Topological Narayana polynomials and interlacing conjectures

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1 From monotone Hurwitz numbers to topological Narayana polynomials

Monotone Hurwitz numbers enumerate factorisations of permutations into transpositions, with a certain monotonicity constraint [3]. They arise naturally as coefficients in the large N topological expansion of the well-studied HCIZ integral over the space of $N \times N$ unitary matrices. We introduce here a deformation of the monotone Hurwitz numbers that produces a generalisation of the Narayana polynomials [1]. Numerical evidence leads to conjectures concerning real-rootedness and interlacing of these so-called topological Narayana polynomials. It is then natural to investigate whether these polynomials admit multivariate generalisations that exhibit some form of stability.

Definition 1. Let g be a non-negative integer and let $\mu_1, \mu_2, \dots, \mu_n$ be positive integers, whose sum we denote by $|\mu|$. The *monotone Hurwitz number* $\vec{H}_{g,n}(\mu_1, \mu_2, \dots, \mu_n)$ is defined to be $\frac{1}{|\mu|!}$ times the number of tuples $(\tau_1, \tau_2, \dots, \tau_m)$ of transpositions in the symmetric group $S_{|\mu|}$ such that

- $m = |\mu| + 2g - 2 + n$;
- $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$ has n cycles labelled $1, 2, \dots, n$, such that the length of cycle i is μ_i;
- $\langle \tau_1, \tau_2, \dots, \tau_m \rangle$ is a transitive subgroup of $S_{|\mu|}$; and

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- if we write $\tau_i = (a_i\ b_i)$ with $a_i < b_i$, then $b_1 \leq b_2 \leq \cdots \leq b_m$.

The *t-monotone Hurwitz number* $\vec{H}^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ is defined analogously, although each tuple is now counted with weight t to the power of its *hive number*, which is defined to be the number of distinct values in the tuple $(b_1, b_2, \ldots, b_m)$.

In the case $(g,n) = (0,1)$, one obtains the Narayana polynomials via the equation

$$\mu\, \vec{H}^t_{0,1}(\mu) = \text{Narayana}_{\mu-1}(t).$$

Thus, we refer to the t-monotone Hurwitz numbers collectively as *topological Narayana polynomials*. The word "topological" here refers to the fact that the t-monotone Hurwitz numbers fit into a large family of enumerative problems governed by the general theory of topological recursion [2]. In that context, the pair (g,n) can be thought of as recording the genus and number of boundary components of a surface.

Example 1. The t-monotone Hurwitz number $\vec{H}^t_{0,2}(2,1)$ counts certain triples (τ_1, τ_2, τ_3) of transpositions in the symmetric group S_3. All such triples appear in the table below and we observe that their compositions always result in a permutation of cycle type $(2,1)$, as required.

~~$(1\,2)\circ(1\,2)\circ(1\,2)$~~	~~$(1\,2)\circ(1\,3)\circ(1\,2)$~~	~~$(1\,2)\circ(2\,3)\circ(1\,2)$~~
$(1\,2)\circ(1\,2)\circ(1\,3)$	$(1\,2)\circ(1\,3)\circ(1\,3)$	$(1\,2)\circ(2\,3)\circ(1\,3)$
$(1\,2)\circ(1\,2)\circ(2\,3)$	$(1\,2)\circ(1\,3)\circ(2\,3)$	$(1\,2)\circ(2\,3)\circ(2\,3)$
~~$(1\,3)\circ(1\,2)\circ(1\,2)$~~	~~$(1\,3)\circ(1\,3)\circ(1\,2)$~~	~~$(1\,3)\circ(2\,3)\circ(1\,2)$~~
~~$(1\,3)\circ(1\,2)\circ(1\,3)$~~	~~$(1\,3)\circ(1\,3)\circ(1\,3)$~~	$(1\,3)\circ(2\,3)\circ(1\,3)$
~~$(1\,3)\circ(1\,2)\circ(2\,3)$~~	$(1\,3)\circ(1\,3)\circ(2\,3)$	$(1\,3)\circ(2\,3)\circ(2\,3)$
~~$(2\,3)\circ(1\,2)\circ(1\,2)$~~	~~$(2\,3)\circ(1\,3)\circ(1\,2)$~~	~~$(2\,3)\circ(2\,3)\circ(1\,2)$~~
~~$(2\,3)\circ(1\,2)\circ(1\,3)$~~	$(2\,3)\circ(1\,3)\circ(1\,3)$	$(2\,3)\circ(2\,3)\circ(1\,3)$
~~$(2\,3)\circ(1\,2)\circ(2\,3)$~~	$(2\,3)\circ(1\,3)\circ(2\,3)$	~~$(2\,3)\circ(2\,3)\circ(2\,3)$~~

