Pachner's Theorem

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Abstract Pachner's Theorem is a purely combinatorial theorem which plays an important role in low-dimensional topology. We give a topology-free self-contained proof of Pachner's Theorem and we outline why Pachner's Theorem is such an essential tool in low-dimensional topology.

1 Introduction

We start out with our definition of an abstract simplicial complex.

Definition 1.

1. An *abstract simplicial complex* is a pair $K = (V,S)$, where V is a finite set, and S is a subset of the power set $\mathscr{P}(V)$ of V satisfying the following three conditions:

   (a) Each element $s \in S$ is a non-empty subset of V.
   (b) Given $s \in S$ and $t \subseteq s$ with $t \neq \varnothing$, we have $t \in S$.
   (c) For all $v \in V$, the singleton $\{v\}$ lies in S.

   We call elements in V *vertices of* K, elements in S *simplices of* K, and we refer to V as the *vertex set of* K and to S as the *simplex set of* K.
2. Let $K = (V,S)$ and $L = (W,T)$ be abstract simplicial complexes. A *simplicial map from* K *to* L is a map $f \colon V \to W$ on the vertex sets such that for all $s \in S$

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we have $f(s) := \{f(v) | v \in s\} \in T$. Such a simplicial map is called a *simplicial isomorphism* if it is a bijection on the vertex sets and on the simplex sets. (One way to view simplicial isomorphisms is simply as renaming the vertices, of course in a way that fully preserves all the simplices.)

As an example we introduce two families of abstract simplicial complexes which play an important role throughout. Let $n \in \mathbb{N}_0$. We consider the set $V_n := \{0, \ldots, n\}$. We set

$$\begin{aligned} D_n &:= (V_n, \mathscr{P}(V_n) \setminus \{\varnothing\}) && (\textit{standard simplicial n-disk}), \\ S_n &:= (V_{n+1}, \mathscr{P}(V_{n+1}) \setminus \{\varnothing, V_{n+1}\}) && (\textit{standard simplicial n-sphere}). \end{aligned}$$

We furthermore define D_{-1} and S_{-1} to be the empty abstract simplicial complex $(\varnothing, \varnothing)$.

Fig. 1: A visualisation of the abstract simplicial complexes D_2 and S_1

In Chapter 4 we introduce the purely combinatorial notion of a stellar subdivision σ_s of an abstract simplicial complex K along a simplex s. Loosely speaking this is defined as follows: we add the barycenter of s to the vertex set and we subdivide all simplices that have s as a face accordingly. In Figure 2 we illustrate the stellar subdivision along a 1-dimensional simplex.

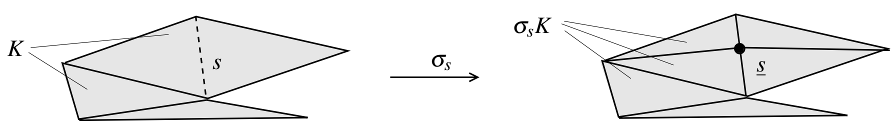

Fig. 2: Stellar subdivision at a 1-simplex

Two abstract simplicial complexes are called *stellar equivalent* if one can go from one to the other through a finite sequence of stellar subdivisions, inverses of stellar subdivisions and simplicial isomorphisms.

To any abstract simplicial complex K one can associate its topological realization $|K|$ and given two such topological realizations there is a natural notion of a PL-homeomorphism between them, which is now a topological notion. We do not give the precise definitions, instead we refer to [Gla70, Lic99, Fri23] for details. The following theorem, which we do not use otherwise, now gives one reason why the notion of stellar equivalence is so important.

Theorem 1. (Alexander-Newman Theorem) *Let K and L be two abstract simplicial complexes. The following two statements are equivalent:*

(1) *The abstract simplicial complexes K and L are stellar equivalent.*
(2) *The topological realizations $|K|$ and $|L|$ are* PL-*homeomorphic.*

Proof. The "(1)⇒(2)"-direction follows easily from the definitions. The much harder "(2)⇒(1)"-direction was first proved by James Alexander [Ale30, Theorem 15:1] in 1930, building on work of Max Newman [New26]. More modern expositions of the proof are given in [Gla70, Theorem II.17] and in [Lic99, Theorem 4.5].

In Chapter 4 we introduce the purely combinatorial notion of a closed combinatorial n-manifold. In a nutshell a closed combinatorial n-manifold is an abstract simplicial complex K such that for every vertex v the link $\mathrm{Lk}(K,v)$ is stellar equivalent to the standard simplicial n-sphere S_n.

Next, let $K=(V,S)$ be a closed combinatorial n-manifold. Furthermore let s be a k-simplex such the link $\mathrm{Lk}(K,s)$ is simplicially isomorphic to the standard simplicial $(n-k-1)$-sphere S_{n-k-1} and such that the vertices of $\mathrm{Lk}(K,s)$ do not form a k-simplex of K. Given such s we introduce in Chapter 6 the *bistellar move* along s to obtain a new closed combinatorial n-manifold $\tau_s K$. At this point we do not give the precise definition, but we hope that Figure 3 gives the flavor of the definition.

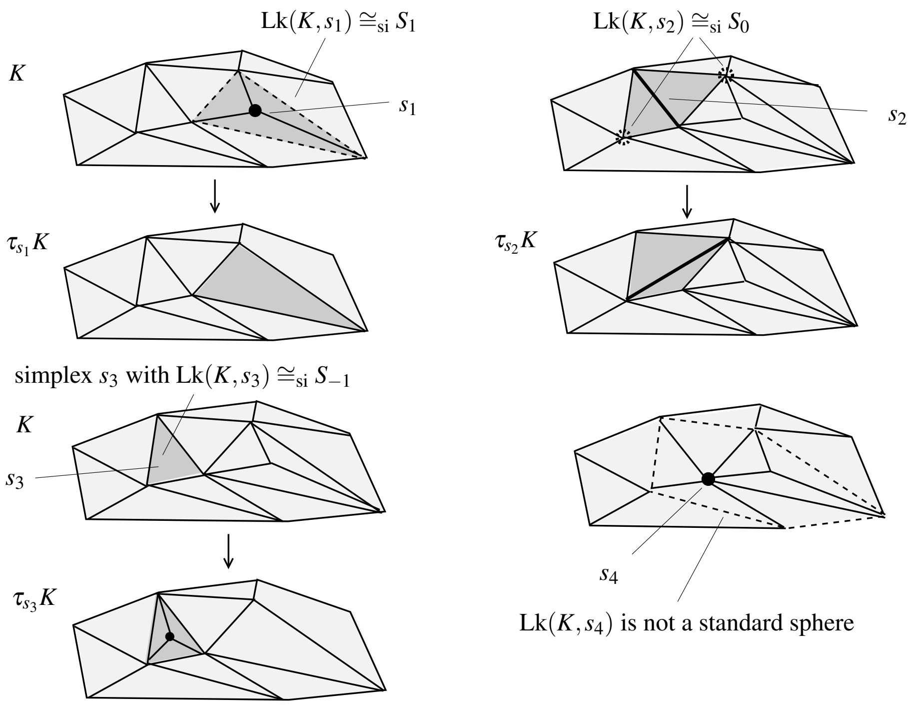

Fig. 3: Examples for three bistellar moves and one impossible move

We say that two closed combinatorial n-manifolds are *bistellar equivalent* if one can go from one to the other through a finite sequence of bistellar moves and simplicial isomorphisms. It follows immediately from the definitions that if two closed combinatorial n-manifolds are bistellar equivalent, then they are also stellar equivalent. Amazingly the converse to this statement also holds:

Theorem 2. (Pachner's Theorem) *Let $K = (V,S)$ and $K' = (V',S')$ be closed combinatorial n-manifolds. Then K and K' are bistellar equivalent if and only if they are stellar equivalent.*

The theorem was first proved by Udo Pachner [Pac91] in a seminal paper. Pachner's original paper is a highly cited and extremely influential paper in combinatorial topology. The significance of Pachner's theorem lies in the fact that given an n-dimensional combinatorial manifold there are infinitely many different types of stellar moves, whereas there are only $n+1$ different type of bistellar moves.

The following combines some of the previous results with results on the topology of smooth manifolds.

Theorem 3. *Let $n \in \mathbb{N}$. There exist natural bijections between the following sets:*

(1) *Closed combinatorial n-manifolds up to bistellar equivalence.*
(2) *Closed combinatorial n-manifolds up to stellar equivalence.*
(3) *Closed PL-manifolds up to PL-homeomorphisms.*

If $n \in \{1, \dots, 6\}$ there exists also a bijection of (1), (2) *and* (3) *to the following set:*

(4) *Closed smooth n-manifolds up to diffeomorphism.*

Proof. The equivalence of (1) and (2) is precisely Pachner's Theorem 2. The equivalence of (2) and (3) follows from the Alexander-Newman Theorem 1. The equivalence of (3) and (4) in dimensions ≤ 6 is a consequence of many deep results in geometric topology, we refer to [Cer68, p. IX] and [Sco05, p. 220] for an exposition and precise references.

Theorem 3 gives us an approach to defining invariants of closed n-dimensional PL-manifolds (or equivalently smooth manifolds for $n \leq 6$): we "just" need to define an invariant for closed n-dimensional stellar manifolds that is invariant under the $n+1$ bistellar moves. This approach to defining invariants has been used by Vladimir Turaev and Oleg Viro [TV92] and John Barrett and Bruce Westbury [BW96] in the 3-dimensional setting and by Christopher Douglas and David Reutter [DR18] in the 4-dimensional setting.

Organization

Chapters 2 and 3 introduce the concept of abstract simplicial complexes, as well as some of the standard notions related to them. Their main goal is to provide an overview for people who are unfamiliar with these concepts. An effort will be made to keep everything free of topology, save for the pictures that illustrate the definitions.

In Chapter 4, we turn our attention to introducing stellar equivalence, and follow it up with an introduction to combinatorial balls, spheres, and combinatorial manifolds. A lot of the properties whose proofs are commonly omitted in literature are proved in great detail.

We will use the results of Chapter 4 in Chapter 5 to introduce the concept of starrability, and subsequently prove that all combinatorial balls are starrable.

At this point, the last two key ingredients required for the proof of Pachner's theorem, namely bistellar moves and elementary shellings, will be introduced in Chapter 6. We also provide proof for some statements that will play a vital role in proving Pachner's theorem in Chapter 7.

Remark

These notes are based on the master thesis of the second author and on lecture notes by the first author. Our goal was to write a detailed topology-free completely combinatorial proof of Pachner's theorem. In the near future we hope to implement this proof in the proof assistant `lean`.

Acknowledgments

We wish to thank Lars Munser for crucial help in proving a particular lemma. We also wish to thank the referee for helpful feedback. The first author was supported by the SFB 1085 "higher invariants" funded by the DFG.

2 Preliminaries

In order to formulate and consequently prove the main results, these first two chapters aim to introduce some of the basic concepts about simplicial complexes. Most of the definitions and lemmas in this chapter are included for rigour's sake, and can therefore be skipped by readers who are already familiar with the notion of (abstract) simplicial complexes. The less well-versed reader is invited to come up with their own examples in order to acquaint themselves with the definitions.

The following definitions and conventions play a major role in the remainder of these notes. It may look like a lot, but all of them are pretty much what one would naively define them as.

Definition 2. Let $K = (V,S)$ be an abstract simplicial complex.

(1) The *dimension of a simplex* $s \in S$ is defined to be

$$\dim s := \#s - 1 \in \mathbb{N}_0.$$

The *dimension of K* is defined as

$$\dim K := \sup_{s\in S} \dim s \in \mathbb{N}_0$$

if K is not the empty abstract simplicial complex $(\varnothing,\varnothing)$, and $\dim K := -1$ if it is.

(2) If $t \in S$ is a simplex and $s \subseteq t$ is a non-empty subset, then we call s a *face* of t.
(3) An *abstract subcomplex of K* is an abstract simplicial complex $L = (W,T)$ such that we have inclusions $W \subseteq V$ and $T \subseteq S$.
(4) Let $n \in \mathbb{N}_0$. The *n-skeleton of K,* defined as

$$K^n := \Big(V, \{s \in S\colon\ \dim s \leq n\}\Big).$$

This is an abstract subcomplex of K. For readers unfamiliar with abstract simplicial complexes, it can be a worthwhile exercise to prove this fact.

The following is another important example of abstract subcomplexes.

Remark 1. Let $K = (V,S)$ be an abstract simplicial complex, and let $s \in S$ be a k-simplex. Then we often view $(s, \mathscr{P}(s) \setminus \{\varnothing\})$ as an abstract subcomplex, and thus as an abstract simplicial complex in its own right, which we also refer to as s by an abuse of notation. It is furthermore simplicially isomorphic[1] to D_k.

3 Some More Constructions on Abstract Simplicial Complexes

Before we can start with the actual topics, we need more methods of constructing new abstract simplicial complexes from existing ones. We start out with an operation that at first may seem rather "out of the blue" but will shortly become one of the core concepts of these notes.

Definition 3.

1. Let V and W be two sets. We define their disjoint union as

$$V \sqcup W \ := \ (V \times \{1\}) \cup (W \times \{2\})$$

and we view V and W as subsets of $V \sqcup W$ in the obvious way.

2. Let $K = (V,S)$ and $L = (W,T)$ be abstract simplicial complexes. We define their *join* to be the abstract simplicial complex

$$K * L := \left(V \sqcup W, \left\{ s \sqcup t \,\middle|\, \begin{array}{l} s \in S \cup \{\varnothing\}, t \in T \cup \{\varnothing\}, \\ \text{at least one of } s \text{ and } t \text{ non-empty} \end{array} \right\} \right).$$

[1] This concept will be defined in the subsequent definition.

Pictorially, the join operation means "connecting every point of K to every point of L". However, we are technically talking about an operation on **abstract** simplicial complexes so the reader should take that intuition as well as the following suggestive pictures with the usual grain of salt.

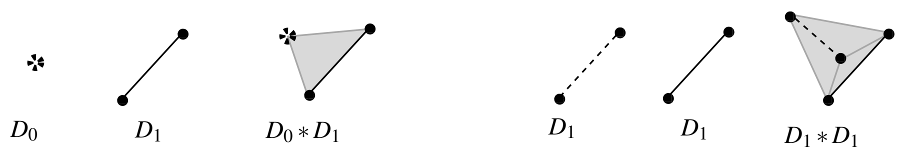

Fig. 4: What one might imagine the joins $D_0 * D_1$ and $D_1 * D_1$ to look like.

Remark 2. Many an author will move on to say that the join operation is clearly associative and commutative. However, one has to be careful when taking disjoint unions as it is only commutative up to a bijection. Therefore, it is only reasonable to claim that the join operation is commutative up to a simplicial isomorphism. However, since this would be quite a hassle, we will ignore this problem for the remainder of these notes, and only take joins of abstract simplicial complexes whose vertex sets are disjoint already. In that case, commutativity and associativity follow quite easily from the definition.
It also follows immediately from the definition that the empty complex $(\varnothing,\varnothing)$ is a neutral element with respect to taking joins, i.e., that for any abstract simplicial complex K, we have $K * (\varnothing,\varnothing) = K$.

The following elementary statement will be used a lot, often without even explicitly mentioning it. Its proof is an easy yet refreshing exercise in set theory.

Lemma 1. *Let $K = (V,S)$ and $L = (W,T)$ be abstract simplicial complexes. Then*

$$\dim(K * L) = \dim K + \dim L + 1.$$

Recall that we had defined the dimension of the empty complex $(\varnothing,\varnothing)$ to be -1.

Proof. Let $K = (V,S)$ and $L = (W,T)$ be abstract simplicial complexes. If one of K or L is the empty complex, then the equation is a tautology. Now assume neither K nor L are empty. Denote by k and l the dimensions of K and L, respectively. By definition, that means there is a simplex $s \in S$ with $\#s = k+1$, and a simplex $t \in T$ with $\#t = l+1$ and no simplices in S or T that are of higher dimension than s or t, respectively. But then

$$\#(s \sqcup t) = \#s + \#t = k + l + 2,$$

and $\dim(s \sqcup t) = \#(s \sqcup t) - 1 = k + l + 1$, and there are no simplices in $S \sqcup T$ that are of higher dimension. Therefore, $\dim(K * L) = \dim K + \dim L + 1$.

Remark 3. There are natural injective simplicial maps $K \to K * L$ and $L \to K * L$. We will use these maps to view K and L as abstract subcomplexes of $K * L$. Looking at Figure 4 above, it should become clear what we mean by that.

One important example for joins is the cone of an abstract simplicial complex.

Example 1. Let $K=(V,S)$ be an abstract simplicial complex. We define the *cone of* K to be

$$\mathrm{Cone}(K) := D_0 * K,$$

where D_0 is the abstract simplicial complex $(\{0\},\{\{0\}\})$ that we had defined back in Chapter 2. From the remark above, it follows that we can view K as an abstract subcomplex of $\mathrm{Cone}(K)$. Furthermore, it follows from Lemma 1 that $\dim \mathrm{Cone}(K) = \dim K + 1$.

Definition 4. Let $K=(V,S)$ be an abstract simplicial complex, let $s \in S$ be a simplex. We define the following abstract subcomplexes:

$$\begin{aligned}
\mathrm{St}(K,s) &:= \left(\bigcup_{t\in S,\, s\subseteq t} t,\ \{t \in S | s \cup t \in S\} \right)\\
\mathrm{Lk}(K,s) &:= \left(\bigcup_{t\in S\, s\subseteq t} t \setminus s,\ \{t \in S | s \cup t \in S \text{ and } s \cap t = \varnothing\} \right)\\
K \setminus \mathring{\mathrm{St}}(K,s) &:= \begin{cases} (V\setminus\{v\}, \{t \in S | s \not\subseteq t\}), & \text{if } s=\{v\} \text{ is a 0-simplex}\\ (V, \{t\in S | s \not\subseteq t\}), & \text{else}\end{cases}\\
\partial s &:= \begin{cases} (\varnothing,\varnothing), & \text{if } s \text{ is a 0-simplex}\\ (s, \mathscr{P}(s)\setminus\{\varnothing, s\}), & \text{else}\end{cases}
\end{aligned}$$

They are called the *(closed) star of s in K,* the *link of s in K,* the *complement*[2] *of the star of s in K,* and the *boundary of the simplex s,* respectively.

