Tree-Partitions with Bounded Degree Trees

Marc Distel and David R. Wood

Abstract A *tree-partition* of a graph G is a partition of $V(G)$ such that identifying the vertices in each part gives a tree. It is known that every graph with treewidth k and maximum degree Δ has a tree-partition with parts of size $O(k\Delta)$. We prove the same result with the extra property that the underlying tree has maximum degree $O(\Delta)$.

1 Introduction

For a graph G and a tree T, a *T-partition* of G is a partition $(V_x : x \in V(T))$ of $V(G)$ indexed by the nodes of T, such that for every edge vw of G, if $v \in V_x$ and $w \in V_y$, then $x = y$ or $xy \in E(T)$. The *width* of a T-partition is $\max\{|V_x| : x \in V(T)\}$. The *tree-partition-width*[1] of a graph G is the minimum width of a tree-partition of G.

Tree-partitions were independently introduced by Seese [31] and Halin [24], and have since been widely investigated [6–8, 15, 16, 21, 32, 33]. Applications of tree-partitions include graph drawing [12, 14, 19, 20, 34], graphs of linear growth [11], nonrepetitive graph colouring [2], clustered graph colouring [1, 27], monadic second-order logic [26], network emulations [3, 4, 9, 22], statistical learning theory [35], and the edge-Erdős-Pósa property [13, 23, 28]. Tree-partitions are also related to graph product structure theory since a graph G has a T-partition of width at most k if and only if G is isomorphic to a subgraph of $T \boxtimes K_k$ for some tree T; see [10, 17, 18] for example.

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[1] Tree-partition-width has also been called *strong treewidth* [7, 31].

Bounded tree-partition-width implies bounded treewidth[2], as noted by Seese [31]. In particular, for every graph G,

$$\operatorname{tw}(G) \leqslant 2\operatorname{tpw}(G) - 1.$$

Of course, $\operatorname{tw}(T) = \operatorname{tpw}(T) = 1$ for every tree T. But in general, $\operatorname{tpw}(G)$ can be much larger than $\operatorname{tw}(G)$. For example, fan graphs on n vertices have treewidth 2 and tree-partition-width $\Omega(\sqrt{n})$. On the other hand, the referee of [15] showed that if the maximum degree and treewidth are both bounded, then so is the tree-partition-width, which is one of the most useful results about tree-partitions. A graph G is *trivial* if $E(G) = \varnothing$. Let $\Delta(G)$ be the maximum degree of G.

Theorem 1 ([15])**.** *For any non-trivial graph G,*

$$\operatorname{tpw}(G) \leqslant 24(\operatorname{tw}(G)+1)\Delta(G).$$

Theorem 1 is stated in [15] with "$\operatorname{tw}(G)$" instead of "$\operatorname{tw}(G)+1$", but a close inspection of the proof shows that "$\operatorname{tw}(G)+1$" is needed. Wood [33] showed that Theorem 1 is best possible up to the multiplicative constant, and also improved the constant 24 to $9+6\sqrt{2} \approx 17.48$.

This paper considers the maximum degree of T in a T-partition. Consider a tree-partition $(B_x : x \in V(T))$ of a graph G with width k. For each node $x \in V(T)$, there are at most $\sum_{v \in B_x} \deg(v)$ edges between B_x and $G - B_x$. Thus we may assume that $\deg_T(x) \leqslant |B_x|\Delta(G) \leqslant k\Delta(G)$, otherwise delete an 'unused' edge of T and add an edge to T between leaf vertices of the resulting component subtrees. It follows that if $\operatorname{tpw}(G) \leqslant k$ then G has a T-partition of width at most k for some tree T with maximum degree at most $\max\{k\Delta(G), 2\}$. By Theorem 1, every graph G has a T-partition of width at most $24(\operatorname{tw}(G)+1)\Delta(G)$ for some tree T with maximum degree at most $24(\operatorname{tw}(G)+1)\Delta(G)^2$. This fact has been used in several applications of Theorem 1 (see [12, 20] for example). The following theorem improves this upper bound on $\Delta(T)$. Indeed, $\Delta(T)$ is independent of $\operatorname{tw}(G)$.

