same colour.

Bérczi and Schwarcz [6, Theorem 4] also show that it is possible to characterize binary matroids as exactly those matroids which do not admit a specific type of colouring with no rainbow circuits. They also prove that if a simple graph G has an edge-colouring with no rainbow cycle, where each colour class has size at most 2, then G is independent in the 2-dimensional rigidity matroid.[6]

3 Proof of Main Theorem

In this section, we prove our main theorem.

Theorem 35. *Every simple edge-coloured graph with n vertices, n colours, and each colour class of size at least 3, contains a rainbow cycle of length at most $\frac{4n}{9}+7$.*

3.1 Excess-k Graphs

We begin by establishing some basic properties about graphs which have at least k more edges than vertices, which we call *excess-k graphs*. The first property follows from Theorem 9: for all $n \geqslant 4$ and $k \geqslant 2$, every n-vertex, excess-k graph has a cycle of length at most $\frac{2(n+k)}{3k}(\log k + \log\log k + 4)$. We require the following tighter bounds when $k \leqslant 2$.

Lemma 36. *Every n-vertex, excess-1 graph has a cycle of length at most $\frac{2n}{3}+1$.*

The proof is easy and is omitted (see [10] for a proof of a stronger claim).

Lemma 37. *Every n-vertex, excess-2 graph H has a cycle of length at most $\frac{n}{2}+1$.*

Proof. A *block* of H is a 2-connected subgraph of H which is maximal under the subgraph relation. If H contains at least two blocks, then H contains two cycles which meet in at most one vertex. Hence, one of these cycles has length at most $\frac{n}{2}+1$. So, we may assume H contains exactly one block B. Note that B is excess-2, since H is excess-2. Let $C \cup P_1 \cup \cdots \cup P_k$ be an ear-decomposition of B. Note that $k \geqslant 2$, since B is excess-2. Let $B' = C \cup P_1 \cup P_2$. Either B' contains two cycles which meet in at most two vertices, or B' is a subdivision of K_4. In the first case, one of the two cycles has length at most $\frac{n}{2}+1$. In the second case, B' contains four cycles $C_1, \dots, C_4$ such that $\sum_{i\in[4]} |V(C_i)| = 2|V(B')|+4$. Thus, one of these four cycles has length at most $\frac{|V(B')|}{2}+1 \leqslant \frac{n}{2}+1$. □

We now prove that the largest stable set of an excess-k graph is roughly half the number of vertices.

[6] Equivalently, for every $X \subseteq V(G)$ with $|X| \geqslant 2$, $|E(G[X])| \leqslant 2|X|-3$.

Lemma 38. *Let H be a simple excess-k graph of minimum degree at least 2. Then a maximum stable set of H has size at most $\frac{|V(H)|+k}{2}$.*

Proof. We proceed by induction on $|V(H)|+|E(H)|$. We may assume that H has exactly k more edges than vertices. If H is the disjoint union of $H_1,\dots,H_\ell$ and H_i has excess k_i, then $k_1+\dots+k_\ell=k$. Therefore, we are done by applying induction to each H_i. So, we may assume H is connected. If $k=0$, then H is a cycle, so the lemma clearly holds. Now suppose $k \geqslant 1$. Let X denote the set of vertices of H of degree at least 3. If $|X|=1$, then H consists of $k+1$ cycles which meet at the same vertex. It is easy to see that the lemma holds in this case. So, we may assume $|X| \geqslant 2$. Suppose $u,v \in X$ and $uv \in E(G)$. By induction, a maximum stable set of $H-uv$ has size at most $\frac{|V(H)|+k-1}{2}$. Hence, a maximum stable set of H has size at most $\frac{|V(H)|+k-1}{2}$. So, we may assume that no two vertices of X are adjacent.