The sketches below show some of the definitions. For better understanding, the reader could go ahead and draw their own abstract simplicial complexes. Note that[3]

$$\mathrm{St}(K,s) \cap \big(K \setminus \mathring{\mathrm{St}}(K,s)\big) = \mathrm{Lk}(K,s).$$

This next lemma is of technical nature and is used every so often throughout these notes.

Lemma 2. *Let $K=(V,S)$ and $L=(W,T)$ be abstract simplicial complexes, and let $s \in S$ be a simplex. Then we have*

(1) $L * \mathrm{Lk}(K,s) = \mathrm{Lk}(K*L,s)$,
(2) $L * \mathrm{St}(K,s) = \mathrm{St}(K*L,s)$, *and*
(3) $L * \big(K \setminus \mathring{\mathrm{St}}(K,s)\big) = (K*L)\setminus \mathring{\mathrm{St}}(K*L,s)$,

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[2] It is important to remember that $K \setminus \mathring{\mathrm{St}}(K,s)$ is simply a notation for what one might pictorially draw as the closure of the complement. By itself, "$\mathring{\mathrm{St}}(K,s)$" is in no way an abstract simplicial complex!

[3] The intersection of abstract simplicial complexes will be defined shortly.

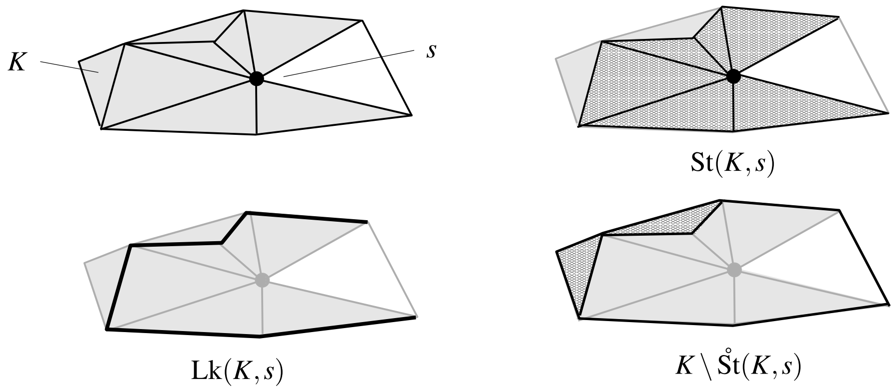

Fig. 5: Star, link, and complement of the star of a 0-simplex

Proof. For the equality involving links, we calculate on the simplex sets

$$\mathrm{Lk}(K*L,s)=\left\{t\sqcup u\,\middle|\,\begin{array}{l}t\in S\cup\{\varnothing\},u\in T\cup\{\varnothing\},(t\sqcup u)\cap s=\varnothing,(t\sqcup u)\cup s\in S\sqcup T,\\ \text{at least one of } t \text{ and } u \text{ non-empty}\end{array}\right\}$$

Now, we use the fact that s is a simplex of K to simplify the conditions, namely we can simplify $(t\sqcup u)\cap s=\varnothing$ to $t\cap s=\varnothing$, and $(t\sqcup u)\cup s\in S\sqcup T$ to $t\cup s\in S$. Doing so yields

$$\mathrm{Lk}(K*L,s)=\left\{t\sqcup u\,\middle|\,\begin{array}{l}t\in S\cup\{\varnothing\},u\in T\cup\{\varnothing\},t\cap s=\varnothing,t\cup s\in S,\\ \text{at least one of } t \text{ and } u \text{ non-empty}\end{array}\right\}=L*\mathrm{Lk}(K,s).$$

The proof of the other two equalities follows essentially the same strategy.

Since working with links in joins is so much fun, we also formulate the following statement, which will be very useful later on.

Lemma 3. *Let $K=(V,S)$ and $L=(W,T)$ be abstract simplicial complexes, and let $s\in S$ and $t\in T$ be simplices. Then we have*

$$\mathrm{Lk}(K*L,s*t)=\mathrm{Lk}(K,s)*\mathrm{Lk}(L,t).$$

Proof. We perform the following elementary calculation

$$\begin{aligned}\mathrm{Lk}(K*L,s*t)&=\{u\in K*L|u\cap(s\sqcup t)=\varnothing,u\cup(s\sqcup t)\in S\sqcup T\}=\\&=\{u_1\sqcup u_2\in K*L|(u_1\sqcup u_2)\cap(s\sqcup t)=\varnothing,(u_1\sqcup u_2)\cup(s\sqcup t)\in S\sqcup T\}=\\&=\{u_1\in S|u_1\cap s=\varnothing,u_1\cup s\in S\}*\{u_2\in T|u_2\cap t=\varnothing,u_2\cup t\in T\}=\\&=\mathrm{Lk}(K,s)*\mathrm{Lk}(L,t).\square\end{aligned}$$

The union and intersection of abstract simplicial complexes are exactly what one would expect it to be, to the point where most authors do not even bother defining them.

Definition 5. Let $K = (V,S)$ and $L = (W,T)$ be simplicial complexes. Their *union* is defined as the abstract simplicial complex[4]

$$K \cup L := (V \cup W, S \cup T),$$

their *intersection* as

$$K \cap L := (V \cap W, S \cap T).$$

The following lemma states a sort of distributivity law for joins with unions and intersections. We will not mention it every time we use it.

Lemma 4. *Let $K = (V,S)$, $L_1 = (W_1,T_1)$, and $L_2 = (W_2,T_2)$ be abstract simplicial complexes. Then we have*

$$K * (L_1 \cup L_2) = K * L_1 \cup K * L_2$$

and

$$K * (L_1 \cap L_2) = K * L_1 \cap K * L_2.$$

Proof. We leave this proof as an exercise to the reader.

The following definition is our way to create vertices that are not in a given abstract simplicial complex.

Definition 6. Let $K = (V,S)$ be an abstract simplicial complex, let $s \in S$ be a simplex. We define the *barycentre* of s to be $\underline{s} := \frac{1}{\#s} \sum_{v \in s} v \in \mathbb{R}^V$.

For the remainder of these notes, the following slogan works: “If something is underlined, it is a barycentre and therefore a 0-simplex not in the vertex set of whatever abstract simplicial complex we are currently working with.” Pictorially, we draw the barycentre of a simplex at the actual “physical” barycentre of the simplex.

4 Stellar Moves and Combinatorial Manifolds

Having finished two chapters that were riddled with definitions and conventions, we now start actually proving statements. This chapter loosely follows the approach pursued by Lickorish (cf. [Lic99, Chapter 3]), but in much more detail, and with an added focus on mathematical rigour.

We start out with the following definition, the second half of which is mostly about semantics.

[4] Note that here, we explicitly do not ask for the vertex (or simplex) sets to be disjoint.

Definition 7. Let $K = (V,S)$ be an abstract simplicial complex, let $s \in S$ be a simplex. We call the abstract simplicial complex[5]

$$\sigma_s K := \Big(K \setminus \mathring{\mathrm{St}}(K,s)\Big) \cup \underline{s} * \partial s * \mathrm{Lk}(K,s)$$

the *abstract simplicial complex obtained from K by stellar subdivision at s.* We also refer to the operation of going from K to $\sigma_s K$ as a *stellar subdivision.*
If an abstract simplicial complex K' is obtained from K via a stellar subdivision, then we say *K is obtained from K' via a stellar weld,* and the operation of going from K' to K is called likewise.
A *stellar move* is either a stellar subdivision or a stellar weld.

The intuition one should have about stellar subdivision is, at least for simplices of dimension at least 1, to add a vertex at the centre of that simplex, and subsequently split its entire star along that point in order for it to "become a simplicial complex again." If that description seems confusing at first, the reader is invited to refer to Figures 2 in the introduction and Figure 6 below, or come up with and draw their own examples.

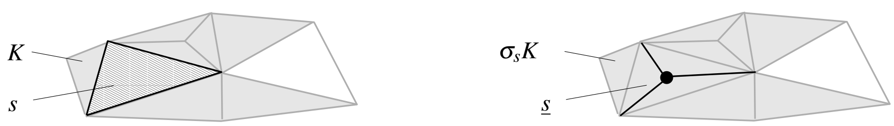

Fig. 6: Stellar subdivision at a 2-simplex

It follows immediately from the definition that stellar subdivision at a 0-simplex creates an abstract simplicial complex that is simplicially isomorphic to the original complex.

Definition 8. Let $K = (V,S)$ and $K' = (V',S')$ be abstract simplicial complexes. We call K and K' *stellar equivalent* if there exists a sequence $K_i = (V_i,S_i)$, $i = 1,\dots,n$, such that $K_1 = K$, $K_n = K'$, and for each $i \in \{1,\dots,n-1\}$ we have that K_{i+1} is a stellar subdivision of, a stellar weld of, or simplicially isomorphic to, K_i. This is clearly an equivalence relation, and we write $K \approx_{\mathrm{st}} K'$ if K and K' are stellar equivalent.

Now that we have some shiny new definitions, we want to put them to use. The following lemma probably comes as no big surprise, and might be used in the future without explicit mention.

Lemma 5. *Let K, K', L, and L' be abstract simplicial complexes. If $K \approx_{\mathrm{st}} K'$ and $L \approx_{\mathrm{st}} L'$, then $K * L \approx_{\mathrm{st}} K' * L'$.*

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[5] Note that while the barycentre $\underline{s}$ is inspired by the topological setting, it is ultimately just a **natural** way to choose a vertex that is not already in V. Most authors just define it the latter way.

Proof. From the commutativity of the join and the fact that joins clearly commute with simplicial isomorphisms it follows that it suffices to show that for abstract simplicial complexes $K = (V,S)$ and $L = (W,T)$, and a simplex $s \in S$, we have the equality

$$(\sigma_s K) * L = \sigma_s(K * L).$$

By definition and using Lemma 4 and Lemma 2 (1) and (3), one calculates

$$\begin{aligned}\sigma_s(K*L) &= \underbrace{(K*L)\setminus \mathring{\mathrm{St}}(K*L,s)}_{=(K\setminus\mathring{\mathrm{St}}(K,s))*L} \cup \underline{s}*\partial s*\underbrace{\mathrm{Lk}(K*L,s)}_{=\mathrm{Lk}(K,s)*L} = \\ &= \Big(K\setminus \mathring{\mathrm{St}}(K,s)\cup \underline{s}*\partial s*\mathrm{Lk}(K,s)\Big)*L = \\ &= (\sigma_s K)*L.\end{aligned}$$

This concludes the proof.

Next, we want to take a look at abstract simplicial complexes that have an especially nice relationship with stellar equivalences.

Definition 9. Let $K = (V,S)$ be an abstract simplicial complex, let $n \in \mathbb{N}_0$.

(1) We call K a *combinatorial n-ball* if $K \approx_{\mathrm{st}} D_n$.
(2) We call K a *combinatorial n-sphere* if $K \approx_{\mathrm{st}} S_n$.
(3) We call K a *combinatorial n-manifold* if for every vertex $v \in V$, its link $\mathrm{Lk}(K,v)$ is a combinatorial $(n-1)$-ball or a combinatorial $(n-1)$-sphere.

The reader is invited to refer to Figure 7 below in order to acquaint themselves with the definition of a combinatorial manifold. We leave the formal proof that this abstract simplicial complex is a combinatorial manifold to the reader.

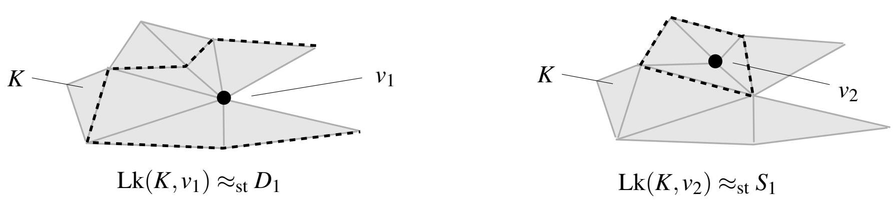


Fig. 7: A combinatorial 2-manifold with the different types of links

Lemma 6. *Let $n \in \mathbb{N}_0$, then D_n and S_n are combinatorial n-manifolds.*

In order to allow the reader to grow used to this topic, we shall be especially thorough in proving this statement.

Proof. Let v be a vertex of D_n. Its link is by definition the subcomplex of D_n that consists of all simplices whose union with $\{v\}$ is again a simplex and whose intersection with $\{v\}$ is empty. Since the simplex set of D_n is just $\mathscr{P}(\{0,\ldots,n\})\setminus\{\varnothing\}$, it follows that

$$\mathrm{Lk}(D_n, v) = \Big(V_n \setminus \{v\}, \mathscr{P}\big(V_n \setminus \{v\}\big) \setminus \{\varnothing\}\Big),$$

which is simplicially isomorphic to D_{n-1}. But by definition, a simplicial isomorphism is in particular also a stellar equivalence, so the link of v is in fact a combinatorial $(n-1)$-ball.
Now let v be a vertex of S_n. Using the same logic as before, we arrive at the conclusion that the link of v in S_n is given by

$$\mathrm{Lk}(S_n, v) = \Big(V_{n+1} \setminus \{v\}, \mathscr{P}\big(V_{n+1} \setminus \{v\}\big) \setminus \big\{\varnothing, V_{n+1} \setminus \{v\}\big\}\Big),$$

where we had to exclude $V_{n+1} \setminus \{v\}$ as a simplex because $V_{n+1} = \{v\} \cup (V_{n+1} \setminus \{v\})$ is not a simplex of S_n. But now we can, analogously to the case of D_n, write down a simplicial isomorphism from $\mathrm{Lk}(S_n, v)$ to S_{n-1}, therefore the link of v is in fact a combinatorial $(n-1)$-sphere.

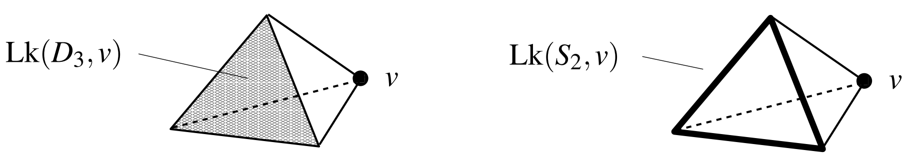


Fig. 8: The links of a vertex in D_3 and S_2 look a lot like D_2 and S_1

In order to get used to the concept of combinatorial balls and spheres, we start out by proving the following lemma about their behaviour under the join operation.

Lemma 7. *Let $m, n \in \mathbb{N}_0$, and let K and L be abstract simplicial complexes. Then the following are true.*

(1) *If K is a combinatorial m-ball and L is a combinatorial n-ball, then $K * L$ is a combinatorial $(m+n+1)$-ball.*
(2) *If K is a combinatorial m-sphere and L is a combinatorial n-sphere, then $K * L$ is a combinatorial $(m+n+1)$-sphere.*
(3) *If K is a combinatorial m-ball and L is a combinatorial n-sphere, then $K * L$ is a combinatorial $(m+n+1)$-ball.*

Proof. Let $m, n \in \mathbb{N}_0$, and let K and L be abstract simplicial complexes.

(1) Assume $K \approx_{\mathrm{st}} D_m$ and $L \approx_{\mathrm{st}} D_n$. From Lemma 5, it follows that

$$K * L \approx_{\mathrm{st}} D_m * D_n.$$

We can give an explicit simplicial isomorphism from $D_m * D_n$ to D_{m+n+1} as follows[6]

[6] Remember that the disjoint union of two sets A and B is defined as $(\{0\} \times A) \cup (\{1\} \times B)$.

$$\begin{aligned}\{0\}\times\{0,\dots,m\}\cup\{1\}\times\{0,\dots,n\} &\to \{0,\dots,m+n+1\}\\ (0,i) &\mapsto i\\ (1,i) &\mapsto m+i+1\end{aligned}$$

Since simplicial isomorphisms are stellar equivalences, this implies that $K*L$ is a combinatorial $(m+n+1)$-ball.

(2) Assume $K\approx_{\mathrm{st}} S_m$ and $L\approx_{\mathrm{st}} S_n$. As in the proof of (1), it follows from Lemma 5 that $K*L\approx_{\mathrm{st}} S_m*S_n$. It remains to show that $S_m*S_n\approx_{\mathrm{st}} S_{m+n+1}$. For this, we inductively prove that for any $k\in\mathbb{N}_0$, we have

$$S_k\approx_{\mathrm{st}}\underbrace{S_0*\cdots*S_0}_{(k+1)\text{-times}}.$$

The case $k=0$ is a tautology. Now let $k\in\mathbb{N}$. Let $s=\{0,\dots,k\}$ be one of the k-simplices of S_k. Now consider

$$\sigma_s S_k=\underbrace{S_k\setminus\mathring{\mathrm{S}}\mathrm{t}(S_k,s)}_{=\{k+1\}*\partial s}\cup\underline{s}*\partial s*\underbrace{\mathrm{Lk}(S_k,s)}_{=(\varnothing,\varnothing)}=\underbrace{(\{k+1\}\cup\underline{s})}_{\cong_{\mathrm{si}}S_0}*\underbrace{\partial s}_{=S_{k-1}}\cong_{\mathrm{si}}S_0*S_{k-1}.$$

Applying the induction hypothesis yields

$$S_k\approx_{\mathrm{st}}S_0*S_{k-1}\approx_{\mathrm{st}}\underbrace{S_0*\cdots*S_0}_{(k+1)\text{-times}}.$$

This result implies that S_m*S_n is stellar equivalent to the join of

$$(m+1)+(n+1)=m+n+2$$

copies of S_0, but so is S_{m+n+1}. In Figure 9, the idea behind the induction step is shown for $k=2$. It may not look like it, but it works the same for all dimensions, as the calculation above shows.