Theorem 2. *Every non-trivial graph G has a T-partition of width at most*

$$18(\operatorname{tw}(G)+1)\Delta(G)$$

for some tree T with $\Delta(T) \leqslant 6\Delta(G)$.

Theorem 2 enables a $\operatorname{tw}(G)\Delta(G)^2$ term to be replaced by a $\Delta(G)$ term in various results [12, 20].

[2] A *tree-decomposition* of a graph G is a collection $(B_x \subseteq V(G) : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the nodes of a tree T, such that: (a) for every edge $uv \in E(G)$, some bag B_x contains both u and v; and (b) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty subtree of T. The *width* of a tree-decomposition is the size of the largest bag, minus 1. The *treewidth* of a graph G, denoted by $\operatorname{tw}(G)$, is the minimum width of a tree-decomposition of G. Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [25, 29**?**] for surveys.

As mentioned above, Wood [33] improved the constant 24 to $9+6\sqrt{2}$ in Theorem 1. By tweaking the constants in the proof of Theorem 2, we match this constant with a small increase in the bound on $\Delta(T)$; see Section 3. We choose to first present the proof with integer coefficients for ease of understanding.

Our final result shows that the linear upper bound on $\Delta(T)$ in Theorem 2 is best possible even for trees.

Proposition 3. *For any integer $\Delta \geqslant 3$ there exist $\alpha > 0$ such that there are infinitely many trees X with maximum degree Δ such that for every tree T with maximum degree less than Δ, every T-partition of X has width at least $|V(X)|^{\alpha}$. Moreover, if $\Delta = 3$ then α can be taken to be arbitrarily close to* 1.

2 Main Proofs

Theorem 2 is implied by the following lemma. The proof is identical to the proof of Theorem 1, except that we pay attention to $\Delta(T)$. Let $\mathbb{N} = \{1,2,\dots\}$.

Lemma 4. *Fix $k,d \in \mathbb{N}$. Let G be a graph with $\operatorname{tw}(G) \leqslant k-1$ and $\Delta(G) \leqslant d$. Then G has a tree-partition $(B_x : x \in V(T))$ of width at most $18kd$ such that $\Delta(T) \leqslant 6d$. Moreover, for any set $S \subseteq V(G)$ with $4k \leqslant |S| \leqslant 12kd$, there exists a tree-partition $(B_x : x \in V(T))$ of G with width at most $18kd$, such that $\Delta(T) \leqslant 6d$ and there exists $z \in V(T)$ such that:*
- *$S \subseteq B_z$,*
- *$|B_z| \leqslant \frac{3}{2}|S| - 2k$,*
- *$\deg_T(z) \leqslant \frac{|S|}{2k} - 1$.*

Proof. We proceed by induction on $|V(G)|$.

Case 1. $|V(G)| < 4k$: Then S is not specified. Let T be the 1-vertex tree with $V(T) = \{x\}$, and let $B_x := V(G)$. Then $(B_x : x \in V(T))$ is the desired tree-partition, since $|B_x| = |V(G)| < 4k \leqslant 18kd$ and $\Delta(T) = 0 \leqslant 6d$.

Now assume that $|V(G)| \geqslant 4k$. If S is not specified, then let S be any set of $4k$ vertices in G.

Case 2. $|V(G-S)| \leqslant 18kd$: Let T be the 2-vertex tree with $V(T) = \{y,z\}$ and $E(T) = \{yz\}$. Note that $\Delta(T) = 1 \leqslant 6d$ and $\deg_T(z) = 1 \leqslant \frac{|S|}{2k} - 1$. Let $B_z := S$ and $B_y := V(G-S)$. Thus $|B_z| = |S| \leqslant \frac{3}{2}|S| - 2k \leqslant 18kd$ and $|B_y| \leqslant |V(G-S)| \leqslant 18kd$. Hence $(B_x : x \in V(T))$ is the desired tree-partition of G.