The triples violating the transitivity and monotonicity conditions of Definition 1 are crossed out. Of the remaining twelve triples, the six in the top third of the table have $(b_1, b_2, b_3) = (2,2,3)$ or $(2,3,3)$ and hence hive number 2, while the remaining six in the bottom two-thirds of the table have $(b_1, b_2, b_3) = (3,3,3)$ and hence hive number 1. So with our normalisation, we have $\vec{H}^t_{0,2}(2,1) = \frac{1}{3!}(6t^2 + 6t) = t^2 + t$.

2 Recursions

It is natural to ask whether properties of the usual monotone Hurwitz numbers or of the usual Narayana polynomials can be generalised to the setting of topological Narayana polynomials. For example, the known linear recursions for monotone Hurwitz numbers and for Narayana polynomials have the following common generalisation, for which the original results can be recovered by specialising to $t = 1$ and $g = 0$, respectively.

Proposition 1 (Linear recursion). *For $g \geq 0$ and $\mu \geq 1$ such that $(g,\mu) \neq (0,1)$ or $(0,2)$, the "one-point" t-monotone Hurwitz numbers satisfy the recursion*

$$\mu^2 \vec{H}^t_{g,1}(\mu) = (\mu-1)(2\mu-3)(t+1)\vec{H}^t_{g,1}(\mu-1) \\ -(\mu-2)(\mu-3)(t-1)^2\vec{H}^t_{g,1}(\mu-2) + \mu^2(\mu-1)^2\vec{H}^t_{g-1,1}(\mu).$$

In this equation, we set $\vec{H}^t_{g,1}(\mu) = 0$ if $g < 0$ or $\mu < 1$.

Similarly, the known quadratic recursions for monotone Hurwitz numbers and for Narayana polynomials have the following common generalisation, for which the original results can be recovered by specialising to $t = 1$ and $g = 0$, respectively.

Proposition 2 (Quadratic recursion). *Let $S = \{2,3,\ldots,n\}$. For all $g \geq 0$ and $\mu_1 + \mu_2 + \cdots + \mu_n > 1$, the t-monotone Hurwitz numbers satisfy the recursion*

$$\mu_1 \vec{H}^t_{g,n}(\mu_1,\mu_S) = \sum_{i=2}^{n}(\mu_1+\mu_i)\vec{H}^t_{g,n-1}(\mu_1+\mu_i,\mu_{S\setminus\{i\}}) + \sum_{\alpha+\beta=\mu_1}\alpha\beta\,\vec{H}^t_{g-1,n+1}(\alpha,\beta,\mu_S) \\ + \sum_{\alpha+\beta=\mu_1}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}\alpha\beta\,\vec{H}^t_{g_1,|I|+1}(\alpha,\mu_I)\vec{H}^t_{g_2,|J|+1}(\beta,\mu_J) + (t-1)(\mu_1-1)\vec{H}^t_{g,n}(\mu_1-1,\mu_S).$$

In this equation, we set $\vec{H}^t_{g,n}(\mu_1,\mu_2,\ldots,\mu_n) = 0$ if $g < 0$ or $\mu_i < 1$ for some i and, for $I = \{i_1,i_2,\ldots,i_k\}$, we use μ_I to denote the sequence $\mu_{i_1},\mu_{i_2},\ldots,\mu_{i_k}$.

Along with the base cases $\vec{H}^t_{0,1}(1) = 1$ and $\vec{H}^t_{g,1}(1) = 0$ for $g \geq 1$, the quadratic recursion uniquely recovers all t-monotone Hurwitz numbers.

Observe that although $\vec{H}^t_{g,n}(\mu_1,\ldots,\mu_n)$ is inherently symmetric in its arguments, the quadratic recursion is manifestly asymmetric. This provides an instance of a much more general phenomenon exhibited by problems governed by the topological recursion. Although we have already hinted at the fact that t-monotone Hurwitz numbers are indeed governed by the topological recursion, we will not state the result explicitly nor consider the consequences thereof in the present note.

3 Real-rootedness, interlacing, and stability?

The usual Narayana polynomials are known to have only real roots and we conjecture that the same is true for topological Narayana polynomials more generally.

Conjecture 1 (Real-rootedness). Each topological Narayana polynomial $\vec{H}^t_{g,n}(\mu_1,\ldots,\mu_n)$ only has real roots.