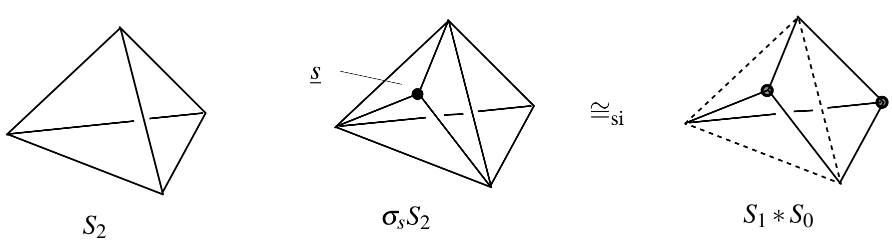


Fig. 9: The subdivision of S_2 is simplicially isomorphic to the join S_1*S_0

(3) Assume $K\approx_{\mathrm{st}} D_m$ and $L\approx_{\mathrm{st}} S_n$. As above, this implies that $K*L\approx_{\mathrm{st}} D_m*S_n$, and it suffices to show $D_m*S_n\approx_{\mathrm{st}} D_{m+n+1}$. We first show that $D_0*S_n\approx_{\mathrm{st}} D_{n+1}$.

For this, let $s := V_{n+1} = \{0,\dots,n+1\}$ be the maximal simplex of D_{n+1}. Then[7]

$$\sigma_s D_{n+1} = \underbrace{D_{n+1} \setminus \mathring{\mathrm{St}}(D_{n+1},s)}_{=(\varnothing,\varnothing)} \cup \underline{s} * \underbrace{\partial s}_{=S_n} * \underbrace{\mathrm{Lk}(D_{n+1},s)}_{=(\varnothing,\varnothing)} \cong_{\mathrm{si}} D_0 * S_n.$$

Now we can use the result from part (1) to calculate

$$D_m * S_n \cong_{\mathrm{si}} D_{m-1} * \underbrace{D_0 * S_n}_{\approx_{\mathrm{st}} D_{n+1}} \approx_{\mathrm{st}} D_{(m-1)+(n+1)+1} = D_{m+n+1},$$

which concludes the proof of the third statement.

Thus, all three statements have been proved.

The following proposition aims to summarise two nice properties of combinatorial manifolds. Both the statement and its proof are based on [Lic99, Lemma 3.2], however steps were taken to fill in some of the gaps.

Proposition 1. *Let $n \in \mathbb{N}_0$, and let $K = (V,S)$ be a combinatorial n-manifold. Then the following two statements hold true:*

(1) *For any k-simplex $s \in S$, its link $\mathrm{Lk}(K,s)$ is a combinatorial $(n-k-1)$-ball or a combinatorial $(n-k-1)$-sphere.*
(2) *If $K' = (V',S')$ is an abstract simplicial complex that is stellar equivalent to K, then K' is a combinatorial n-manifold.*

Remark 4. It shall be stressed that somewhat counter-intuitively, the second statement does *not* immediately follow from the definitions. In the future, however, we use part (2) without explicitly mentioning it.

We need the following technical lemma.

Lemma 8. *Let $K = (V,S)$ be an abstract simplicial complex, let $s \in S$ be a simplex. If $t \subseteq s$ is a face of s, then we have the equality*

$$\mathrm{Lk}(K,s) = \mathrm{Lk}\left(\mathrm{Lk}(K,t), s \setminus t\right).$$

Proof. If $s = t$, then the equality holds since we had set the link of the empty set in an abstract simplicial complex to be that complex. Therefore, we now assume that t is a proper face of s.
We prove this claim set-theoretically for the simplex sets

$$\begin{aligned}\mathrm{Lk}\left(\mathrm{Lk}(K,t), s \setminus t\right) &= \left\{u \in \mathrm{Lk}(K,t) \middle| u \cap (s\setminus t) = \varnothing \wedge u \cup (s \setminus t) \in \mathrm{Lk}(K,t)\right\} = \\ &= \{u \in S | u \cap t = \varnothing \wedge u \cap (s \setminus t) = \varnothing \wedge u \cup s \in S\} = \\ &= \{u \in S | u \cap s = \varnothing \wedge u \cup s \in S\} = \\ &= \mathrm{Lk}(K,s). \square\end{aligned}$$

[7] Both $D_{n+1} \setminus \mathring{\mathrm{St}}(D_{n+1},s)$ and $\mathrm{Lk}(D_{n+1},s)$ are empty because the star of s is already all of D_{n+1}.

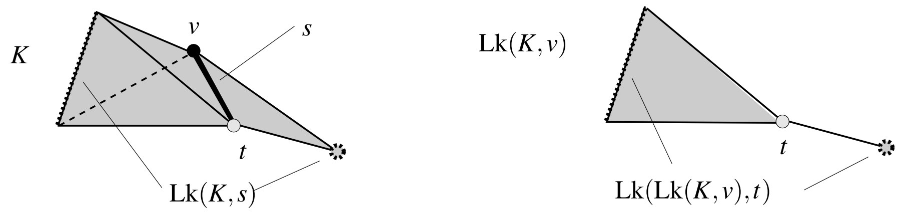


Fig. 10: Sketch of Lemma 8

Now that this is out of the way, we can turn to the proof of Proposition 1.

Proof. We perform induction on the dimension n. The case $n = 0$ is rather trivial: a combinatorial 0-manifold is a finite collection of 0-simplices, because every vertex $v \in V$ has an empty link $\mathrm{Lk}(K,v)$, so K cannot have any 1-simplices. Since every simplex is a 0-simplex, the first property follows immediately from the definition of a combinatorial manifold. The second property follows from the fact that any stellar move on a 0-simplex is trivial, therefore the stellar equivalence between K and K' is essentially a simplicial isomorphism, which restricts to the links and therefore turns any link in K' into a combinatorial (-1)-ball.
Now assume that K is a combinatorial n-manifold for some $n \in \mathbb{N}$, and both (1) and (2) are true for dimensions less than n.

(1) Let $s \in S$ be a k-simplex of K. If $k = 0$, then (1) follows immediately from the definition since any 0-simplex is a vertex. If $k > 0$, we write $s = \{v\} \sqcup t$, for v a vertex of s and $t := s \setminus \{v\}$. Note that t is a $(k-1)$-simplex of K. From Lemma 8, we obtain that
$$\mathrm{Lk}(K,s) = \mathrm{Lk}\left(\mathrm{Lk}(K,v),t\right).$$
Since K is a combinatorial n-manifold, the link $\mathrm{Lk}(K,v)$ is a combinatorial $(n-1)$-ball or a combinatorial $(n-1)$-sphere. By Lemma 6 and property (2), in dimension $n-1$, this implies that $\mathrm{Lk}(K,v)$ is a combinatorial $(n-1)$-manifold. Now we can use property (1), in dimension $n-1$, to obtain that $\mathrm{Lk}\left(\mathrm{Lk}(K,v),t\right)$ is a combinatorial ball or a combinatorial sphere of dimension
$$\Big(\underbrace{(n-1)}_{\dim \mathrm{Lk}(v)} - \underbrace{(k-1)}_{\dim(t)} - 1\Big) = (n-k-1).$$
Therefore, it follows from the claim that $\mathrm{Lk}(K,s)$ is a combinatorial $(n-k-1)$-ball or a combinatorial $(n-k-1)$-sphere. Since s was chosen arbitrarily, this concludes the proof of (1) in dimension n.

(2) Clearly, an abstract simplicial complex that is simplicially isomorphic to a combinatorial n-manifold is itself a combinatorial manifold. Thus, it suffices to show that if $K' = \sigma_s K$ for some k-simplex $s \in S$ and K is a combinatorial mani-

fold, then K' is a combinatorial manifold. For this, let $v \in V'$ be a vertex of K'. We consider the following cases[8]:

<u>Case 1:</u> $v \notin \mathrm{St}(K,s)$. In this case, it follows from the definition of stellar subdivision that we have the equality $\mathrm{Lk}(K',v) = \mathrm{Lk}(K,v)$, because the stellar subdivision σ_s does not change the abstract simplicial complex outside of the star of s. Since K is a combinatorial n-manifold, the link $\mathrm{Lk}(K,v)$ is a combinatorial $(n-1)$-ball or a combinatorial $(n-1)$-sphere, and hence so is $\mathrm{Lk}(K',v)$.

<u>Case 2:</u> $v \in \mathrm{Lk}(K,s)$. In this case, we know that $s \in \mathrm{Lk}(K,v)$. It follows from set theory, similar to the arguments used at the end of chapter 3, that

$$\mathrm{Lk}(K',v) \cong_{\mathrm{si}} \sigma_s \,\mathrm{Lk}(K,v).$$

In other words, a stellar move connects the links of v in K and K', respectively. Therefore, $\mathrm{Lk}(K,v)$ is a combinatorial $(n-1)$-ball or a combinatorial $(n-1)$-sphere if and only if $\mathrm{Lk}(K',v)$ is.

<u>Case 3:</u> $v \in s$. We let $t := s \setminus \{v\}$, thus $s = \{v\} * t$. Just as in the second case, it follows that $\mathrm{Lk}(K',v) \cong_{\mathrm{si}} \sigma_t \,\mathrm{Lk}(K,v)$. Therefore, the links again differ by a stellar move.

<u>Case 4:</u> $v = \underline{s}$. By definition of $K' = K \setminus \mathring{\mathrm{St}}(K,s) \cup \underline{s} * \partial s * \mathrm{Lk}(K,s)$, we have

$$\mathrm{Lk}(K',\underline{s}) = \partial s * \mathrm{Lk}(K,s).$$

Again, $\mathrm{Lk}(K,s)$ is a combinatorial $(n-k-1)$-ball or a combinatorial $(n-k-1)$-sphere by property (1), in dimension n. But since ∂s is the boundary of a k-simplex, it is simplicially isomorphic to S_{k-1}, and thus in particular a combinatorial $(k-1)$-sphere. By Lemma 7 (2) and (3), we obtain that $\mathrm{Lk}(K',\underline{s})$ is a combinatorial ball or a combinatorial sphere of dimension $(k-1)+(n-k-1)+1 = n-1$.

In each of the cases, $\mathrm{Lk}(K',v)$ is a combinatorial $(n-1)$-ball or a combinatorial $(n-1)$-sphere, and therefore, K' is a combinatorial manifold.

This concludes the proof by induction.

Next, we shall define the boundary of a combinatorial manifold.

Definition 10. Let $K = (V,S)$ be a combinatorial n-manifold. By Proposition 1, we know that the link of every simplex is a combinatorial ball or a combinatorial sphere. We define the *boundary ∂K of K* as the abstract subcomplex consisting of all the vertices and simplices of K whose link in K is not a combinatorial sphere[9]. Often, we slightly abuse notation and also denote the simplex set of ∂K by ∂K.
We say a combinatorial manifold K is *closed* if its boundary is the empty simplicial complex, in other words if the links of all simplices are combinatorial spheres.

[8] It can be a helpful exercise to think about why these four cases are the ones one has to consider.

[9] It is important to define it this way, because a (-1)-ball is the same as a (-1)-sphere.

Example 2. As one would expect, we have $\partial D_n = S_{n-1}$. This becomes evident as follows: If s is a k-simplex of D_n, then by definition of the link we have

$$\mathrm{Lk}(D_n, s) = \Big(V_n \setminus s, \mathscr{P}\big(V_n \setminus s\big) \setminus \{\varnothing\}\Big).$$

This is a fancy way of writing down what one would intuitively call "the opposite face of s." Due to the nature of this abstract simplicial complex, it is easy to write down a simplicial isomorphism to D_{n-k-1}. However, this is a combinatorial sphere if and only if $k = n$, thus the abstract subcomplex ∂D_n consists of all the simplices of D_n that are of dimension strictly less than n, which just happens to be S_{n-1}.

We use the following technical lemma on multiple occasions.

Lemma 9. *Let $K = (V, S)$ be a combinatorial n-manifold, let $s \in S$ be a k-simplex with $s \in \partial K$. Then we have*[10] $\mathrm{Lk}(\partial K, s) = \partial \mathrm{Lk}(K, s)$.

Proof. Using Lemma 8, we have[11]

$$\begin{aligned}\partial\,\mathrm{Lk}(K,s) &= \{u \in \mathrm{Lk}(K,s) \,|\, \mathrm{Lk}(\mathrm{Lk}(K,s),u) \approx_{\mathrm{st}} D_l\} = \\ &= \{u \in K | u \cap s = \varnothing, u \cup s \in K, \mathrm{Lk}(K, u \cup s) \approx_{\mathrm{st}} D_l, \mathrm{Lk}(K,u) \approx_{\mathrm{st}} D_m\} = \\ &= \{u \in \partial K | u \cap s = \varnothing, u \cup s \in \partial K\} = \\ &= \mathrm{Lk}(\partial K, s).\end{aligned}$$

Note that for legibility's sake, we did notationally not distinguish between simplicial complexes and their simplex sets. The dimensions of the combinatorial balls follow immediately from Lemma 7 (1) and (3).

For rigour's sake, it is also important to prove the following unsurprising statement.

Lemma 10. *Let $K = (V, S)$ be a combinatorial n-manifold. Then its boundary is in fact an abstract subcomplex, and it is furthermore a closed combinatorial $(n-1)$-manifold.*

Proof. Let $K = (V, S)$ be a combinatorial n-manifold, and let $s \in S$ be a k-simplex of K whose link in K is a combinatorial $(n-k-1)$-ball but not a combinatorial $(n-k-1)$-sphere. Since for $k = n$, this combinatorial (-1)-ball would also be a combinatorial (-1)-sphere, we can assume for the remainder of the proof that $k < n$.
For the first claim, we have to show that the link in K of any face of s is also a combinatorial ball. So let $t \subsetneq s$ be a proper face of s. From Proposition 1, we know that $\mathrm{Lk}(K, t)$ is a combinatorial ball or a combinatorial sphere, so it suffices to

[10] Here, we implicitly use that the link of s in K is a combinatorial ball, hence in particular a combinatorial manifold by Lemma 6 and Proposition 1 (1).

[11] Here, we set $l = l(u) := n-k-2-\dim u$ and $m = m(u) := n-1-\dim u$. The diligent reader can check for themselves that this is in fact the correct dimension for those links.

exclude the possibility of it being a sphere. Let us assume that it is a combinatorial sphere. Note that it follows from Lemma 8 that

$$\mathrm{Lk}(K,s) = \mathrm{Lk}\left(\mathrm{Lk}(K,t), s\setminus t\right).$$

Since we had assumed $\mathrm{Lk}(K,t)$ to be a combinatorial sphere, this is a combinatorial sphere of dimension[12]

$$\dim \mathrm{Lk}(K,t) - \dim(s\setminus t) - 1 = n-l-1-(k-l-1)-1 = n-k-1 \geq 0,$$

in contradiction to $\mathrm{Lk}(K,s)$ being a combinatorial ball.
For the second claim, we have to show that the link of s in ∂K is a combinatorial $(n-k-2)$-sphere. Since by Corollary 1[13], the boundary $\partial\,\mathrm{Lk}(K,s)$ of the combinatorial $(n-k-1)$-ball $\mathrm{Lk}(K,s)$ is a combinatorial $(n-k-2)$-sphere, it follows from Lemma 9 that the links of all simplices of ∂K in ∂K are combinatorial $(n-k-2)$-spheres, which implies that the boundary of ∂K is empty.

The following lemma may not come as a surprise, but many authors fail to acknowledge that it is something that needs proof.

Lemma 11. *Let $K = (V,S)$ be a combinatorial ball, and let $s \in S$ be a simplex. If $s \notin \partial K$, then $\partial(\sigma_s K) = \partial K$.*

Proof. Let $K = (V,S)$ be a combinatorial ball, and let $s \in S$ be a k-simplex not in ∂K. This implies that $\mathrm{Lk}(K,s) \approx_{\mathrm{st}} S_{n-k-1}$. Using the definition of a stellar subdivision, we have

$$\mathrm{Lk}(\sigma_s K, \underline{s} * \partial s) = \mathrm{Lk}(K,s) \approx_{\mathrm{st}} S_{n-k-1},$$

thus $\underline{s} * \partial s \notin \partial(\sigma_s K)$. But since the only way that σ_s could potentially change ∂K is by changing $\underline{s} * \partial s$, it follows that the boundary remains unchanged under σ_s, i.e., that $\partial(\sigma_s K) = \partial K$.

In the following we state two nice corollaries, which we will use occasionally.

Corollary 1. *Let $K = (V,S)$ be a combinatorial n-ball. Then ∂K is a combinatorial $(n-1)$-sphere.*

Proof. Let $K = (V,S)$ be a combinatorial n-ball, i.e., there exists a sequence of stellar moves (and simplicial isomorphisms) from K to D_n. Our goal is to observe how a stellar move affects ∂K. If for a stellar move $\sigma_s^{\pm 1}$ the simplex s does not lie in the boundary, then it follows from Lemma 11 that the stellar move does not change the boundary. If on the other hand, the simplex s lies in the boundary, then $\partial(\sigma_s^{\pm 1} K) = \sigma_s^{\pm 1}(\partial K)$, so the boundaries of K and $\sigma_s^{\pm 1} K$ are related by a stellar move. Either way, we have $\partial K \approx_{\mathrm{st}} \partial D_n = S_{n-1}$, where the last equality was shown in the example on page 18.