Now assume that $|V(G-S)| \geqslant 18kd$.

Case 3. $4k \leqslant |S| \leqslant 12k$: Let $S' := \bigcup\{N_G(v) \setminus S : v \in S\}$. Thus $|S'| \leqslant d|S| \leqslant 12kd$. If $|S'| < 4k$ then add $4k - |S'|$ vertices from $V(G-S-S')$ to S', so that $|S'| = 4k$. This is well-defined since $|V(G-S)| \geqslant 18kd \geqslant 4k$, implying $|V(G-S-S')| \geqslant 4k - |S'|$. By induction, there exists a tree-partition $(B_x : x \in V(T'))$ of $G-S$ with width at most $18kd$, such that $\Delta(T') \leqslant 6d$ and there exists $z' \in V(T')$ such that:
- $S' \subseteq B_{z'}$,

- $|B_{z'}| \leqslant \frac{3}{2}|S'| - 2k \leqslant 18kd - 2k$,
- $\deg_{T'}(z') \leqslant \frac{|S'|}{2k} - 1 \leqslant 6d - 1$.

Let T be the tree obtained from T' by adding one new node z adjacent to z'. Let $B_z := S$. So $(B_x : x \in V(T))$ is a tree-partition of G with width at most $\max\{18kd, |S|\} \leqslant \max\{18kd, 12k\} = 18kd$. By construction, $\deg_T(z) = 1 \leqslant \frac{|S|}{2k} - 1$ and $\deg_T(z') = \deg_{T'}(z') + 1 \leqslant (6d-1) + 1 = 6d$. Every other vertex in T has the same degree as in T'. Hence $\Delta(T) \leqslant 6d$, as desired. Finally, $S = B_z$ and $|B_z| = |S| \leqslant \frac{3}{2}|S| - 2k$.

Case 4. $12k \leqslant |S| \leqslant 12kd$: By the separator lemma of Robertson and Seymour [30, (2.6)], there are induced subgraphs G_1 and G_2 of G with $G_1 \cup G_2 = G$ and $|V(G_1 \cap G_2)| \leqslant k$, where $|S \cap V(G_i)| \leqslant \frac{2}{3}|S|$ for each $i \in \{1,2\}$. Let $S_i := (S \cap V(G_i)) \cup V(G_1 \cap G_2)$ for each $i \in \{1,2\}$.

We now bound $|S_i|$. For a lower bound, since $|S \cap V(G_1)| \leqslant \frac{2}{3}|S|$, we have $|S_2| \geqslant |S \setminus V(G_1)| \geqslant \frac{1}{3}|S| \geqslant 4k$. By symmetry, $|S_1| \geqslant 4k$. For an upper bound, $|S_i| \leqslant \frac{2}{3}|S| + k \leqslant 8kd + k \leqslant 12kd$. Also note that $|S_1| + |S_2| \leqslant |S| + 2k$.

We have shown that $4k \leqslant |S_i| \leqslant 12kd$ for each $i \in \{1,2\}$. Thus we may apply induction to G_i with S_i the specified set. Hence there exists a tree-partition $(B^i_x : x \in V(T_i))$ of G_i with width at most $18kd$, such that $\Delta(T_i) \leqslant 6d$ and there exists $z_i \in V(T_i)$ such that:

- $S_i \subseteq B_{z_i}$,
- $|B_{z_i}| \leqslant \frac{3}{2}|S_i| - 2k$,
- $\deg_{T_i}(z_i) \leqslant \frac{|S_i|}{2k} - 1$.