One approach to proving the real-rootedness of Narayana polynomials is to show the stronger fact that any two consecutive Narayana polynomials interlace. A polynomial of degree d is said to *interlace* with a polynomial of degree $d+1$ if the roots $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d$ of the first and the roots $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{d+1}$ of the second are real and satisfy

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \beta_d \leq \alpha_d \leq \beta_{d+1}.$$

One observes a similar phenomenon for topological Narayana polynomials.

Conjecture 2 (Interlacing). The topological Narayana polynomial $\vec{H}^t_{g,n}(\mu_1,\ldots,\mu_n)$ interlaces with each of the topological Narayana polynomials

$$\vec{H}^t_{g,n}(\mu_1+1,\mu_2,\ldots,\mu_n),\ \vec{H}^t_{g,n}(\mu_1,\mu_2+1,\ldots,\mu_n),\ \ldots,\ \vec{H}^t_{g,n}(\mu_1,\mu_2,\ldots,\mu_n+1).$$

The quadratic recursion gives an effective way to compute topological Narayana polynomials, from which one can then test the above predictions. Real-rootedness and interlacing have been confirmed for all $g+n \leq 5$ and $|\mu| \leq 10$. This amounts to over 300 checks, thus providing strong numerical evidence for the conjectures.

Given the close connection between real-rootedness for polynomials of one variable and stability for polynomials of many, one is led to consider whether the topological Narayana polynomials admit a natural multivariate generalisation that exhibits stability. Recall that a multivariate polynomial is said to be *stable* if it is non-zero whenever its arguments are evaluated at complex numbers with positive imaginary part.

An obvious way to define a multivariate version of the t-monotone Hurwitz numbers is to mimic the enumeration of Definition 1, associating to a tuple of transpositions $((a_1\ b_1),(a_2\ b_2),\ldots,(a_m\ b_m))$ the weight

$$\prod_{k\in(b_1,b_2,\ldots,b_m)} t_k,$$

where we ignore multiplicities in the tuple $(b_1,b_2,\ldots,b_m)$. One then obtains multiaffine polynomials in $\mathbb{Q}[t_2,t_3,\ldots]$ for which the corresponding t-monotone Hurwitz numbers are recovered by setting $t_k \mapsto t$ for all k. It is natural to wonder whether such polynomials are stable.

4 Data

The following table gives some examples of t-monotone Hurwitz numbers.

$(\mu_1,\ldots,\mu_n)$	$\mu_1\cdots\mu_n\vec{H}^t_{0,n}(\mu_1,\ldots,\mu_n)$	$\mu_1\cdots\mu_n\vec{H}^t_{1,n}(\mu_1,\ldots,\mu_n)$	$\mu_1\cdots\mu_n\vec{H}^t_{2,n}(\mu_1,\ldots,\mu_n)$
(1)	1	0	0
(2)	t	t	t
(3)	t^2+t	$5t^2+5t$	$21t^2+21t$
(4)	t^3+3t^2+t	$15t^3+40t^2+15t$	$161t^3+413t^2+161t$
(5)	$t^4+6t^3+6t^2+t$	$35t^4+175t^3+175t^2+35t$	$777t^4+3612t^3+3612t^2+777t$
$(1,1)$	t	t	t
$(2,1)$	$2t^2+2t$	$10t^2+10t$	$42t^2+42t$
$(3,1)$	$3t^3+9t^2+3t$	$45t^3+120t^2+45t$	$483t^3+1239t^2+483t$
$(2,2)$	$4t^3+10t^2+4t$	$50t^3+128t^2+50t$	$504t^3+1278t^2+504t$
$(4,1)$	$4t^4+24t^3+24t^2+4t$	$140t^4+700t^3+700t^2+140t$	$3108t^4+14448t^3+14448t^2+3108t$
$(3,2)$	$6t^4+30t^3+30t^2+6t$	$168t^4+792t^3+792t^2+168t$	$3402t^4+15450t^3+15450t^2+3402t$
$(1,1,1)$	$4t^2+4t$	$20t^2+20t$	$84t^2+84t$
$(2,1,1)$	$10t^3+28t^2+10t$	$140t^3+368t^2+140t$	$1470t^3+3756t^2+1470t$
$(3,1,1)$	$18t^4+102t^3+102t^2+18t$	$588t^4+2892t^3+2892t^2+588t$	$12726t^4+58794t^3+58794t^2+12726t$
$(2,2,1)$	$24t^4+120t^3+120t^2+24t$	$672t^4+3168t^3+3168t^2+672t$	$13608t^4+61800t^3+61800t^2+13608t$

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