[12] Here, we denote the dimension of t by l.

[13] A closer look shows that we indeed do not use the statement that the boundary has no boundary anywhere leading up to Corollary 1, so we can, in fact, use it here.

This next corollary is another well-known statement in the topological setting, but it does require some thought.

Corollary 2. *Let $K = (V,S)$ be a combinatorial n-sphere. Then K is closed, i.e., ∂K is the empty abstract simplicial complex.*

Proof. So let $K = (V,S)$ be a combinatorial n-sphere. Recall that it follows from the example on page 18 and Lemma 10 that ∂S_n is the empty abstract simplicial complex. Therefore, we can apply Lemma 11 for a sequence of stellar moves (and simplicial isomorphisms) from K to S_n in order to obtain that $\partial K = \partial S_n = (\varnothing,\varnothing)$.

We will also often use the following statement, which may look similar to statements from other fields of mathematics.

Proposition 2.

(1) *Let $K = (V,S)$ and $L = (W,T)$ be combinatorial balls. Then*

$$\partial(K * L) = (K * \partial L) \cup (\partial K * L).$$

(2) *Let $K = (V,S)$ be a combinatorial sphere and $L = (W,T)$ be a combinatorial ball. Then*

$$\partial(K * L) = K * \partial L.$$

Proof. We prove both statements individually on a set-theoretic level. Unsurprisingly, one of the main ingredients for the proof is Lemma 7. Note that by a slight abuse of notation, we will sometimes not differentiate between simplicial complexes and their simplex sets.

(1) Let K and L be combinatorial balls as above. We prove this equation by proving both that both sides are included in one another. For the "$\subseteq$" inclusion we let $s * t \in \partial(K * L)$, i.e., we have $s \in S \cup \{\varnothing\}$ and $t \in T \cup \{\varnothing\}$ not both empty, in such a way that[14]

$$\mathrm{Lk}(K * L, s * t) \approx_{\mathrm{st}} D.$$

Our goal is to show that $s * t$ lies in $(K * \partial L) \cup (\partial K * L)$.
If $s = \varnothing$, then

$$s * t = t \in T \subseteq \partial K * L,$$

and if $t = \varnothing$, then

$$s * t = s \in S \subseteq K * \partial L.$$

The final case is that neither s nor t are empty. From Lemma 3, we obtain that

$$\mathrm{Lk}(K,s) * \mathrm{Lk}(L,t) = \mathrm{Lk}(K * L, s * t) \approx_{\mathrm{st}} D,$$

which then implies by Lemmas 7, 6, and 1 (2) that at least one of the links of s and t is a combinatorial ball; in other words, we have $s \in \partial K$ or $t \in \partial L$, which

[14] In the remainder of this proof, we omit the dimensions of our combinatorial balls and spheres, and briefly write "$\approx_{\mathrm{st}} D$" and "$\approx_{\mathrm{st}} S$" for combinatorial balls and spheres, respectively.

concludes this inclusion.

For the opposite direction, it suffices to show that $\partial K * L \subseteq \partial(K * L)$, because the expression on the right-hand side is symmetric in K and L. Thus, let $s \in \partial K \cup \{\varnothing\}$ and $t \in L \cup \{\varnothing\}$ not both empty. We want to show that $s * t \in \partial(K * L)$.

If $s = \varnothing$, then $s * t = t \in L \subseteq K * L$, and we have

$$\mathrm{Lk}(K * L, s * t) = \mathrm{Lk}(K * L, t) \underset{2\,(1)}{=} \underbrace{\mathrm{Lk}(L,t)}_{\approx_{\mathrm{st}} D \text{ or } \approx_{\mathrm{st}} S} * \underbrace{K}_{\approx_{\mathrm{st}} D} \approx_{\mathrm{st}} D$$

by Lemma 7 (1) or (3). But by definition, this means that $s * t \in \partial(K * L)$.

If $t = \varnothing$, then $s * t = s \in \partial K \subset K * L$, and we have

$$\mathrm{Lk}(K * L, s * t) = \mathrm{Lk}(K * L, s) \underset{2\,(1)}{=} \underbrace{\mathrm{Lk}(K,s)}_{\approx_{\mathrm{st}} D} * \underbrace{L}_{\approx_{\mathrm{st}} D} \approx_{\mathrm{st}} D$$

by Lemma 7 (1). But by definition, this means that $s * t \in \partial(K * L)$.

If neither s nor t are empty, we can apply Lemma 3 in order to obtain

$$\mathrm{Lk}(K * L, s * t) = \underbrace{\mathrm{Lk}(K,s)}_{\approx_{\mathrm{st}} D} * \underbrace{\mathrm{Lk}(L,t)}_{\approx_{\mathrm{st}} D \text{ or } \approx_{\mathrm{st}} S} \approx_{\mathrm{st}} D$$

by Lemma 7 (1) or (3). But by definition, this means that $s * t \in \partial(K * L)$.

As mentioned above, the inclusion $K * \partial L \subseteq \partial(K * L)$ follows analogously, which concludes the proof of the statement.

(2) Now assume that K is a combinatorial sphere and L is a combinatorial ball. We again prove both inclusions, starting with "$\subseteq$". So let $s * t \in \partial(K * L)$, i.e., $s \in S \cup \{\varnothing\}$ and $t \in T \cup \{\varnothing\}$ not both empty, with

$$\mathrm{Lk}(K * L, s * t) \approx_{\mathrm{st}} D.$$

We want to show that $s * t \in K * \partial L$.

If $s = \varnothing$, then $s * t = t$, and from

$$D \approx_{\mathrm{st}} \mathrm{Lk}(K * L, s * t) = \mathrm{Lk}(K * L, t) = \underbrace{\mathrm{Lk}(L,t)}_{\approx_{\mathrm{st}} D \text{ or } \approx_{\mathrm{st}} S} * \underbrace{K}_{\approx_{\mathrm{st}} S}$$

and Lemma 7 (2) and (3), we obtain that $\mathrm{Lk}(L,t)$ must be a combinatorial ball; in other words, $s * t = t \in \partial L \subseteq K * \partial L$.

If $t = \varnothing$, then we immediately obtain that $s * t = s \in K \subseteq K * \partial L$.

If neither s nor t are empty, then we calculate using Lemma 3 that

$$D \approx_{\mathrm{st}} \mathrm{Lk}(K * L, s * t) = \underbrace{\mathrm{Lk}(K,s)}_{\approx_{\mathrm{st}} S \text{ by } 2} * \underbrace{\mathrm{Lk}(L,t)}_{\approx_{\mathrm{st}} D \text{ or } \approx_{\mathrm{st}} S}.$$

But by Lemma 7 (2) and (3), this again implies that $\mathrm{Lk}(L,t)$ must be a combinatorial ball; in other words, we have $s * t \in K * \partial L$.

For the converse inclusion, we let $s * t \in K * \partial L$, i.e., $s \in S \cup \{\varnothing\}$ and $t \in \partial L \cup \{\varnothing\}$ not both empty. Our goal is to show that we have $s * t \in \partial(K * L)$.
If $s = \varnothing$, then $s * t = t$, and we obtain from Lemma 7 (3) that

$$\mathrm{Lk}(K * L, s * t) = \mathrm{Lk}(K * L, t) = \underbrace{\mathrm{Lk}(L,t)}_{\approx_{\mathrm{st}} D} * \underbrace{K}_{\approx_{\mathrm{st}} S} \approx_{\mathrm{st}} D.$$

But this just means that $s * t \in \partial(K * L)$.
If $t = \varnothing$, then $s * t = s$, and we obtain from Lemma 7 (3) that

$$\mathrm{Lk}(K * L, s * t) = \mathrm{Lk}(K * L, s) = \underbrace{\mathrm{Lk}(K,s)}_{\approx_{\mathrm{st}} S \text{ by } 2} * \underbrace{L}_{\approx_{\mathrm{st}} D} \approx_{\mathrm{st}} D.$$

But this again just means that $s * t \in \partial(K * L)$.
The final case is that neither s nor t are empty. We use Lemmas 3 and 7 (3) in order to obtain that

$$\mathrm{Lk}(K * L, s * t) = \underbrace{\mathrm{Lk}(K,s)}_{\approx_{\mathrm{st}} S \text{ by } 2} * \underbrace{\mathrm{Lk}(L,t)}_{\approx_{\mathrm{st}} D} \approx_{\mathrm{st}} D.$$

This shows that in all cases, we have $s * t \in \partial(K * L)$. We have therefore seen both inclusions, which concludes the proof. □

The next Corollary, which is a direct consequence of Proposition 2 (1), will be used in a lot of the proofs of the main statements of the next chapter.

Corollary 3. *Let K be a combinatorial ball. Then* $\partial\mathrm{Cone}(K) = \mathrm{Cone}(\partial K) \cup K$.

Proof. Let K be a combinatorial ball. Remember that $\mathrm{Cone}(K) = D_0 * K$. (In particular, the cone of K is in fact a combinatorial manifold by Lemmas 7 (1) and 6.) It follows from Proposition 2 (1) that

$$\partial\mathrm{Cone}(K) = (D_0 * \partial K) \cup (\underbrace{\partial D_0}_{=(\varnothing,\varnothing)} * K) = \mathrm{Cone}(\partial K) \cup K,$$

where we used that taking the join with the empty abstract simplicial complex preserves any abstract simplicial complex.

5 Starrability of Stellar Balls

Now that we know a fair bit about some of the properties of stellar moves, balls, spheres, and manifolds, we want to introduce a concept of "especially nice" combinatorial balls, only to then prove that all combinatorial balls have this nice property. Just like the previous one, this chapter is also modelled after the approach made by Lickorish (cf. [Lic99, Chapter 3]), again however with a greater emphasis on details.

Definition 11. Let $K = (V,S)$ be a combinatorial n-ball. Denote by $s := \{0,\dots,n\}$ the maximal simplex of D_n. We have the following chain of stellar equivalences

$$K \approx_{\mathrm{st}} D_n \approx_{\mathrm{st}} \sigma_s D_n = \underline{s} * S_{n-1} \approx_{\mathrm{st}} \mathrm{Cone}(\partial K),$$

where the last stellar equivalence is a consequence of Corollary 1 and Lemma 5. By transitivity, this implies that there exists a sequence of stellar moves and simplicial isomorphisms from K to $\mathrm{Cone}(\partial K)$.

(1) A stellar subdivision σ_t is called *internal* if $t \notin \partial K$.
(2) A stellar weld σ_t^{-1} is called *internal* if the stellar subdivision that is its inverse is an internal stellar subdivision.
(3) A stellar move $\sigma_t^{\pm 1}$ is called an *internal move* if it is an internal stellar subdivision or an internal stellar weld.
(4) We call K *starrable in r moves* if there exists a sequence of r internal moves (and any number of simplicial isomorphisms) that transforms K into $\mathrm{Cone}(\partial K)$. Such a sequence is called a *starring of K*.

Remark 5. One of the main goals of this chapter is to prove that *all* combinatorial balls are starrable. Put another way, this means that any abstract simplicial complex K that is stellar equivalent to D_n can be transformed into $\mathrm{Cone}(\partial K)$ without "changing the boundary."

The following lemma may look harmless, but as mathematicians we know that looks can be deceiving. It can be found in [Lic99, Lemma 3.4].

Lemma 12. *Let $K = (V,S)$ be a starrable n-ball. If L is a stellar subdivision of K, then L is also starrable.*

Proof. Let $K = (V,S)$ be a starrable n-ball, i.e., there is a sequence of internal moves (and simplicial isomorphisms) $K \approx_{\mathrm{st}} \underline{v} * \partial K$, and let $L := \sigma_s K$ for some k-simplex $s \in S$. If $s \notin \partial K$, then $\partial L = \partial K$ by Lemma 11, and therefore

$$L = \sigma_s K \approx_{\mathrm{st}} K \approx_{\mathrm{st}} \underline{v} * \partial K = \underline{v} * \partial L,$$

where all the stellar moves are internal since $s \notin \partial K$. Hence, L is also starrable. For the remainder of the proof, we can therefore assume that $s \in \partial K$.
We shall perform induction on r, which we define to be the minimum number of stellar moves in a sequence $K \approx_{\mathrm{st}} \underline{v} * \partial K$ of internal moves and simplicial isomorphisms.
The case $r = 0$ is relatively straightforward: up to simplicial isomorphism, we have the equation $K = \underline{v} * \partial K$, and therefore

$$L = \sigma_s K \cong_{\mathrm{si}} \sigma_s(\underline{v} * \partial K) = \underline{v} * \sigma_s(\partial K),$$

where the last equality follows from the fact that $s \in \partial K$. But since again $s \in \partial K$, we have $\sigma_s(\partial K) = \partial L$, thus it follows in fact that $L \cong_{\mathrm{si}} \underline{v} * \partial L$, and L is (trivially) starrable.

Now assume that stellar subdivisions of combinatorial n-balls that are starrable in less than r moves are themselves starrable. We have to consider two cases:

Case 1: The first of the r internal moves is a stellar subdivision (as opposed to a stellar weld). We denote the result of this first subdivision by $K_1 := \sigma_t K$ for some $t \in S$ with $t \notin \partial K$. Remember that K_1 is, by construction, starrable in $r-1$ moves. Since σ_t is an internal move, we have $\partial K_1 = \partial K \ni s$. Hence, we can perform a stellar subdivision of K_1 at s, and we denote the resulting abstract simplicial complex by $L_1 := \sigma_s K_1$. Since L_1 is a stellar subdivision of K_1, which is starrable in $r-1$ moves, it follows from the induction hypothesis that L_1 is starrable. For clarity's sake, it makes sense to consider the following three cases:

Case 1a: If $s \cap t = \varnothing$, then up to a simplicial isomorphism, σ_s and σ_t commute, hence

$$L_1 = \sigma_s K_1 = \sigma_s \sigma_t K \cong_{\mathrm{si}} \sigma_t \sigma_s K = \sigma_t L.$$

But L_1 is starrable and σ_t is an internal move, so L is starrable.

Case 1b: If there does not exist a simplex in K that has both s and t as its faces, then the open stars of s and t in K do not intersect, and therefore σ_s and σ_t still commute. The claim follows as in case 1a.

If one wanted to summarise cases 1a and 1b in a succinct way, the commutativity of the following diagram would be the key element.

$$\begin{tikzcd} K \arrow[r, "\sigma_t"] \arrow[d, "\sigma_s"] & K_1 \arrow[d, "\sigma_s"] \\ L \arrow[r, "\sigma_t"] & L_1 \end{tikzcd}$$

Case 1c: It remains to show the claim for the case that s and t are faces of a common simplex, and that $s \cap t = u$ for some simplex $u \in S$. If we define $s_0 := s \setminus u$ and $t_0 := t \setminus u$, we have $s = s_0 * u$ and $t = t_0 * u$, as well as $s_0 * t_0 * u \in S$. The situation is illustrated at the very left of Figure 11. We want to investigate the relationship between $\sigma_s \sigma_t K$ and $\sigma_t \sigma_s K$. Note that by definition of stellar subdivisions, the difference between these two abstract simplicial complexes has to lie entirely in what happens on the star $\mathrm{St}(K, s_0 * t_0 * u)$. First we shall focus on the simplex $s_0 * t_0 * u$ itself. Using associativity and commutativity of the simplicial join operation, one obtains

$$\begin{aligned} \sigma_t \sigma_s (\underbrace{s_0 * u}_{=s} * t_0) &= \sigma_t (\underline{s} * \partial s * t_0) \overset{2\,(1)}{=} \\ &= \sigma_t ((\underline{s} * s_0 * \partial u * t_0) \cup (\underline{s} * \partial s_0 * \underbrace{u * t_0}_{=t})) = \\ &= (\underline{s} * s_0 * \partial u * t_0) \cup (\underline{s} * \partial s_0 * \underline{t} * \partial t) \overset{2\,(1)}{=} \\ &= (\underline{s} * (s_0 \cup \partial s_0 * \underline{t}) * t_0 * \partial u) \cup (\underline{s} * \partial s_0 * \underline{t} * \partial t_0 * u) \overset{2\,(1)}{=} \\ &= (\underline{s} * \partial (\underline{t} * s_0) * t_0 * \partial u) \cup (\underline{s} * \underline{t} * \partial s_0 * \partial t_0 * u) \end{aligned}$$

where in the third step, the stellar subdivision σ_t does not change the first part because the dimension of the second part is precisely the dimension of $s_0 * t_0 * u$ and t is entirely contained in the second part[15]. In the last step, we used the fact that $\partial \underline{t} = (\varnothing, \varnothing)$ since it is a 0-simplex.
Figure 11 attempts to summarise the effects of the composition of two stellar subdivisions $\sigma_t \sigma_s$.