Let T be the tree obtained from the disjoint union of T_1 and T_2 by merging z_1 and z_2 into a vertex z. Let $B_z := B^1_{z_1} \cup B^2_{z_2}$. Let $B_x := B^i_x$ for each $x \in V(T_i) \setminus \{z_i\}$. Since $G = G_1 \cup G_2$ and $V(G_1 \cap G_2) \subseteq B^1_{z_1} \cap B^2_{z_2} \subseteq B_z$, we have that $(B_x : x \in V(T))$ is a tree-partition of G. By construction, $S \subseteq B_z$ and since $V(G_1 \cap G_2) \subseteq B^i_{z_i}$ for each i,

$$\begin{aligned}
|B_z| &\leqslant |B^1_{z_1}| + |B^2_{z_2}| - |V(G_1 \cap G_2)| \\
&\leqslant (\tfrac{3}{2}|S_1| - 2k) + (\tfrac{3}{2}|S_2| - 2k) - |V(G_1 \cap G_2)| \\
&= \tfrac{3}{2}(|S_1| + |S_2|) - 4k - |V(G_1 \cap G_2)| \\
&\leqslant \tfrac{3}{2}(|S| + 2|V(G_1 \cap G_2)|) - 4k - |V(G_1 \cap G_2)| \\
&\leqslant \tfrac{3}{2}|S| + 2|V(G_1 \cap G_2)| - 4k \\
&\leqslant \tfrac{3}{2}|S| - 2k \\
&< 18kd.
\end{aligned}$$

Every other part has the same size as in the tree-partition of G_1 or G_2. So this tree-partition of G has width at most $18kd$. Note that

$$\begin{aligned}
\deg_T(z) = \deg_{T_1}(z_1) + \deg_{T_2}(z_2) &\leqslant (\tfrac{|S_1|}{2k} - 1) + (\tfrac{|S_2|}{2k} - 1) \\
&= \tfrac{|S_1| + |S_2|}{2k} - 2 \\
&\leqslant \tfrac{|S| + 2k}{2k} - 2
\end{aligned}$$

$$= \tfrac{|S|}{2k} - 1$$
$$< 6d.$$

Every other node of T has the same degree as in T_1 or T_2. Thus $\Delta(T) \leqslant 6d$. This completes the proof. □

We now prove the lower bound. For $\Delta, d \in \mathbb{N}$ with $\Delta \geqslant 2$, let $X_{\Delta,d}$ be the tree rooted at a vertex r such that every leaf is at distance d from r and every non-leaf vertex has degree Δ. Observe that $X_{\Delta,d}$ has the maximum number of vertices in a tree with maximum degree Δ and radius d, where

$$|V(X_{\Delta,d})| = 1 + \Delta \sum_{i=0}^{d-1} (\Delta - 1)^i.$$

Note that $|V(X_{2,d})| = 2d + 1$, and if $\Delta \geqslant 3$ then

$$(\Delta - 1)^d \leqslant |V(X_{\Delta,d})| = 1 + \tfrac{\Delta}{\Delta - 2}\big((\Delta - 1)^d - 1\big) \leqslant 3(\Delta - 1)^d.$$

Proposition 3. *For any integer $\Delta \geqslant 3$ there exist $\alpha > 0$ such that there are infinitely many trees X with maximum degree Δ such that for every tree T with maximum degree less than Δ, every T-partition of X has width at least $|V(X)|^{\alpha}$. Moreover, if $\Delta = 3$ then α can be taken to be arbitrarily close to* 1.