Let P be the shortest path between any two vertices of X. By the minimality of P, each internal vertex of P has degree 2. Let $H'=H-I$, where I is the set of internal vertices of P. Note that H' has excess $k-1$ and minimum degree at least 2. Let S be a maximum size stable set in H. Note that $S \cap V(H')$ is a stable set in H'. By induction, $|S\cap V(H')| \leqslant (|V(H')|+k-1)/2$. Also, $|S \cap I| \leqslant (|I|+1)/2$. Thus, $|S| = |S\cap V(H')|+|S\cap I| \leqslant (|V(H)|+k)/2$. □

We finish by establishing the following lemma about 'minimal' excess-k rainbow subgraphs of an edge-coloured graph.

Lemma 39. *Let H be a simple edge-coloured graph where each colour class has size at most r. Let R be an excess-k rainbow subgraph of H such that $V(R)$ is minimal under inclusion. Then H contains a rainbow 2-cycle, or there are at most $\max\{\binom{2k+2}{2}, 6k(r-1)\}$ chords of R in H.*

Proof. We may assume that H does not contain parallel edges; otherwise H contains a rainbow 2-cycle. By the minimality of R, $\deg_R(v) \geqslant 2$ for all $v \in V(R)$. Let $V(R) = X \cup Y$, where X is the set of vertices of degree at least 3, and Y is the set of degree-2 vertices of R. By the Handshaking Lemma, $|X| \leqslant 2k$.

Say that a chord of R is *novel* if its colour does not appear in R, and *plain* otherwise. Suppose $e=uv$ is a novel chord. If $|Y| \geqslant 3$, then there is a vertex $y \in Y$ such $y \notin \{u,v\}$, and so $(R\cup e)-y$ contradicts the minimality of R. Thus, there are either no novel chords, or the total number of chords is at most $\binom{|X|+|Y|}{2} \leqslant \binom{2k+2}{2}$.

We may thus assume that there are no novel chords. Let $e \in E(H)$ be a plain chord of R. Since e is plain, there is an edge $f \in E(R)$ of the same colour as colour e. If both ends of f are in Y, then $R' := (R\cup e)\setminus f$ contains a degree-1 vertex x. But now $R'-x$ contradicts the minimality of R. Therefore, at least one end of f is in X. Let R_0 be the multigraph on vertex set X obtained by suppressing all degree-2 vertices. Note that R_0 also has excess k and at most $2k$ vertices. Therefore, $|E(R_0)| \leqslant 3k$. There are at most $2|E(R_0)|$ edges of R which are incident to a vertex in X. Each of these edges can correspond to at most $r-1$ plain chords. Hence, there are at most $6k(r-1)$ plain chords. □

3.2 The set-up

Let G be a simple edge-coloured graph with n vertices, n colours, and each colour class of size exactly 3. An *r-star* is a star with r edges. A colour class of G is a *star class* if it is a 3-star. A vertex of G is a *star vertex* if it is the centre of a star class, and is otherwise a *non-star vertex*. Let S denote the set of star vertices of G, and N denote the set of non-star vertices of G. Let N' be the set of non-star classes. Since star classes may be centred at the same vertex, we have $|N'| \leqslant |N|$. We will do a case analysis depending on whether $|N| \geqslant 8$ or $|N| \leqslant 7$.

3.3 Many non-star vertices

Throughout this section we suppose $|N| \geqslant 8$. Let $\{x,y\} \subseteq N$. We say that a colour class A *dominates* $\{x,y\}$ if every edge in A has an end in $\{x,y\}$.

Claim 40. *There exists $\{x,y\} \subseteq N$ such that no colour class dominates $\{x,y\}$.*

Proof. Suppose A is a star class. Then A is a 3-star centred at a vertex not in N. Therefore, for all $\{u,v\} \subseteq N$, A does not dominate $\{u,v\}$ since at least one leaf of A is not in $\{u,v\}$.