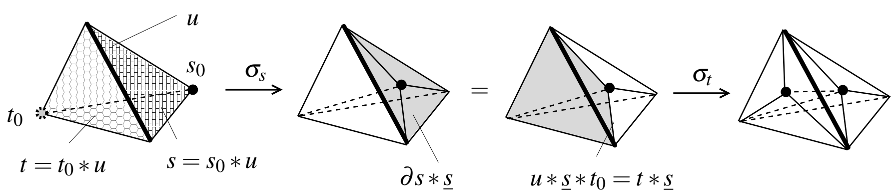


Fig. 11: The effects of $\sigma_t \sigma_s$ in case 1c

If we define $v := \underline{s} * t_0$, then stellar subdivision yields

$$\begin{aligned}\sigma_v \sigma_t \sigma_s (s_0 * u * t_0) &= \sigma_v((\partial(\underline{t} * s_0) * v * \partial u) \cup (\underline{s} * \underline{t} * \partial s_0 * \partial t_0 * u)) = \\ &= (\partial(\underline{t} * s_0) * \underline{v} * \partial v * \partial u) \cup (\underline{s} * \underline{t} * \partial s_0 * \partial t_0 * u) = \\ &= (\partial(\underline{t} * s_0) * \underline{v} * \partial(\underline{s} * t_0) * \partial u) \cup (\underline{s} * \underline{t} * \partial s_0 * \partial t_0 * u).\end{aligned}$$

However, since this expression is symmetric in s and t, we can use the same logic in order to perform another three subdivisions on the original $s_0 * u * t_0$ to obtain a simplicially isomorphic abstract simplicial complex[16]:

$$\sigma_{v'} \sigma_s \sigma_t (s_0 * u * t_0) = (\partial(\underline{t} * s_0) * \underline{v'} * \partial(\underline{s} * t_0) * \partial u) \cup (\underline{s} * \underline{t} * \partial s_0 * \partial t_0 * u).$$

Here, we set $v' := \underline{t} * s_0$ after performing $\sigma_s \sigma_t$.
We now take the join in K with $\mathrm{Lk}(K, s_0 * u * t_0)$ and obtain that

$$\sigma_v \sigma_t L = \sigma_v \sigma_t \sigma_s K \cong_{\mathrm{si}} \sigma_{v'} \sigma_s \sigma_t K = \sigma_{v'} L_1.$$

It now suffices to show that both v and v' are internal moves. Since $u \subseteq s \in \partial K$ implies $u \in \partial K$, and $t_0 * u = t \notin \partial K$, the simplex t_0 cannot lie in ∂K. Therefore, $v = \underline{s} * t_0 \notin \partial K$. Since $t \notin \partial K$, the 0-simplex $\underline{t}$ cannot lie in ∂K, thus also $v' = \underline{t} * s_0 \notin \partial K$.
Therefore, since σ_t is also still an internal move, L and L_1 differ by internal moves, proving that L is starrable.

---

[15] In Figure 11, this argument effectively means that only the magenta portion experiences a change from the stellar subdivision σ_t.

[16] If one were to draw $\sigma_s \sigma_t$, the result would essentially be Figure 11, but mirrored along a vertical axis.

Case 2: The first of the r internal moves is a stellar weld. If we again denote the result of this stellar weld by $K_1 := \sigma_t^{-1} K$, this time for some simplex t of K_1 with $t \notin \partial K_1$, we can use the same logic as before to conclude that $s \in \partial K_1$, and again define $L_1 := \sigma_s K_1$. Again, since L_1 is a subdivision of K_1, which is starrable in $r-1$ moves, L_1 is starrable. But we have already shown in Case 1 that L and L_1 are related by internal stellar moves, so L is also starrable. □

Our next step on the way to proving that all combinatorial balls are starrable is to consider the cones of starrable balls. As a reminder, the cone of an abstract simplicial complex is its join with $D_0 = (\{0\}, \{\{0\}\})$. Note that as a consequence of Lemma 7 (1), the cone of a combinatorial n-ball is indeed a combinatorial $(n+1)$-ball. Both the lemma and its proof can be found (albeit with less details) in [Lic99, Lemma 3.5].

Lemma 13. *Let $K = (V,S)$ be a starrable n-ball. Then* $\mathrm{Cone}(K)$ *is starrable.*

Proof. So let $K = (V,S)$ be a starrable n-ball, that means $K \approx_{\mathrm{st}} \underline{v} * \partial K$ by internal moves. As in the previous proofs, we shall perform induction on r, the minimum number of internal moves in such a sequence.
The case $r = 0$ means that $K \cong_{\mathrm{si}} \underline{v} * \partial K$. Using associativity of the join operation, we obtain

$$\mathrm{Cone}(K) = D_0 * K \cong_{\mathrm{si}} D_0 * \underline{v} * \partial K.$$

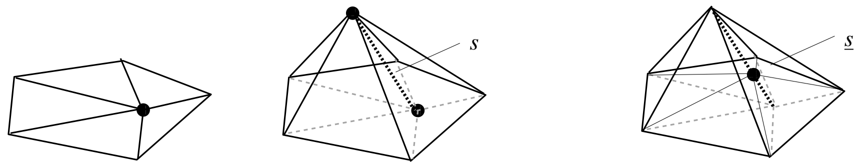
We now define a 1-simplex $s := D_0 * \underline{v}$. Note that since by construction $\underline{v} \notin \partial K$, we have $s = D_0 * \underline{v} \notin \partial\mathrm{Cone}(K) = \mathrm{Cone}(\partial K) \cup K$, where the last equality is just Corollary 3, so performing a stellar subdivision along s in $\mathrm{Cone}(K)$ is an internal move. By a slight abuse of notation, we also refer to the preimage of s under the simplicial isomorphism as s. We then have

$$\sigma_s \mathrm{Cone}(K) \cong_{\mathrm{si}} \sigma_s(s * \partial K) = \underline{s} * \underbrace{\partial s}_{=D_0 \cup \underline{v}} * \underbrace{\mathrm{Lk}(\mathrm{Cone}(K), s)}_{=\partial K}.$$

By Corollary 3, we still have the identity $\partial\mathrm{Cone}(K) = \mathrm{Cone}(\partial K) \cup K$. Using this, we can resume the above calculation to obtain

$$\sigma_s \mathrm{Cone}(K) \cong_{\mathrm{si}} \underline{s} * \Big(\underbrace{D_0 * \partial K}_{=\mathrm{Cone}(\partial K)} \cup \underbrace{\underline{v} * \partial K}_{\cong_{\mathrm{si}} K} \Big) \cong_{\mathrm{si}} \underline{s} * \partial\mathrm{Cone}(K).$$

We have thus transformed $\mathrm{Cone}(K)$ into $\underline{s} * \partial\mathrm{Cone}(K)$ using the internal move σ_s (together with some simplicial isomorphisms), so $\mathrm{Cone}(K)$ is starrable. This concludes the case that $r = 0$. Figure 12 hopefully gives some insight on what happened. Note that we intentionally avoided filling in the 2- and 3-simplices in order to keep it legible.
Now let us assume that $K \approx_{\mathrm{st}} \underline{v} * \partial K$ in r moves, and that the cones of combinatorial n-balls that are starrable in less than r moves are themselves starrable. Based on the nature of the first of the r internal moves, we distinguish two cases:



$K \cong_{\mathrm{si}} \underline{v} * \partial K$ $\qquad$ $\mathrm{Cone}(K)$ $\qquad$ $\sigma_s \mathrm{Cone}(K) \cong_{\mathrm{si}} \underline{s} * \partial \mathrm{Cone}(K)$

Fig. 12: The case $r = 0$ in the proof of Lemma 13

Case 1: If the first move is a stellar weld $K_1 = \sigma_s^{-1} K$ for some internal simplex s of K_1, then $K = \sigma_s K_1$ and $\mathrm{Cone}(K) \cong_{\mathrm{si}} \sigma_{\mathrm{Cone}(s)} \mathrm{Cone}(K_1)$. Since K_1 is starrable in $r-1$ moves, $\mathrm{Cone}(K_1)$ is starrable. By Lemma 12, $\mathrm{Cone}(K)$ is also starrable as it is a stellar subdivision of $\mathrm{Cone}(K_1)$.

Case 2: Now assume that the first of the r moves is a stellar subdivision $K_1 = \sigma_s K$ for some simplex $s \in S$ with $s \notin \partial K$. We have

$$K = (K \setminus \mathring{\mathrm{St}}(K,s)) \cup \mathrm{St}(K,s) = (K \setminus \mathring{\mathrm{St}}(K,s)) \cup s * \mathrm{Lk}(K,s),$$

so taking the cone according to Lemma 4 yields

$$\mathrm{Cone}(K) = \mathrm{Cone}(K \setminus \mathring{\mathrm{St}}(K,s)) \cup \mathrm{Cone}(s * \mathrm{Lk}(K,s)).$$

Now we want to perform stellar subdivision at the simplex $t := \mathrm{Cone}(s) = D_0 * s$ in the subcomplex $\mathrm{Cone}(\mathrm{St}(K,s)) = \mathrm{St}(\mathrm{Cone}(K), t)$ as follows:

$$\begin{aligned} \sigma_t \mathrm{Cone}(s * \mathrm{Lk}(K,s)) &= \underline{t} * \partial t * \mathrm{Lk}(K,s) = \\ &= \underline{t} * \mathrm{Cone}(\partial s) * \mathrm{Lk}(K,s) \cup \underline{t} * s * \mathrm{Lk}(K,s) = \\ &= \mathrm{Cone}(\underline{t} * \partial s * \mathrm{Lk}(K,s)) \cup \underline{t} * s * \mathrm{Lk}(K,s) \end{aligned}$$

where we used that $\partial t = \partial \mathrm{Cone}(s) = \mathrm{Cone}(\partial s) \cup s$ by Corollary 3, and the commutativity of the join operation, which allows us to "pull out the cone." Note that by definition of $K_1 = \sigma_s K$, and by using Lemma 4, we have

$$\mathrm{Cone}(K_1) = \mathrm{Cone}(K \setminus \mathring{\mathrm{St}}(K,s)) \cup \mathrm{Cone}(\underline{s} * \partial s * \mathrm{Lk}(K,s)).$$

Now recall that, by our induction assumption, the cone on K_1 is starrable as K_1 is starrable in $r-1$ moves, so $\mathrm{Cone}(K_1) \approx_{\mathrm{st}} \mathrm{Cone}(\partial \mathrm{Cone}(K_1))$ by internal moves. Using Corollary 3 to rewrite the boundary of the cone on K_1, we are now able to put everything together[17]:

[17] From the induction hypothesis and the fact that $s \notin \partial K$, thus $t \notin \partial \mathrm{Cone}(K)$, it follows that all the stellar moves are internal.

$$\begin{aligned}
\sigma_t \mathrm{Cone}(K) &= \mathrm{Cone}(K) \setminus \mathring{\mathrm{St}}(\mathrm{Cone}(K),t)) \cup \underline{t} * \partial t * \mathrm{Lk}(\mathrm{Cone}(K),t) \cong_{\mathrm{si}} \\
&\cong_{\mathrm{si}} \mathrm{Cone}(K \setminus \mathring{\mathrm{St}}(K,s)) \cup \mathrm{Cone}(\underline{t} * \partial s * \mathrm{Lk}(K,s)) \cup \underline{t} * s * \mathrm{Lk}(K,s) \cong_{\mathrm{si}} \\
&\cong_{\mathrm{si}} \mathrm{Cone}(K_1) \cup \underline{t} * s * \mathrm{Lk}(K,s) \approx_{\mathrm{st}} \\
&\approx_{\mathrm{st}} \mathrm{Cone}(\partial \mathrm{Cone}(K_1)) \cup \underline{t} * s * \mathrm{Lk}(K,s) = \\
&= \mathrm{Cone}\Big(\mathrm{Cone}(\partial K) \cup K \setminus \mathring{\mathrm{St}}(K,s) \cup \underline{s} * \partial s * \mathrm{Lk}(K,s)\Big) \cup \underline{t} * s * \mathrm{Lk}(K,s) \cong_{\mathrm{si}} \\
&\cong_{\mathrm{si}} \mathrm{Cone}\Big(\underbrace{\mathrm{Cone}(\partial K) \cup K}_{=\partial \mathrm{Cone}(K)} \cup \underline{s} * \partial s * \mathrm{Lk}(K,s)\Big) \approx_{\mathrm{st}} \\
&\approx_{\mathrm{st}} \mathrm{Cone}(\partial \mathrm{Cone}(K)).
\end{aligned}$$

In the fifth step, we again used Lemma 11 to replace ∂K_1 by ∂K, as well as the definition of $K_1 = \sigma_s K = K \setminus \mathring{\mathrm{St}}(K,s) \cup \underline{s} * \partial s * \mathrm{Lk}(K,s)$. We have thus shown that $\mathrm{Cone}(K)$ is starrable by internal moves.

In both cases, we have thus performed the induction step, which concludes the proof.

After this extremely technical proof, we can reap the fruit of our labour in form of the following theorem, which is essentially [Lic99, Theorem 3.6].

Theorem 4. *Let $n \in \mathbb{N}_0$, and let $K = (V,S)$ be a combinatorial n-ball. Then K is starrable.*

Proof. Unsurprisingly, we perform induction on n. The case $n = 0$ is rather trivial: if K is a combinatorial 0-ball, it is by definition an abstract simplicial complex stellar equivalent to D_0. But stellar subdivisions and welds on D_0 are just simplicial isomorphisms, and therefore K is starrable.
Now suppose that combinatorial balls of dimension less than n are starrable. We perform another induction, this time on r, the number of stellar moves[18] in $K \approx_{\mathrm{st}} D_n$. If $r = 0$, then there exists a simplicial isomorphism $f\colon D_n \to K$. Let $s := \{0,\ldots,n\}$ be the maximal simplex of D_n. Since $\mathrm{Lk}(K, f(s)) \cong_{\mathrm{si}} \mathrm{Lk}(D_n, s) = (\varnothing, \varnothing)$, we have $f(s) \notin \partial K$, so

$$\sigma_{f(s)} K \cong_{\mathrm{si}} \sigma_s D_n = \underline{s} * S_{n-1} \cong_{\mathrm{si}} \underline{f(s)} * \partial K,$$

using an internal move, and K is starrable.
Now suppose $K \approx_{st} D_n$ in r stellar moves, and that combinatorial n-balls that are stellar equivalent to D_n in less than r moves are starrable. We denote the result of the first move by K_1. By our induction hypothesis, K_1 is starrable.
If the first move is internal, then K is starrable. If the first move is a weld $K_1 = \sigma_s^{-1} K$, then $K = \sigma_s K_1$ is a stellar subdivision of a starrable n-ball, and therefore itself starrable by Lemma 12. We shall therefore assume for the remainder of the proof that $K_1 = \sigma_s K$ for some k-simplex $s \in \partial K$.
Our plan now is as follows:

Step 1: Show that the star of s in K is starrable, and perform said starring.

[18] Remember that these stellar moves are in general not internal.

Step 2: From the resulting abstract simplicial complex, isolate a copy of
Step 3: Use the induction hypothesis to obtain a starring of this copy of K_1.
Step 4: Perform the starring from Step 1 backwards to recover a starring of K.

We choose a vertex $v \in s$ and call the "opposite face" $t := s \setminus \{v\}$. For the star of s in K, the following equation holds:

$$\mathrm{St}(K,s) = s * \mathrm{Lk}(K,s) = \{v\} * t * \mathrm{Lk}(K,s) \cong_{\mathrm{si}} \mathrm{Cone}\big(t * \mathrm{Lk}(K,s)\big).$$

We make the following observations:

1) Since t is a $(k-1)$-simplex, it is in particular a combinatorial $(k-1)$-ball.
2) The link $\mathrm{Lk}(K,s)$ is the link of a k-simplex from the boundary of a combinatorial manifold, therefore it is a combinatorial $(n-k-1)$-ball.
3) By Lemma 7 (1), the join of two combinatorial balls is a combinatorial ball, of dimension

$$\dim\big(t * \mathrm{Lk}(K,s)\big) = (k-1) + (n-k-1) + 1 = n-1.$$

By our induction hypothesis, this combinatorial $(n-1)$-ball is starrable.
4) The star $\mathrm{St}(K,s) \cong_{\mathrm{si}} \mathrm{Cone}\big(t * \mathrm{Lk}(K,s)\big)$ is therefore starrable by Lemma 13.

In order to show that, in fact, the entire combinatorial n-ball K is starrable, we perform the following calculation[19]

$$\begin{aligned}
K &= \mathrm{St}(K,s) \cup \big(K \setminus \mathring{\mathrm{St}}(K,s)\big) \overset{4)}{\approx}_{\mathrm{st}} \\
&\approx_{\mathrm{st}} \underline{w} * \partial(\mathrm{St}(K,s)) \cup \big(K \setminus \mathring{\mathrm{St}}(K,s)\big) \overset{2\,(1)}{=} \\
&= \underline{w} * s * \mathrm{Lk}(\partial K,s) \cup \underbrace{\underline{w} * \partial s * \mathrm{Lk}(K,s) \cup \big(K \setminus \mathring{\mathrm{St}}(K,s)\big)}_{\cong_{\mathrm{si}} \sigma_s K = K_1 \approx_{\mathrm{st}} \underline{u} * \partial K_1} \approx_{\mathrm{st}} \\
&\approx_{\mathrm{st}} \underline{u} * \underbrace{\partial\Big(\underline{w} * \partial s * \mathrm{Lk}(K,s) \cup \big(K \setminus \mathring{\mathrm{St}}(K,s)\big)\Big)}_{= \underline{w} * \partial s * \mathrm{Lk}(\partial K,s) \cup (\partial K \setminus \mathring{\mathrm{St}}(\partial K,s))} \cup \underline{w} * s * \mathrm{Lk}(\partial K,s) = \\
&= \underline{u} * \Big(\underline{w} * \partial s * \mathrm{Lk}(\partial K,s) \cup \big(\partial K \setminus \mathring{\mathrm{St}}(\partial K,s)\big)\Big) \cup \underline{w} * s * \mathrm{Lk}(\partial K,s) = \\
&= \underline{u} * \underline{w} * \partial s * \mathrm{Lk}(\partial K,s) \cup \underline{u} * \big(\partial K \setminus \mathring{\mathrm{St}}(\partial K,s)\big) \cup \underline{w} * s * \mathrm{Lk}(\partial K,s).
\end{aligned}$$

In the second to last step, we used the fact that $s \in \partial K$ to calculate the boundary of K_1.
Now we perform a stellar subdivision on $\underline{u} * \mathrm{St}(\partial K,s) = \underline{u} * s * \mathrm{Lk}(\partial K,s)$ as follows:

$$\begin{aligned}
\sigma_{\underline{u}*s}\big(\underline{u} * \mathrm{St}(\partial K,s)\big) &\cong_{\mathrm{si}} \underline{w} * \partial(\underline{u} * s) * \underbrace{\mathrm{Lk}(\underline{u} * \mathrm{St}(\partial K,s),s)}_{=\mathrm{Lk}(\partial K,s)} \overset{2\,(1)}{=} \\
&= \underline{u} * \underline{w} * \partial s * \mathrm{Lk}(\partial K,s) \cup \underline{w} * s * \mathrm{Lk}(\partial K,s).
\end{aligned}$$

[19] We leave it to the reader to verify that all the stellar moves are, in fact, internal.