Proof. First suppose that $\Delta \geqslant 4$. Let $d_0 \in \mathbb{N}$ be sufficiently large so that $(\frac{\Delta-1}{\Delta-2})^{d_0} > 3$. Let $\alpha := 1 - \log_{\Delta-1}(3^{1/d_0}(\Delta - 2))$, which is positive by the choice of d_0. Let $d \in \mathbb{N}$ with $d \geqslant d_0$. It follows that $(\Delta - 1)^{(1-\alpha)d} \geqslant 3(\Delta - 2)^d$. Consider any tree-partition $(B_u : u \in V(T))$ of $X_{\Delta,d}$, where T is any tree with maximum degree at most $\Delta - 1$. Let z be the vertex of T such that the root $r \in B_z$. Since adjacent vertices in $X_{\Delta,d}$ belong to adjacent parts or the same part in T, every vertex in T is at distance at most d from z. Thus T has radius at most d, and

$$|V(T)| \leqslant |V(X_{\Delta-1,d})| \leqslant 3(\Delta - 2)^d \leqslant (\Delta - 1)^{(1-\alpha)d} \leqslant |V(X_{\Delta,d})|^{1-\alpha}.$$

By the pigeon-hole principle, there is a vertex $u \in V(T)$ such that $|B_u| \geqslant \frac{|V(X_{\Delta,d})|}{|V(T)|} \geqslant |V(X_{\Delta,d})|^{\alpha}$.

Now assume that $\Delta = 3$. Let $\alpha \in (0,1)$, let $d_0 \in \mathbb{N}$ be sufficiently large so that $2d_0 + 1 \leqslant 2^{(1-\alpha)d_0}$, and let $d \geqslant d_0$. So $2d + 1 \leqslant 2^{(1-\alpha)d}$. Consider any tree-partition $(B_u : u \in V(T))$ of $X_{3,d}$, where T is any tree with maximum degree at most 2. By the argument above, T has radius at most d, implying

$$|V(T)| \leqslant |V(X_{2,d})| = 2d + 1 \leqslant 2^{(1-\alpha)d} \leqslant |V(X_{3,d})|^{1-\alpha}.$$

Again, there is a vertex $u \in V(T)$ such that $|B_u| \geqslant \frac{|V(X_{3,d})|}{|V(T)|} \geqslant |V(X_{3,d})|^{\alpha}$. □

3 Tweaking the Constants

Consider the following generalisation of Lemma 4.

Lemma 5. *Fix $k,d \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ with $\alpha > 2$. Let G be a graph with $\operatorname{tw}(G) \leqslant k-1$ and $\Delta(G) \leqslant d$. Then G has a tree-partition $(B_x : x \in V(T))$ of width at most*

$$\tfrac{3\alpha(\alpha-1)}{\alpha-2}kd - \tfrac{\alpha}{\alpha-2}k$$

such that $\Delta(T) \leqslant \frac{3\alpha}{\alpha-2}d + \frac{\alpha-4}{\alpha-2}$. Moreover, for any set $S \subseteq V(G)$ with $\alpha k \leqslant |S| \leqslant 3\alpha kd$, there exists a tree-partition $(B_x : x \in V(T))$ of G with width at most $\frac{3\alpha(\alpha-1)}{\alpha-2}kd - \frac{\alpha}{\alpha-2}k$, such that $\Delta(T) \leqslant \frac{3\alpha}{\alpha-2}d + \frac{\alpha-4}{\alpha-2}$ and there exists $z \in V(T)$ such that:

- *$S \subseteq B_z$,*
- *$|B_z| \leqslant \frac{\alpha-1}{\alpha-2}|S| - \frac{\alpha}{\alpha-2}k$,*
- *$\deg_T(z) \leqslant \frac{1}{(\alpha-2)k}|S| - \frac{2}{\alpha-2}$.*

Proof. We proceed by induction on $|V(G)|$.

Case 1. $|V(G)| < \alpha k$: Then S is not specified. Let T be the 1-vertex tree with $V(T) = \{x\}$, and let $B_x := V(G)$. Then $(B_x : x \in V(T))$ is the desired tree-partition, since $|B_x| = |V(G)| < \alpha k \leqslant \frac{3\alpha(\alpha-1)}{\alpha-2}kd - \frac{\alpha}{\alpha-2}k$ and $\Delta(T) = 0 \leqslant \frac{3\alpha}{\alpha-2}d + \frac{\alpha-4}{\alpha-2}$.