Suppose A is a non-star class. Up to isomorphism, $A \in \{K_3, P_3, P_2 \sqcup P_1, P_1 \sqcup P_1 \sqcup P_1\}$, where P_i is a path with i edges, and $\sqcup$ denotes disjoint union. Let $\gamma(A)$ be the number of vertex covers of A of size 2. Observe that the number of pairs $\{u,v\} \subseteq N$ which A dominates is exactly equal to $\gamma(A)$. We have $\gamma(K_3) = 3, \gamma(P_3) = 1, \gamma(P_2 \sqcup P_1) = 2$, and $\gamma(P_1 \sqcup P_1 \sqcup P_1) = 0$. Thus, every non-star class dominates at most 3 pairs of vertices in N. So, the number of pairs dominated by non-star classes is at most $3|N'| \leqslant 3|N| < \binom{N}{2}$, since $|N| \geqslant 8$. Therefore, at least one pair $\{x,y\} \subseteq N$ is not dominated by any colour class. $\square$

Claim 41. *G contains an excess-2 rainbow subgraph R.*

Proof. A *transversal* of G is a subgraph consisting of exactly one edge of each colour. By Claim 40, G has a transversal R such that at least two vertices x,y are not in $V(R)$. Since R has exactly n edges, R is excess-2 (and clearly rainbow). $\square$

We now choose R to be an excess-2 rainbow subgraph of G such that $V(R)$ is minimal under inclusion.

Claim 42. *G contains a rainbow cycle C' such that $E(R) \cap E(C') = \emptyset$.*

Proof. Choose a transversal R' which is edge-disjoint from R. Since R' has n vertices and n edges, R' contains a cycle C'. Clearly, C' is rainbow since R' is rainbow. $\square$

Claim 43. *G contains a rainbow cycle C of length at most $\frac{2n}{5} + 7$.*

Proof. Let $n_1 = |V(R) \setminus V(C')|$, $n_2 = |V(R) \cap V(C')|$, and $n_3 = |V(C') \setminus V(R)|$. First suppose $n_3 \geqslant \frac{n}{5} - 12$. Then, $|V(R)| \leqslant \frac{4n}{5} + 12$. By Lemma 37, R contains a rainbow cycle of length at most $\frac{1}{2} \cdot (\frac{4n}{5} + 12) + 1 = \frac{2n}{5} + 7$.

Thus, we may assume that $n_3 < \frac{n}{5} - 12$. Let A be the subset of edges of C' which are chords of R. Applying Lemma 39 to R in $C' \cup R$ (so $r = 2$), we have $|A| \leqslant 15$. Note that $V(R) \cap V(C')$ is a stable set of $C' \setminus A$. The maximum stable set of C' has size at most $\frac{|V(C')|}{2}$. Deleting one edge from a graph can increase the size of a maximum stable set by at most 1. Therefore, we conclude that $n_2 \leqslant n_3 + 2|A| \leqslant n_3 + 30$. Thus,

$$|V(C')| = n_2 + n_3 \leqslant 2n_3 + 30 < 2(\tfrac{n}{5} - 12) + 30 = \tfrac{2n}{5} + 6,$$

and so we may take $C = C'$ in this case. $\square$

This completes the case $|N| \geqslant 8$, since we have found a rainbow cycle of length at most $\frac{2n}{5} + 7$, which is better than the bound of $\frac{4n}{9} + 7$ required by Theorem 35.

3.4 Few non-star vertices

We complete the proof by considering the case $|N| \leqslant 7$.

Claim 44. *At least one vertex of G is a non-star vertex.*

Proof. Suppose every vertex of G is a star vertex. Since G has the same number of vertices as colours, this implies that at each $v \in V(G)$ there is a star class S_v centred at v. Let D be the digraph obtained from G by orienting the edges of S_v away from v for all $v \in V(G)$. By the $r = 3$ case of the Caccetta-Häggkvist conjecture [12], D contains a directed cycle $\vec{C}$ of length at most $\lceil \frac{n}{3} \rceil$. Note that $\vec{C}$ corresponds to a rainbow cycle C in G. $\square$

Fix a non-star vertex $z \in V(G)$. Since z is a non-star vertex, for every colour a, there is an edge e_a coloured a such that e_a is not incident to z. Thus, there is a transversal R of G such that $z \notin V(R)$. In particular, R is an excess-1 rainbow subgraph of G such that $z \notin V(R)$. Let R_1 be an excess-1 rainbow subgraph of $G - z$ such that $V(R_1)$ is minimal under inclusion.