Since $\underline{u} \notin \partial K$, and thus $\underline{u} * s \notin \partial K$, we can continue our sequence of internal stellar equivalences from above by adding

$$\begin{aligned} K &\approx_{\mathrm{st}} \underline{u} * \underline{w} * \partial s * \mathrm{Lk}(\partial K, s) \cup \underline{u} * \left(\partial K \setminus \mathring{\mathrm{St}}(\partial K, s)\right) \cup \underline{w} * s * \mathrm{Lk}(\partial K, s) \approx_{\mathrm{st}} \\ &\approx_{\mathrm{st}} \underline{u} * \mathrm{St}(\partial K, s) \cup \underline{u} * \left(\partial K \setminus \mathring{\mathrm{St}}(\partial K, s)\right) = \\ &= \underline{u} * \partial K. \end{aligned}$$

This finally completes our induction argument. The sketch below hopefully does a good job at illustrating what happened, as much as our dimensionally limited view allows.

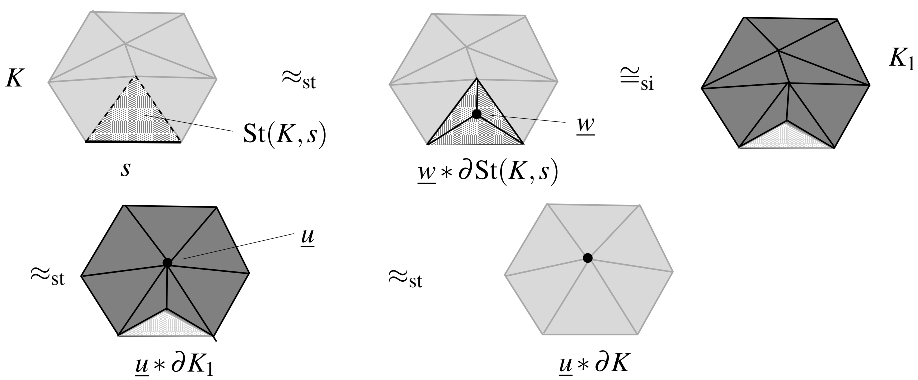


Fig. 13: Sketch of the induction step in the proof of Theorem 4

We want to continue with a technical lemma about "gluing on the cone of a nice piece of the boundary". It can be found in [Lic99, Lemma 3.7].

Lemma 14. *Let $n \in \mathbb{N}$. Let $K = (V, S)$ be a combinatorial n-manifold. If $L = (W, T)$ is an abstract subcomplex of ∂K that is a combinatorial $(n-1)$-ball, then there is a stellar equivalence*[20]

$$K \approx_{st} K \cup \mathrm{Cone}(L) = K \cup \underline{u} * L.$$

The idea for the proof is to use Theorem 4 in order to find a combinatorial n-manifold that is stellar equivalent to K and that contains a "second copy of $\mathrm{Cone}(L)$". Then, these two copies of $\mathrm{Cone}(L)$, which intersect in L, are a stellar subdivision of a single $\mathrm{Cone}(L)$ if we also show that L is a cone in itself.

Proof. Let $n \in \mathbb{N}$. Let $K = (V, S)$ be a combinatorial n-manifold, and let $L = (W, T)$ be an abstract subcomplex of ∂K with $L \approx_{\mathrm{st}} D_{n-1}$. Then L is starrable by Theorem 4, so we can choose a starring sequence for L, i.e., a sequence of internal stellar

[20] Here, we assume that the cone point $\underline{u}$ does not lie in V.

moves (and simplicial isomorphisms, which we ignore as usual) from L to $\underline{v} * \partial L$. This leads us to the following claim:

Claim. We can extend this starring sequence to all of $K \cup \mathrm{Cone}(L)$, i.e., there is a sequence of stellar moves on $K \cup \mathrm{Cone}(L)$ that changes the abstract subcomplex L to $\underline{v} * \partial L$.

Proof. Each stellar move in the sequence is either an internal stellar subdivision or an internal stellar weld of L.
The stellar subdivisions in the sequence can easily be extended to all of K and to $\mathrm{Cone}(L)$ since L is a subcomplex of both K and $\mathrm{Cone}(L)$, and thus any simplex of L is also a simplex of these two abstract simplicial complexes.
If there is an internal stellar weld in the sequence, then without loss of generality we may assume it is the first of the stellar moves. So we can write

$$L = \sigma_s L_1 = K_1 \setminus \mathring{\mathrm{St}}(L_1, s) \cup \underline{s} * \partial s * \mathrm{Lk}(L_1, s),$$

for some combinatorial $(n-1)$-ball L_1 and some k-simplex s of L_1 with $s \notin \partial L_1$. Clearly, the weld extends to $\mathrm{Cone}(L)$, the difficult problem is proving it also extends to K. By construction, the star of $\underline{s}$ in K is given by

$$\mathrm{St}(K, \underline{s}) = \underline{s} * \mathrm{Lk}(K, \underline{s}).$$

Since $\underline{s}$ is a combinatorial 0-ball and $\mathrm{Lk}(K, \underline{s})$ is a combinatorial $(n-1)$-sphere we can conclude from Lemma 7 (3) that $\mathrm{St}(K, \underline{s})$ is a combinatorial n-ball, hence starrable by Theorem 4. That means there is a stellar equivalence

$$K \approx_{\mathrm{st}} K' := \big(K \setminus \mathring{\mathrm{St}}(K, \underline{s})\big) \cup \underline{s} * \partial\big(\mathrm{St}(K, \underline{s})\big) \cong_{\mathrm{si}} \big(K \setminus \mathring{\mathrm{St}}(K, \underline{s})\big) \cup \mathrm{Cone}\Big(\partial\big(\mathrm{St}(K, \underline{s})\big)\Big).$$

This stellar equivalence to the abstract simplicial complex K', which contains the cone on the boundary of $\mathrm{St}(K, \underline{s})$, allows us to extend the stellar weld to K', and thus also to the stellar equivalent K. This concludes the proof of the claim.

Since the stellar equivalence $L \approx_{\mathrm{st}} \underline{v} * \partial L$ extends to $K \cup \mathrm{Cone}(L)$, we may, without loss of generality, assume that $L = \underline{v} * \partial L$.
Just as before, the star $\mathrm{St}(K, \underline{v})$ is a combinatorial n-ball, which we star according to Theorem 4 to obtain the result that K contains

$$\underline{w} * \partial\big(\mathrm{St}(K, \underline{v})\big) = \underline{w} * \mathrm{Lk}(K, \underline{v})$$

as an abstract subcomplex. Since ∂L is an abstract subcomplex of $\mathrm{Lk}(K, \underline{v})$ by construction, this implies that $\underline{w} * L = \underline{w} * \underline{v} * \partial L$ is an abstract subcomplex of K.
Now, consider the following subcomplex of $K \cup \mathrm{Cone}(L)$:

$$\mathrm{Cone}(L) \cup \underline{w} * \underline{v} * \partial L = \mathrm{Cone}(\underline{v} * \partial L) \cup \underline{w} * \underline{v} * \partial L = \underline{u} * \underline{v} * \partial L \cup \underline{w} * \underline{v} * \partial L.$$

These two copies of $\mathrm{Cone}(\underline{v} * \partial L)$ are stellar equivalent to a single copy by welding at $\underline{v} * (\underline{u} \cup \underline{w})$. Hence,

$$K \cup \mathrm{Cone}(L) \approx_{\mathrm{st}} K \setminus \mathring{\mathrm{St}}(K,s) \cup \underline{w} * \underline{v} * \partial L \cup \mathrm{Cone}(L) \approx_{\mathrm{st}}$$
$$\approx_{\mathrm{st}} K \setminus \mathring{\mathrm{St}}(K,s) \cup \underline{w} * \underline{v} * \partial L \approx_{\mathrm{st}} K.$$

This concludes the proof. A sketch of the strategy for the proof when L consists of a single simplex can be found below.

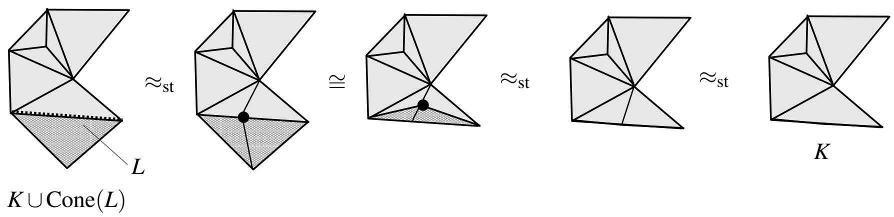


Fig. 14: Strategy for the proof of Lemma 14

The next theorem, which is due to Newman (cf. [New26]), has a very intuitive topological meaning, but is surprisingly tricky to prove in the abstract setting. It states that "removing" a combinatorial n-ball from a combinatorial n-sphere leaves behind another combinatorial n-ball. First, we have to define what we even mean by that. Most authors would give the more topologically inspired definition of "closure of $K \setminus L$", but we shall try and avoid topology at all costs in this chapter.

Definition 12. Let $K = (V,S)$ be a combinatorial n-sphere, and let $L = (W,T)$ be an abstract subcomplex that is a combinatorial n-ball. We define

$$K \setminus \mathring{L} := \left(\left\{ v \in V \,\middle|\, \begin{array}{l} \text{there is an } n\text{-simplex} \\ s \in S \setminus T \text{ with } v \in s \end{array} \right\}, \left\{ t \in S \,\middle|\, \begin{array}{l} \text{there is an } n\text{-simplex} \\ s \in S \setminus T \text{ with } t \subseteq s \end{array} \right\} \right).$$

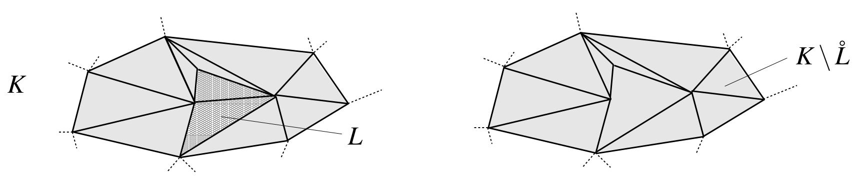


Fig. 15: Removing a 2-ball from a 2-sphere (indicated by the dotted lines)

This is a fancy way of saying to "keep only the vertices and simplices that actually contribute to something outside of L". Using the transitivity of the subset relation, it is an easy exercise to prove that $K \setminus \mathring{L}$ is in fact an abstract simplicial complex, and hence it is also a subcomplex of K by definition of subcomplexes.

Theorem 5. *Let $n \in \mathbb{N}_0$, let $K = (V,S)$ be a combinatorial n-sphere, and let $L = (W,T)$ be an abstract subcomplex of K that is a combinatorial n-ball. Then $K \setminus \mathring{L}$ is a combinatorial n-ball.*

Proof. Let $n \in \mathbb{N}_0$, let $K = (V,S)$ be a combinatorial n-sphere, and let $L = (W,T)$ be an abstract subcomplex of K that is a combinatorial n-ball. We start with an induction on n.
The case $n = 0$ is quite simple: As a combinatorial 0-sphere, K is simplicially isomorphic to S_0 since stellar moves at 0-simplices are just simplicial isomorphisms. On the other hand, L, being a combinatorial 0-ball, is simplicially isomorphic to D_0 for the same reason. So in the end, the statement for $n = 0$ boils down to the fact that if one takes two 0-simplices and removes one of them, they end up with a single 0-simplex. But that just means that $K \setminus \mathring{L}$ is a combinatorial 0-ball.
Now let $n \in \mathbb{N}$, and assume the claim is true for combinatorial spheres (and combinatorial balls) of dimension less than n.
In order to use some of the previous results, we first check that $K \setminus \mathring{L}$ is a combinatorial n-manifold[21]. Let v be a vertex of $K \setminus \mathring{L}$, i.e., there exists an n-simplex $s \in S \setminus T$ with $v \in s$. If for all n-simplices $t \in T$ we have $v \notin t$, then all n-simplices of K that v is a part of lie in $K \setminus \mathring{L}$, thus

$$\mathrm{Lk}(K \setminus \mathring{L}, v) = \mathrm{Lk}(K, v) \approx_{\mathrm{st}} S_{n-1},$$

because K is a combinatorial n-manifold. If, on the other hand, there does exist an n-simplex $t \in T$ with $v \in t$, then we have[22]

$$\mathrm{Lk}(K \setminus \mathring{L}, v) = \mathrm{Lk}(K, v) \setminus \mathrm{Lk}(\mathring{L}, v).$$

Here, $\mathrm{Lk}(K,v)$ is a combinatorial $(n-1)$-sphere as v is a vertex of the combinatorial n-sphere K, and $\mathrm{Lk}(L,v)$ is a combinatorial $(n-1)$-ball since v is a vertex in the boundary of the combinatorial n-ball L. By our induction hypothesis for the case $n-1$, the link $\mathrm{Lk}(K \setminus \mathring{L}, v)$ is a combinatorial $(n-1)$-ball. Therefore, $K \setminus \mathring{L}$ is a combinatorial n-manifold.
Now we shall start proving the actual claim. Since L is starrable by Theorem 4, we will, without loss of generality, assume for the remainder of the proof that $L = \mathrm{St}(K,v)$ for some vertex $v \in V$. This also has the convenient side effect that the definition of $K \setminus \mathring{L}$ coincides with the definition of $K \setminus \mathring{\mathrm{S}}\mathrm{t}(K,v)$ from page 8.[23]
We start another induction, on the number r of stellar moves in $K \approx_{\mathrm{st}} S_n$.
In the case $r = 0$, we have $K \cong_{\mathrm{si}} S_n$, so $L \cong_{\mathrm{si}} \mathrm{St}(S_n, v)$ is the star of a vertex of S_n, so $K \setminus \mathring{L}$ is the "n-simplex opposite to that vertex". But this is clearly simplicially

[21] This is something we actually have to prove because we defined $K \setminus \mathring{L}$ in by "removing vertices and simplices" from a combinatorial sphere, which does not always result in a combinatorial manifold.

[22] Note that the little circle on the right-hand side applies to all of $\mathrm{Lk}(L,v)$.

[23] The proof that these two definitions coincide is a simple exercise in set theory that shall be left to the reader. Intuitively however, there is no doubt about it as they are both defined in terms of removing simplices that are in $\mathrm{St}(K,v)$.

isomorphic to D_n, so $K \setminus \mathring{L}$ is a combinatorial n-ball.
Now let $K \approx_{\text{st}} S_n$ in r moves, and assume that the claim is true for n-spheres that are stellar equivalent to S_n in less than r stellar moves. As so often, there are two cases for us to distinguish:

Case 1: If the first of the r moves is a stellar subdivision, denote its result by $K_1 = \sigma_s K$, for some k-simplex $s \in S$. We are interested in the relationship between $K \setminus \mathring{\mathrm{St}}(K, v)$ and $K_1 \setminus \mathring{\mathrm{St}}(K_1, v)$. Depending on v and s, we have to distinguish another two cases:

Case 1a: If $v \notin s$, then it follows from the definition of stellar moves that

$$K_1 \setminus \mathring{\mathrm{St}}(K_1, v) = \sigma_s\big(K \setminus \mathring{\mathrm{St}}(K, v)\big).$$

The expression on the left-hand side is a combinatorial n-ball by induction on r, thus $K \setminus \mathring{\mathrm{St}}(K, v)$ is as well, because it is related to the left-hand side via a stellar move.

Case 1b: If $v \in s$, we set $t := s \setminus \{v\}$ and obtain

$$K_1 \setminus \mathring{\mathrm{St}}(K_1, v) = K \setminus \mathring{\mathrm{St}}(K, v) \cup \underline{s} * t * \mathrm{Lk}(K, s).$$

Note that on the right-hand side, $K \setminus \mathring{\mathrm{St}}(K, v)$ is a combinatorial n-manifold, and $t * \mathrm{Lk}(K, s)$ is a combinatorial ball of dimension $(k-1) + (n-k-1) + 1 = n-1$ that also is a subcomplex of $K \setminus \mathring{\mathrm{St}}(K, v)$. Furthermore, the cone point $\underline{s}$ does not lie in K, so we can apply Lemma 14 to obtain

$$K_1 \setminus \mathring{\mathrm{St}}(K_1, v) \cong_{\text{si}} K \setminus \mathring{\mathrm{St}}(K, v) \cup \mathrm{Cone}\big(t * \mathrm{Lk}(K, s)\big) \overset{14}{\approx}_{\text{st}} K \setminus \mathring{\mathrm{St}}(K, v)$$

Just as in Case 1a, we can therefore conclude that $K \setminus \mathring{\mathrm{St}}(K, v)$ is a combinatorial n-ball.