Now assume that $|V(G)| \geqslant \alpha k$. If S is not specified, then let S be any set of $\lceil \alpha k \rceil$ vertices in G (implying $|S| \leqslant 3\alpha kd$).

Case 2. $|V(G-S)| \leqslant \frac{3\alpha(\alpha-1)}{\alpha-2}kd - \frac{\alpha}{\alpha-2}k$: Let T be the 2-vertex tree with $V(T) = \{y,z\}$ and $E(T) = \{yz\}$. Note that $\Delta(T) = 1 \leqslant \frac{3\alpha}{\alpha-2}d + \frac{\alpha-4}{\alpha-2}$ and $\deg_T(z) = 1 \leqslant \frac{1}{(\alpha-2)k}|S| - \frac{2}{\alpha-2}$. Let $B_z := S$ and $B_y := V(G-S)$. Thus $|B_z| = |S| \leqslant \frac{\alpha-1}{\alpha-2}|S| - \frac{\alpha}{\alpha-2}k \leqslant \frac{3\alpha(\alpha-1)}{\alpha-2}kd - \frac{\alpha}{\alpha-2}k$ and $|B_y| \leqslant |V(G-S)| \leqslant \frac{3\alpha(\alpha-1)}{\alpha-2}kd$. Hence $(B_x : x \in V(T))$ is the desired tree-partition of G.

Now assume that $|V(G-S)| \geqslant \frac{3\alpha(\alpha-1)}{\alpha-2}kd - \frac{\alpha}{\alpha-2}k$.

Case 3. $\alpha k \leqslant |S| \leqslant 3\alpha k$: Let $S' := \bigcup_{v \in S} N_G(v) \setminus S$. Thus $|S'| \leqslant d|S| \leqslant 3\alpha kd$. If $|S'| < \alpha k$ then add $\alpha k - |S'|$ vertices from $V(G-S-S')$ to S', so that $|S'| = \alpha k$. This is well-defined since $|V(G-S)| \geqslant \frac{3\alpha(\alpha-1)}{\alpha-2}kd - \frac{\alpha}{\alpha-2}k \geqslant \alpha k$, implying $|V(G-S-S')| \geqslant \alpha k - |S'|$. By induction, there exists a tree-partition $(B_x : x \in V(T'))$ of $G-S$ with width at most $\frac{3\alpha(\alpha-1)}{\alpha-2}kd - \frac{\alpha}{\alpha-2}k$, such that $\Delta(T') \leqslant \frac{3\alpha}{\alpha-2}d + \frac{\alpha-4}{\alpha-2}$ and there exists $z' \in V(T')$ such that:

- $S' \subseteq B_{z'}$,
- $|B_{z'}| \leqslant \frac{\alpha-1}{\alpha-2}|S'| - \frac{\alpha}{\alpha-2}k \leqslant \frac{3\alpha(\alpha-1)}{\alpha-2}kd - \frac{\alpha}{\alpha-2}k$,
- $\deg_{T'}(z') \leqslant \frac{1}{(\alpha-2)k}|S'| - \frac{2}{\alpha-2} \leqslant \frac{3\alpha}{\alpha-2}d - \frac{2}{\alpha-2}$.

Let T be the tree obtained from T' by adding one new node z adjacent to z'. Let $B_z := S$. So $(B_x : x \in V(T))$ is a tree-partition of G with width at most

$$\max\{\tfrac{3\alpha(\alpha-1)}{\alpha-2}kd - \tfrac{\alpha}{\alpha-2}k, |S|\} \leqslant \max\{\tfrac{3\alpha(\alpha-1)}{\alpha-2}kd - \tfrac{\alpha}{\alpha-2}k, 3\alpha k\} = \tfrac{3\alpha(\alpha-1)}{\alpha-2}kd - \tfrac{\alpha}{\alpha-2}k.$$