Claim 45. *There exists a transversal R_2' of G such that $z \notin V(R_2')$ and $|E(R_1) \cap E(R_2')| \leqslant 7$.*

Proof. Let A be a star colour class. Observe that there are at least 2 edges of A which are not incident to z. Thus, there exists an edge $e_A \in A$ such that $e_A \notin E(R_1)$ and e_A is not incident to z. If A is a non-star colour class, there is an edge $e_A \in A$ which is not incident to z. Let $R_2' = \bigcup_A \{e_A\}$, where the union is over all colour classes. Since there are at most $|N'| \leqslant |N| \leqslant 7$ non-star colour classes, $|E(R_1) \cap E(R_2')| \leqslant 7$. $\square$

Since $z \notin V(R_2')$, R_2' is excess-1 (and rainbow). Let $R_2 \subseteq R_2'$ be excess-1 and rainbow such that $V(R_2)$ is minimal under inclusion. In particular, R_2 has minimum degree at least 2.

Claim 46. *G contains a rainbow cycle C of length at most $\frac{4n}{9}+7$.*

Proof. Let $n_1 = |V(R_1) \setminus V(R_2)|$, $n_2 = |V(R_1) \cap V(R_2)|$, and $n_3 = |V(R_2) \setminus V(R_1)|$. First suppose $n_3 \geqslant \frac{n}{3} - 9$. Thus, $|V(R_1)| \leqslant \frac{2n}{3} + 9$. By Lemma 36, R_1 contains a rainbow cycle of length at most $\frac{2}{3} \cdot (\frac{2n}{3} + 9) + 1 = \frac{4n}{9} + 7$.

Thus, we may assume that $n_3 < \frac{n}{3} - 9$. Let A be the subset of edges of R_2 which are chords or edges of R_1. By Lemma 39 applied to R_1 in $R_1 \cup R_2$ (so $r = 2$), and Claim 45, $|A| \leqslant 6 + 7 = 13$. Note that $V(R_1) \cap V(R_2)$ is a stable set of $R_2 \setminus A$. By Lemma 38, the maximum stable set of R_2 has size at most $\frac{|V(R_2)|+1}{2}$. Deleting one edge from a graph can increase the size of a maximum stable set by at most 1. Therefore, we conclude that $n_2 \leqslant n_3 + 2|A| + 1 \leqslant n_3 + 27$. By Lemma 36, R_2 contains a rainbow cycle of length at most

$$\frac{2}{3} \cdot (n_2 + n_3) + 1 \leqslant \frac{2}{3} \cdot (2n_3 + 27) + 1 < \frac{2}{3} \cdot (2(\frac{n}{3} - 9) + 27) + 1 = \frac{4n}{9} + 7. \qquad \square$$

Claim 46 completes the proof of Theorem 35.

4 Generalising our Approach

For the $r = 3$ case of Aharoni's conjecture, we proved that there is a rainbow cycle of length at most $\frac{4n}{9} + 7$. The additive constant can be easily improved since there are stronger versions of Lemma 39 for $k \leqslant 2$. However, we opted to instead prove Lemma 39 for all r and k as we suspected our approach could be generalised. To that end, the previous version of this section in [9] outlined two strategies to achieve this. Combining one of the strategies with some new ideas, Hompe and Huynh [16] subsequently succeeded in generalising our result, by proving the following theorem for all values of r.

Theorem 47. *For all $r \geqslant 1$, there exists a constant c_r such that if G is a simple n-vertex edge-coloured graph with n colour classes of size at least r, then G contains a rainbow cycle of length at most*

$$\frac{n}{r} + c_r.$$

As our previous discussion is now out-dated, we have removed it from this version of the paper. Interested readers can find the old discussion in [9].

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