Case 2: If the first of the r moves is a stellar weld, denote its result by $K_1 = \sigma_s^{-1} K$, i.e., $K = \sigma_s K_1$ for some k-simplex s of K_1. We are still interested in showing that $K \setminus \mathring{\mathrm{St}}(K, v)$ is a combinatorial n-ball.
If $v \neq \underline{s}$, then the same arguments as in Case 1 apply, and $K \setminus \mathring{\mathrm{St}}(K, v)$ is a combinatorial n-ball.
Now let us consider the case that $v = \underline{s}$. By construction, we have

$$K \setminus \mathring{\mathrm{St}}(K, v) = K_1 \setminus \mathring{\mathrm{St}}(K_1, s).$$

We pick a vertex $w \in s$, and again set $t := s \setminus \{w\}$. Since $K_1 \approx_{\text{st}} S_n$ in $r-1$ moves, we know from our induction hypothesis that $K_1 \setminus \mathring{\mathrm{St}}(K_1, w)$ is a combinatorial n-ball. Thus, its boundary

$$\partial\big(K_1 \setminus \mathring{\mathrm{St}}(K_1, w)\big) = \mathrm{Lk}(K_1, w)$$

is a combinatorial $(n-1)$-sphere by Corollary 1. Since by construction $w \notin t * \mathrm{Lk}(K_1, s)$, and $\{w\} * t * \mathrm{Lk}(K_1, s) = \mathrm{St}(K_1, s)$ is a subcomplex of K_1, it follows that $t * \mathrm{Lk}(K_1, s)$ is a subcomplex of $\partial\big(K_1 \setminus \mathring{\mathrm{St}}(K_1, w)\big)$. However, we also have

$$t * \mathrm{Lk}(K_1, s) \approx_{\mathrm{st}} D_{k-1} * S_{n-k-1} \overset{7\,(3)}{\approx}_{\mathrm{st}} D_{n-1},$$

and therefore we can again apply the induction hypothesis for the case $n-1$ to conclude that

$$M := \partial\big(K_1 \setminus \mathring{\mathrm{St}}(K_1, w)\big) \setminus \big(t * \mathrm{L}\mathring{\mathrm{k}}(K_1, s)\big)$$

is a combinatorial $(n-1)$-ball. Connecting all these deliberations, we can now finally calculate

$$\begin{aligned} K \setminus \mathring{\mathrm{St}}(K, v) = K_1 \setminus \mathring{\mathrm{St}}(K_1, s) &= \\ &= \big(K_1 \setminus \mathring{\mathrm{St}}(K_1, w)\big) \cup \big(\mathrm{St}(K_1, w) \setminus \mathring{\mathrm{St}}(K_1, s)\big) = \\ &= \big(K_1 \setminus \mathring{\mathrm{St}}(K_1, w)\big) \cup \{w\} * \Big(\mathrm{Lk}(K_1, w) \setminus \big(t * \mathrm{L}\mathring{\mathrm{k}}(K_1, s)\big)\Big) = \\ &= \big(K_1 \setminus \mathring{\mathrm{St}}(K_1, w)\big) \cup \{w\} * M \cong_{\mathrm{si}} \\ &\cong_{\mathrm{si}} \big(K_1 \setminus \mathring{\mathrm{St}}(K_1, w)\big) \cup \mathrm{Cone}(M) \overset{14}{\approx}_{\mathrm{st}} \\ &\approx_{\mathrm{st}} \big(K_1 \setminus \mathring{\mathrm{St}}(K_1, w)\big). \end{aligned}$$

As mentioned before, the very right-hand side is a combinatorial n-ball, and thus, so is $K \setminus \mathring{\mathrm{St}}(K, v)$.

In all cases, we showed that $K \setminus \mathring{\mathrm{St}}(K, v)$ is a combinatorial n-ball. This concludes both the induction on r and the one on n, and thus the proof.

The final theorem of this chapter, which is due to Alexander (cf. [Ale30]), is another gluing statement. It is, however, more of a corollary to the main statements of this chapter.

Theorem 6. *Let $K = (V, S)$ be a combinatorial n-manifold, and let $L = (W, T)$ be a combinatorial n-ball. We assume that the intersection $K \cap L$ satisfies the following two properties:*

(1) *We have $K \cap L = \partial K \cap \partial L$.*
(2) *The intersection $K \cap L$ is a combinatorial $(n-1)$-ball.*

Then $K \cup L \approx_{st} K$.

Proof. By Theorem 4, the combinatorial n-ball L is starrable. Therefore, without loss of generality, we can assume that $L = \underline{v} * \partial L$. Furthermore, ∂L is a combinatorial $(n-1)$-sphere by Corollary 1, and $K \cap L$ is a combinatorial $(n-1)$-ball that is an abstract subcomplex of ∂L by property (1). It now follows from Theorem 5 that

$$M := \partial L \setminus (K \mathring{\cap} L)$$

is a combinatorial $(n-1)$-ball, which is starrable by Theorem 4, i.e., $M \approx_{\mathrm{st}} \underline{w} * \partial M$. By Lemma 10, we have $\partial M = \partial(K \cap L)$, hence

$$M \approx_{\mathrm{st}} \underline{w} * \partial(K \cap L).$$

This starring can be extended onto $\mathrm{Cone}(M) \cong_{\mathrm{si}} \underline{v} * M$, yielding

$$\underline{v} * M \approx_{\mathrm{st}} \underline{v} * \underline{w} * \partial M = \underline{w} * \underline{v} * \partial(K \cap L).$$

Therefore, one obtains

$$\begin{aligned} K \cup L &= K \cup \underline{v} * \partial L = \\ &= K \cup \underline{v} * \big((K \cap L) \cup M\big) \stackrel{4}{=} \\ &= K \cup \underline{v} * (K \cap L) \cup \underline{v} * M. \end{aligned}$$

Since both $K \cap L$ and M are combinatorial $(n-1)$-balls that are abstract subcomplexes of ∂K, we can apply Lemma 14 twice in order to find a stellar equivalence from that final abstract simplicial complex to K, finishing the proof.

We end this chapter with the following lemma, which we will actually use in the proof of Pachner's Theorem 7. It can be found in [Lic99, Lemma 4.6].

Lemma 15. *Let $k \in \mathbb{N}$, and let $K = (V, S)$ be an abstract simplicial complex.*
If $K * S_{k-1}$ is a combinatorial n-sphere or a combinatorial n-ball, then K is a combinatorial $(n-k)$-sphere or a combinatorial $(n-k)$-ball.

Proof. Let $k \in \mathbb{N}$, and let K be an abstract simplicial complex such that $K * S_{k-1}$ is a combinatorial n-sphere or a combinatorial n-ball. Consider the abstract simplicial complex

$$\mathrm{Cone}(K * S_{k-1}).$$

It follows from Lemma 7 (1) or (3) that this cone is a combinatorial $(n+1)$-ball. It follows from the associativity and commutativity of the join operation that $K * D_k \cong_{\mathrm{si}} K * \mathrm{Cone}(S_{k-1})$ is also a combinatorial $(n+1)$-ball. Using Proposition 1 (1), we conclude that

$$K = \mathrm{Lk}(K * D_k, D_k)$$

is a combinatorial sphere or a combinatorial ball of dimension

$$\dim(K * D_k) - \dim D_k - 1 = (n+1) - k - 1 = n - k. \square$$

6 Bistellar Moves and Elementary Shellings

Having now spent a lot of time on stellar moves, we want to turn to a second type of move on combinatorial manifolds. Despite initially sounding more complicated, there will be an extremely nice finiteness property to them, which is one of the reasons why Pachner's theorem is so important in the first place. This chapter follows the approach taken in [Lic99, Chapter 5].

Definition 13. Let $K = (V,S)$ be an abstract simplicial complex, let $s \in S$ be a simplex, and let $m \in \mathbb{N}_0 \cup \{-1\}$. We denote the link of s in K by $(U,R) := \mathrm{Lk}(K,s)$. We call (U,R) a *non-trivial m-sphere in K* if $(U,R) \cong_{\mathrm{si}} S_m$ and U itself is not a simplex of K.
Note that this definition makes sense for $m = -1$ since we had defined $S_{-1} := (\varnothing, \varnothing)$, and the empty set is not a simplex by definition of abstract simplicial complexes.
In this case, we also define

$$B(K,s) := \begin{cases} (U, \mathscr{P}(U) \setminus \{\varnothing\}), & \text{if } m \in \mathbb{N}_0, \\ (\{\underline{s}\}, \{\{\underline{s}\}\}), & \text{if } m = -1. \end{cases}$$

This abstract simplicial complex $B(K,s)$ is simplicially isomorphic to D_{m+1} by construction, and we call it the *dual simplex of s*.[24]

Figure 16 hopefully does a decent job at illustrating the situation. Essentially, we want the link of s to be the boundary of a simplex that is not in K.

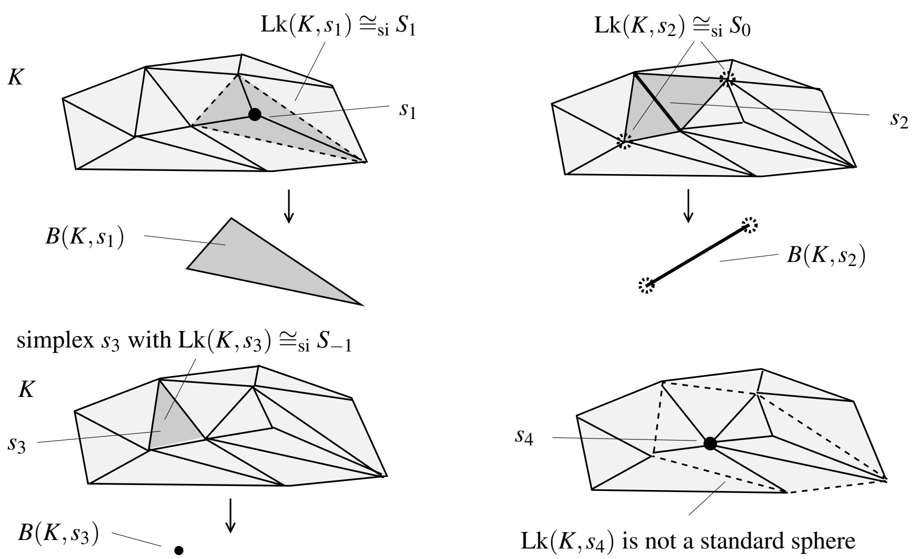


Fig. 16: From left to right, examples for a non-trivial 1-sphere, a non-trivial 0-sphere, a non-trivial (-1)-sphere, and a link that is not a non-trivial sphere

Definition 14. Let $K = (V,S)$ be a closed combinatorial n-manifold, $s \in S$ be a k-simplex such that $\mathrm{Lk}(K,s)$ is a non-trivial $(n-k-1)$-sphere in K. We call

$$\tau_s K := (K \setminus \mathring{\mathrm{St}}(K,s)) \cup \partial s * B(K,s)$$

[24] We will shortly see a justification for the name "dual simplex", more specifically in Lemma 16.

(the result of) a *bistellar move along s*.

Essentially, a bistellar move replaces the star $\operatorname{St}(K,s) = s * \operatorname{Lk}(K,s)$ by $\partial s * B(K,s)$. We refer to Figure 3 in the introduction for an illustration of the definition.

The following lemma summarises some of the main properties of bistellar moves. Another reason to include it is that proving it allows the reader to grow used to this new concept.

Lemma 16. *Let $K = (V,S)$ be a combinatorial n-manifold, $s \in S$ be a k-simplex such that $\operatorname{Lk}(K,s)$ is a non-trivial $(n-k-1)$-sphere in K. If we denote the dual simplex $B(K,s)$ by t, then we have the following useful properties:*

(1) *The dual simplex t is an $(n-k)$-simplex of $\tau_s K$, and its boundary is given by $\partial t = \operatorname{Lk}(K,s)$.*
(2) *The link of t in $\tau_s K$ is given by $\operatorname{Lk}(\tau_s K,t) = \partial s$.*
(3) *We have*

$$B(\tau_s K,t) = \begin{cases} s, & \text{if } k > 0, \\ (\{\underline{t}\},\{\{\underline{t}\}\}), & \text{if } k = 0. \end{cases}$$

(4) *Removing the open star of t from $\tau_s K$ yields*

$$\tau_s K \setminus \mathring{\operatorname{St}}(\tau_s K,t) = K \setminus \mathring{\operatorname{St}}(K,s).$$

(5) *The abstract simplicial complexes $\sigma_s K$ and $\sigma_t(\tau_s K)$ are simplicially isomorphic. In particular we have, up to a simplicial isomorphism, $\tau_s = \sigma_t^{-1}\sigma_s$.*
(6) *There is a simplicial isomorphism between $\tau_t(\tau_s K)$ and K.*

Proof. Let $K = (V,S)$ be a combinatorial n-manifold, $s \in S$ be a k-simplex such that $\operatorname{Lk}(K,s)$ is a non-trivial $(n-k-1)$-sphere in K. Denote the dual simplex $B(K,s)$ by t.

(1) Since $\operatorname{Lk}(K,s)$ is a non-trivial $(n-k-1)$-sphere, the dual simplex $t = B(K,s)$ is an $(n-k)$-simplex by construction, and its boundary is $\operatorname{Lk}(K,s)$ also by construction.
(2) This also follows directly from the definition.
(3) By (2) we have that $\operatorname{Lk}(\tau_s K,t) = \partial s$. For $k = 0$, this is a non-trivial (-1)-sphere, thus $B(\tau_s K,t) = (\{\underline{t}\},\{\{\underline{t}\}\})$. If $k > 0$, then the link is the non-trivial $(k-1)$-sphere ∂s, so in this case, the dual simplex of t in $\tau_s K$ is given by $B(\tau_s K,t) = s$.
(4) Using (2), we calculate

$$\operatorname{St}(\tau_s K,t) = \operatorname{Lk}(\tau_s K,t) * t = \partial s * B(K,s),$$

therefore

$$\begin{aligned}\tau_s K \setminus \mathring{\operatorname{St}}(\tau_s K,t) &= \big((K \setminus \mathring{\operatorname{St}}(K,s)) \cup \partial s * B(K,s)\big) \setminus \mathring{\operatorname{St}}(\tau_s K,t) = \\ &= K \setminus \mathring{\operatorname{St}}(K,s).\end{aligned}$$

(5) Using (1), (2), and (4), one calculates

$$\begin{aligned}\sigma_t(\tau_s K) &= \tau_s K \setminus \mathring{\mathrm{St}}(\tau_s K, t) \cup \underline{t} * \partial t * \mathrm{Lk}(\tau_s K, t) = \\ &= K \setminus \mathring{\mathrm{St}}(K, s) \cup \underline{t} * \mathrm{Lk}(K, s) * \partial s \cong_{\mathrm{si}} \\ &\cong_{\mathrm{si}} K \setminus \mathring{\mathrm{St}}(K, s) \cup \underline{s} * \partial s * \mathrm{Lk}(K, s) = \\ &= \sigma_s K\end{aligned}$$

(6) This follows directly from (3) and (5) if we believe that a stellar subdivision, followed by a stellar weld on "the same" simplex, amount to a simplicial isomorphism.

All six claims have thus been proved.

Lemma 16 (6) allows us to give the following definition. To be more precise, it ensures that bistellar equivalence is, in fact, an equivalence relation.

Definition 15. Let $K = (V, S)$ and $L = (W, T)$ be combinatorial manifolds. We call K and L *bistellar equivalent* if they are connected by a finite sequence of bistellar moves and simplicial isomorphisms. If K and L are bistellar equivalent, we write $K \approx_{\mathrm{bst}} L$.

Now we have all the tools to formulate Pachner's theorem, which we already stated in the introduction, but without giving precise definitions.

Theorem 7. (Pachner's Theorem) *Let $K = (V, S)$ and $K' = (V', S')$ be closed combinatorial n-manifolds. Then*

$$K \approx_{\mathrm{bst}} K' \iff K \approx_{\mathrm{st}} K'.$$

This theorem is deceptively easy to state. To put it briefly, it says that if two combinatorial manifolds are stellar equivalent, then one can find a sequence of bistellar moves between them (and vice versa). At this point, there is one more tool we need to add to our toolbox before we can move onto the path toward proving Pachner's theorem.

Definition 16. Let $K = (V, S)$ be a combinatorial n-manifold, and let $s, t \in S$ be simplices such that the following three conditions are satisfied:

(1) the join $s * t$ is an n-simplex of K,
(2) we have $s \cap \partial K = \partial s$, and
(3) we have $t * \partial s \subseteq \partial K$ is a subcomplex.

Let $k := \dim t$. The *elementary k-shelling of K along t* is the combinatorial manifold[25]

$$K' := \mathrm{sh}_t K := K \setminus \mathring{\mathrm{St}}(K, s * t).$$

[25] Note that the star of the n-simplex $s * t$ is just the abstract simplicial complex consisting of all the faces of $s * t$. Therefore, a shelling can be imagined as "removing" the simplex $s * t$ from K.

Remark 6.

(1) Consider the situation from the above definition. As the notation "sh_t" suggests, the elementary k-shelling is uniquely determined by the k-simplex t. This follows from properties (2) and (3).
(2) In the sketches below, the two different "types" of elementary shellings on a combinatorial 2-manifold are illustrated. Note that for dimensional reasons[26], there cannot be an elementary n-shelling on an elementary n-manifold.

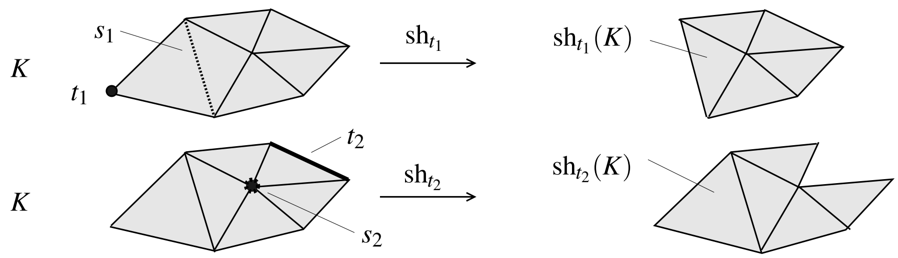


Fig. 17: An elementary 0-shelling (top) and an elementary 1-shelling (bottom) in a 2-manifold

We make the following observation about the boundary of a combinatorial manifold. We refer to the sketch above for some examples.