By construction,

$$\deg_T(z) = 1 \leqslant \tfrac{1}{(\alpha-2)k}|S| - \tfrac{2}{\alpha-2} \quad \text{and}$$
$$\deg_T(z') = \deg_{T'}(z') + 1 \leqslant \tfrac{3\alpha}{\alpha-2}d - \tfrac{2}{\alpha-2} + 1 = \tfrac{3\alpha}{\alpha-2}d + \tfrac{\alpha-4}{\alpha-2}.$$

Every other vertex in T has the same degree as in T'. Hence $\Delta(T) \leqslant \frac{3\alpha}{\alpha-2}d + \frac{\alpha-4}{\alpha-2}$, as desired. Finally, $S = B_z$ and $|B_z| = |S| \leqslant \frac{\alpha-1}{\alpha-2}|S| - \frac{\alpha}{\alpha-2}k$.

Case 4. $3\alpha k \leqslant |S| \leqslant 3\alpha kd$: By the separator lemma of Robertson and Seymour [30, (2.6)], there are induced subgraphs G_1 and G_2 of G with $G_1 \cup G_2 = G$ and $|V(G_1 \cap G_2)| \leqslant k$, where $|S \cap V(G_i)| \leqslant \frac{2}{3}|S|$ for each $i \in \{1,2\}$. Let $S_i := (S \cap V(G_i)) \cup V(G_1 \cap G_2)$ for each $i \in \{1,2\}$.

We now bound $|S_i|$. For a lower bound, since $|S \cap V(G_1)| \leqslant \frac{2}{3}|S|$, we have $|S_2| \geqslant |S \setminus V(G_1)| \geqslant \frac{1}{3}|S| \geqslant \frac{1}{3}3\alpha k \geqslant \alpha k$. By symmetry, $|S_1| \geqslant \alpha k$. For an upper bound, $|S_i| \leqslant \frac{2}{3}|S| + k \leqslant 2\alpha kd + k \leqslant 3\alpha kd$. Also note that $|S_1| + |S_2| \leqslant |S| + 2|V(G_1 \cap G_2)| \leqslant |S| + 2k$.

We have shown that $\alpha k \leqslant |S_i| \leqslant 3\alpha kd$ for each $i \in \{1,2\}$. Thus we may apply induction to G_i with S_i the specified set. Hence there exists a tree-partition $(B^i_x : x \in V(T_i))$ of G_i with width at most $\frac{3\alpha(\alpha-1)}{\alpha-2}kd - \frac{\alpha}{\alpha-2}k$, such that $\Delta(T_i) \leqslant \frac{3\alpha}{\alpha-2}d + \frac{\alpha-4}{\alpha-2}$ and there exists $z_i \in V(T_i)$ such that:

- $S_i \subseteq B_{z_i}$,
- $|B_{z_i}| \leqslant \frac{\alpha-1}{\alpha-2}|S_i| - \frac{\alpha}{\alpha-2}k$,
- $\deg_{T_i}(z_i) \leqslant \frac{1}{(\alpha-2)k}|S_i| - \frac{2}{\alpha-2}$.

Let T be the tree obtained from the disjoint union of T_1 and T_2 by merging z_1 and z_2 into a vertex z. Let $B_z := B^1_{z_1} \cup B^2_{z_2}$. Let $B_x := B^i_x$ for each $x \in V(T_i) \setminus \{z_i\}$. Since $G = G_1 \cup G_2$ and $V(G_1 \cap G_2) = B^1_{z_1} \cap B^2_{z_2} \subseteq B_z$, we have that $(B_x : x \in V(T))$ is a tree-partition of G. By construction, $S \subseteq B_z$ and since $V(G_1 \cap G_2) \subseteq B^i_{z_i}$ for each i,

$$\begin{aligned}
|B_z| &\leqslant |B^1_{z_1}| + |B^2_{z_2}| - |V(G_1 \cap G_2)| \\
&\leqslant \left(\tfrac{\alpha-1}{\alpha-2}|S_1| - \tfrac{\alpha}{\alpha-2}k\right) + \left(\tfrac{\alpha-1}{\alpha-2}|S_2| - \tfrac{\alpha}{\alpha-2}k\right) - |V(G_1 \cap G_2)| \\
&= \tfrac{\alpha-1}{\alpha-2}(|S_1| + |S_2|) - \tfrac{2\alpha}{\alpha-2}k - |V(G_1 \cap G_2)| \\
&\leqslant \tfrac{\alpha-1}{\alpha-2}(|S| + 2|V(G_1 \cap G_2)|) - \tfrac{2\alpha}{\alpha-2}k - |V(G_1 \cap G_2)| \\
&= \tfrac{\alpha-1}{\alpha-2}|S| - \tfrac{2\alpha}{\alpha-2}k + \tfrac{\alpha}{\alpha-2}|V(G_1 \cap G_2)| \\
&\leqslant \tfrac{\alpha-1}{\alpha-2}|S| - \tfrac{2\alpha}{\alpha-2}k + \tfrac{\alpha}{\alpha-2}k \\
&= \tfrac{\alpha-1}{\alpha-2}|S| - \tfrac{\alpha}{\alpha-2}k \\
&\leqslant \tfrac{3\alpha(\alpha-1)}{\alpha-2}kd - \tfrac{\alpha}{\alpha-2}k.
\end{aligned}$$

Every other part has the same size as in the tree-partition of G_1 or G_2. So this tree-partition of G has width at most $\frac{3\alpha(\alpha-1)}{\alpha-2}kd - \frac{\alpha}{\alpha-2}k$. Note that

$$\deg_T(z) = \deg_{T_1}(z_1) + \deg_{T_2}(z_2)$$

$$
\begin{aligned}
&\leqslant \tfrac{1}{(\alpha-2)k}|S_1| - \tfrac{2}{\alpha-2} + \tfrac{1}{(\alpha-2)k}|S_2| - \tfrac{2}{\alpha-2} \\
&\leqslant \tfrac{1}{(\alpha-2)k}(|S_1|+|S_2|) - \tfrac{4}{\alpha-2} \\
&\leqslant \tfrac{1}{(\alpha-2)k}(|S|+2k) - \tfrac{4}{\alpha-2} \\
&\leqslant \tfrac{1}{(\alpha-2)k}|S| - \tfrac{2}{\alpha-2} \\
&\leqslant \tfrac{3\alpha}{\alpha-2}d - \tfrac{2}{\alpha-2} \\
&< \tfrac{3\alpha}{\alpha-2}d + \tfrac{\alpha-4}{\alpha-2}.
\end{aligned}
$$

Every other node of T has the same degree as in T_1 or T_2. Thus $\Delta(T) \leqslant \frac{3\alpha}{\alpha-2}d + \frac{\alpha-4}{\alpha-2}$. This completes the proof. □

Lemma 5 with $\alpha = 4$ implies the following slight strengthening of Theorem 2.

Theorem 6. *Every non-trivial graph G has a T-partition of width at most*

$$2(\mathrm{tw}(G)+1)(9\Delta(G)-1),$$

for some tree T with $\Delta(T) \leqslant 6\Delta(G)$.

Lemma 5 with $\alpha = 2+\sqrt{2}$ (chosen to minimise $\frac{3\alpha(\alpha-1)}{\alpha-2}$) implies the next result.

Theorem 7. *Every non-trivial graph G has a T-partition of width at*

$$(1+\sqrt{2})(\mathrm{tw}(G)+1)(3(1+\sqrt{2})\Delta(G)-1),$$

for some tree T with $\Delta(T) \leqslant (3+3\sqrt{2})\Delta(G) - 3(\sqrt{2}-1)$.

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