Lemma 17. *Let $K = (V,S)$ be a combinatorial manifold, and let $s,t \in S$ be as above. Then* $\mathrm{Lk}(\partial K,t) = \partial s$*, and*
$$\partial(\mathrm{sh}_t K) = \tau_t(\partial K).$$

Proof. From property (3) and Lemma 10, we can conclude that $\mathrm{Lk}(\partial K,t) = \partial s$. Now, property (2) ensures that $s \notin \partial K$, and together with property (1), this implies that the link of t in ∂K is a non-trivial $(n-k-2)$-sphere[27]. Therefore, the right-hand side is well-defined, and we have $B(\partial K,t) = s$.
Using Lemma 16, we calculate

$$\begin{aligned}\partial(\mathrm{sh}_t K) &= \partial K \setminus \mathring{\mathrm{St}}(\partial K,t) \cup \underbrace{\partial(s*t) \setminus (t \mathring{*} \partial s)}_{=s*\partial t} = \\ &= \partial K \setminus \mathring{\mathrm{St}}(\partial K,t) \cup \partial t * B(\partial K,t) = \\ &= \tau_t(\partial K).\end{aligned}$$

This proves the claim.

As the sketches above show, elementary shellings "make the abstract simplicial complex smaller." This intuitive concept shall be expanded upon in the following definition.

[26] More precisely, it follows from $n \geq \dim(s*t) = \dim s + \dim t + 1 > \dim t$.

[27] In detail, this follows from $n = \dim s + k + 1$ and $\dim \partial s = \dim s - 1$.

Definition 17. Let $K = (V, S)$ be a combinatorial n-manifold.

(1) If $L = (W, T)$ is another combinatorial n-manifold, and L can be obtained from K by means of a finite sequence of elementary shellings and simplicial isomorphisms, we write $K \xrightarrow{\text{sh}} L$.
(2) If K is a combinatorial n-ball, then we call K *shellable* if $K \xrightarrow{\text{sh}} D_n$.
(3) If K is a combinatorial n-sphere, then we call K *shellable* if there exists an n-simplex $s \in S$ such that $K \setminus \mathring{\mathrm{St}}(K, s)$ is a shellable combinatorial n-ball[28].

The sketch below illustrates the concept of shellability. In each step, the dark grey simplex is the one that is "being removed" next. It is also evident that there exist many different sequences of elementary shellings that transform K into something simplicially isomorphic to D_2.

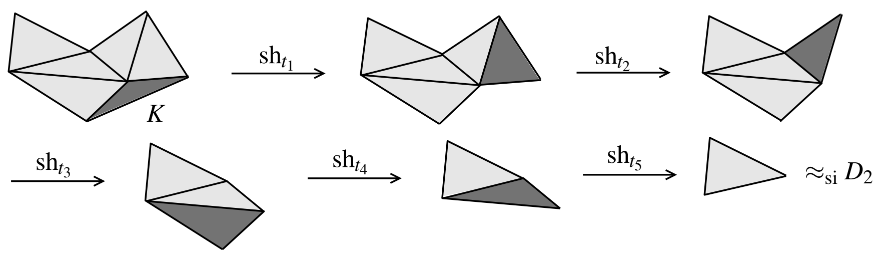


Fig. 18: A combinatorial 2-ball K that is shellable in 5 moves

Remark 7. While the picture makes it seem like every combinatorial n-ball is shellable, there are, in fact, examples of combinatorial n-balls that are not shellable. For instance, Rudin (cf. [Rud58]) found an unshellable combinatorial 3-ball with 41 simplices of dimension 3. Furthermore, for any $n \geq 3$, Lickorish (cf. [Lic91]) gives a combinatorial n-sphere that is not shellable. However, the intuition we have that all combinatorial 2-balls and -spheres are shellable is, in fact, correct. A proof for this statement can be found in Bing (cf. [Bin64]).

For us, the more interesting question is how we can use the concept of shellability in order to prove Pachner's theorem. We start by proving a lemma that deals with cones of shellable combinatorial balls and spheres. It is essentially [Lic99, Lemma 5.4], but as we have many times before, we fill in the details in order to make it more rigorous.

Lemma 18. *Let $K = (V, S)$ be a shellable combinatorial n-ball or a shellable combinatorial n-sphere. Then* $\mathrm{Cone}(K)$ *is shellable*[29].

[28] Note that it follows from Theorem 5 that $K \setminus \mathring{\mathrm{St}}(K, s)$ is in fact a combinatorial n-ball.
[29] Remember that by Lemma 7 (1) or (3), the cone of K is a combinatorial $(n+1)$-ball.

Proof. The idea for the proof is to just "lift" the sequence of elementary shellings (and simplicial isomorphisms) that we have for K to $\mathrm{Cone}(K)$. Of course, there are some technical conditions we will have to check, but this is one of the cases where the intuitive approach actually works.
Let us first consider the case that $K=(V,S)$ is a shellable combinatorial n-ball. For each elementary shelling sh_t in a sequence $K \xrightarrow{\mathrm{sh}} D_n$, we want to lift sh_t to $\mathrm{Cone}(K)$. So let $s,t \in S$ satisfy the three conditions we had required[30]:

(1) The join $s*t$ is an n-simplex of K,
(2) we have $s\cap\partial K=\partial s$, and
(3) we have $t*\partial s\subseteq\partial K$.

Our goal is to show that these conditions still hold if we replace s by $\mathrm{Cone}(s)$, K by $\mathrm{Cone}(K)$, and n by $n+1$. We do so in great detail.

(1') The join $\mathrm{Cone}(s)*t=D_0*s*t$ is an $(n+1)$-simplex of $\mathrm{Cone}(K)=D_0*K$ by the definition of joins and property (1).
(2') Using Lemma 4 and Corollary 3, as well as property (2), we calculate

$$\begin{aligned}\mathrm{Cone}(s)\cap\partial\mathrm{Cone}(K)&=\mathrm{Cone}(s)\cap(K\cup\mathrm{Cone}(\partial K))=\\&=\underbrace{(\mathrm{Cone}(s)\cap K)}_{=s}\cup\underbrace{(\mathrm{Cone}(s)\cap\mathrm{Cone}(\partial K))}_{=\mathrm{Cone}(s\cap\partial K)}\overset{(2)}{=}\\&=s\cup\mathrm{Cone}(\partial s)=\partial\mathrm{Cone}(s).\end{aligned}$$

(3') Using Lemma 4 and Corollary 3, as well as properties (1) and (3) and the commutativity of the join, we calculate

$$\begin{aligned}t*\partial\mathrm{Cone}(s)&=t*(s\cup\mathrm{Cone}(\partial s))=\\&=t*s\cup\underbrace{t*\mathrm{Cone}(\partial s)}_{=\mathrm{Cone}(t*\partial s)}\overset{(1),(3)}{\subseteq}\\&\subseteq K\cup\mathrm{Cone}(\partial K)=\partial\mathrm{Cone}(K).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\mathrm{sh}_t\mathrm{Cone}(K)&=\mathrm{Cone}(K)\setminus\mathring{\mathrm{St}}(\mathrm{Cone}(K),\mathrm{Cone}(s)*t)=\\&=\mathrm{Cone}(K\setminus\mathring{\mathrm{St}}(K,s*t))=\mathrm{Cone}(\mathrm{sh}_tK).\end{aligned}$$

Therefore, we can "lift" an entire sequence $K\xrightarrow{\mathrm{sh}}D_n$ to one of the form

$$\mathrm{Cone}(K)\xrightarrow{\mathrm{sh}}\mathrm{Cone}(D_n)\cong_{\mathrm{si}}D_{n+1},$$

and $\mathrm{Cone}(K)$ is a shellable combinatorial $(n+1)$-ball.
Now consider the case that $K=(V,S)$ is a shellable combinatorial n-sphere. Let

[30] Here, we slightly abuse notation in order to write K instead of something like $\mathrm{sh}_{t_{k-1}}\dots\mathrm{sh}_{t_1}K$.

$s \in S$ be an n-simplex such that $K \setminus \mathring{\mathrm{St}}(K,s)$ is a shellable combinatorial n-ball. Note that $\mathrm{Cone}(s)$ is an $(n+1)$-simplex of $\mathrm{Cone}(K)$, so we can perform the following elementary n-shelling:

$$\mathrm{sh}_s\mathrm{Cone}(K) = \mathrm{Cone}(K) \setminus \mathring{\mathrm{St}}(\mathrm{Cone}(K), \mathrm{Cone}(s)) = \mathrm{Cone}\big(K \setminus \mathring{\mathrm{St}}(K,s)\big).$$

Since $K \setminus \mathring{\mathrm{St}}(K,s)$ is a shellable combinatorial n-ball, $\mathrm{Cone}\big(K \setminus \mathring{\mathrm{St}}(K,s)\big)$ is a shellable combinatorial $(n+1)$-ball by the first case. But since there is an elementary shelling from $\mathrm{Cone}(K)$ onto this shellable combinatorial $(n+1)$-ball, $\mathrm{Cone}(K)$ is a shellable combinatorial $(n+1)$-ball as well.

Next, we want to look into the join of shellable combinatorial balls or spheres with the simplicial complex S_m. It can be found in [Lic99, Lemma 5.5].

Lemma 19. *Let $K = (V,S)$ be a shellable combinatorial n-ball or a shellable combinatorial n-sphere, and let $m \in \mathbb{N}_0 \cup \{-1\}$. Then $K * S_m$ is shellable*[31].

Proof. We start with the case that $K = (V,S)$ is a shellable combinatorial n-ball. We perform induction on $m \in \mathbb{N}_0 \cup \{-1\}$.
The case $m = -1$ is trivial since $S_{-1} = (\varnothing, \varnothing)$, and joining with the empty complex preserves any abstract simplicial complex.
Now let $m \in \mathbb{N}_0$, and suppose the claim is true for balls of dimension less than m. Let v be a vertex of S_m, denote by $s := \{0,\dots,m+1\} \setminus \{v\}$ the "opposite face". Since K is shellable, we can pick a sequence $\mathrm{sh}_{t_1},\dots,\mathrm{sh}_{t_k}$ of elementary shellings for $K \xrightarrow{\mathrm{sh}} D_n$. Then, the join simplices $t_1 * s, \dots, t_k * s$ are part of the boundary of the iterated elementary shellings of $K * S_m$, and we can now perform the sequence $\mathrm{sh}_{t_1 * s}, \dots, \mathrm{sh}_{t_k * s}, \mathrm{sh}_s$ on $K * S_m$ as follows:

$$\mathrm{sh}_s\mathrm{sh}_{t_k * s}\cdots\mathrm{sh}_{t_1 * s}(K * S_m) \cong_{\mathrm{si}} \mathrm{sh}_s(\underbrace{\{v\}}_{\cong_{\mathrm{si}}\{0\}} * K * \underbrace{\partial s}_{\cong_{\mathrm{si}} S_{m-1}} \cup s * D_n) \cong_{\mathrm{si}} \mathrm{Cone}(K * S_{m-1})$$

By our induction hypothesis, $K * S_{m-1}$ is a shellable combinatorial $(n+m)$-ball. Therefore, $\mathrm{Cone}(K * S_{m-1})$ is a shellable combinatorial $(n+m+1)$-ball by Lemma 18. Since there is a sequence of elementary shellings from $K * S_m$ to $\mathrm{Cone}(K * S_{m-1})$, we obtain that $K * S_m$ is a shellable combinatorial $(n+m+1)$-ball. The claim now follows from our induction argument.
The sketch in Figure **??** illustrates what has happened, even if it is a little unsatisfying because it only covers the case $m = 0$. It shall be mentioned that one could draw the case $m = 1$, but then K would have to be a 1-manifold, which is also not a very enlightening sketch.

Now we shall turn to the case that $K = (V,S)$ is a shellable combinatorial n-sphere, and we let $r \in S$ be an n-simplex such that $K \setminus \mathring{\mathrm{St}}(K,r)$ is a shellable combinatorial n-ball. As before, we let v be a vertex of S_m, and we denote by $s := \{0,\dots,m+1\} \setminus \{v\}$ its "opposite face". Since $\partial s \cong_{\mathrm{si}} S_{m-1}$, we can remove

[31] Remember that by Lemma 7 (2) or (3), the join $K * S_m$ is a combinatorial $(n+m+1)$-ball or a combinatorial $(n+m+1)$-sphere.

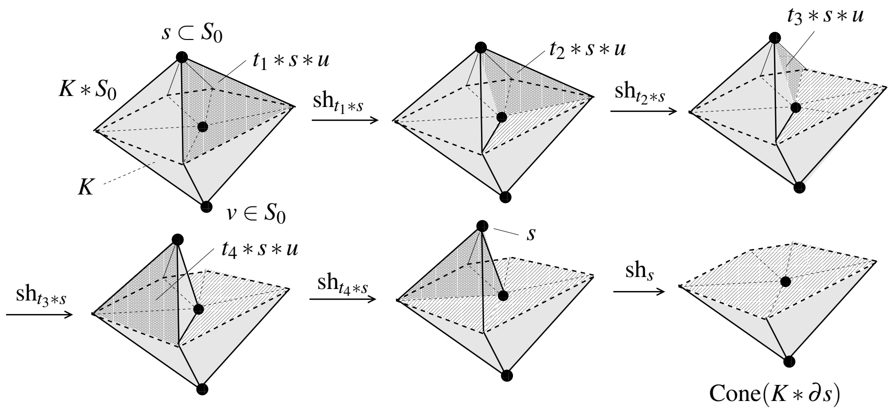


Fig. 19: Proof of Lemma 19 in the case of K being a 2-ball and $m = 0$

an $(m-1)$-simplex from ∂s, and then perform a total of $m-2$ elementary shellings $\mathrm{sh}_{s_1}, \dots, \mathrm{sh}_{s_{m-2}}$ in order to obtain[32]

$$\mathrm{sh}_{s_{m-2}} \cdots \mathrm{sh}_{s_1} \big(\partial s \setminus \mathring{\mathrm{St}}(\partial s, s_0)\big) \cong_{\mathrm{si}} D_{m-1}.$$

Similarly to how we did in the previous case, we now perform the following sequence of elementary shellings:

$$\mathrm{sh}_r \mathrm{sh}_{r * s_{m-2}} \cdots \mathrm{sh}_{r * s_1} \big(K * S_m \setminus \mathring{\mathrm{St}}(K * S_m, r * s)\big) \cong_{\mathrm{si}} \big(K \setminus \mathring{\mathrm{St}}(K, r)\big) * S_m.$$

Since $K \setminus \mathring{\mathrm{St}}(K, r)$ is a shellable combinatorial n-ball, it follows from the first case that $\big(K \setminus \mathring{\mathrm{St}}(K, r)\big) * S_m$ is a shellable combinatorial $n+m+1$-ball. Due to the existence of the sequence of shellings

$$\big(K * S_m \setminus \mathring{\mathrm{St}}(K * S_m, r * s)\big) \xrightarrow{\mathrm{sh}} \big(K \setminus \mathring{\mathrm{St}}(K, r)\big) * S_m,$$

this implies that $\big(K * S_m \setminus \mathring{\mathrm{St}}(K * S_m, r * s)\big)$ also is a shellable combinatorial $n + m + 1$-ball. But by definition, this means that $K * S_m$ is a shellable combinatorial $n+m+1$-sphere, which finishes the proof.

This next Lemma, which is just [Lic99, Lemma 5.7], has a certain similarity to our old friend Theorem 4, albeit with an additional condition and a stronger result.

Lemma 20. *If $K = (V, S)$ is a shellable combinatorial n-ball, then* $\mathrm{Cone}(\partial K) \approx_{\mathrm{bst}} K$.

Proof. Let $K = (V, S)$ be a shellable combinatorial n-ball. We perform induction on the number $r \in \mathbb{N}$ of n-simplices in K.

[32] This might seem like came completely out of the blue, but remember that an abstract simplicial complex simplicially isomorphic to S_{m-1} is just "S_{m-1} with its vertices renamed".

The case $r = 1$ is rather straightforward: If we denote this single n-simplex by s, then we make the following claim:

Claim. In this case, we have $(V,S) = (s, \mathscr{P}(s) \setminus \{\varnothing\})$.

Proof. The inclusion "$\supseteq$" follows from $s \in S$ and property (2) of abstract simplicial complexes. For the other inclusion, assume there was a vertex $v \in V \setminus s$. Then it follows from the definition of a combinatorial manifold[33] that $\mathrm{St}(K,v) = \{v\} * \mathrm{Lk}(K,v)$ is a combinatorial n-ball, and thus contains an n-simplex. But the only n-simplex of K is s, so since by assumption $v \notin s$, we have that s must be an n-simplex in the combinatorial $(n-1)$-ball $\mathrm{Lk}(K,v)$, which is clearly a contradiction.

Since K is just given by s and its faces, we have that $\mathrm{Lk}(K,s) = (\varnothing,\varnothing)$ is a non-trivial (-1)-sphere in K, so we can perform

$$\tau_s K = \underbrace{K \setminus \mathring{\mathrm{St}}(K,s)}_{=(\varnothing,\varnothing)} \cup \partial s * \underbrace{B(K,s)}_{=\underline{s}} = \underline{s} * \partial s \cong_{\mathrm{si}} \mathrm{Cone}(\partial K).$$

Now assume that $r \geq 2$, and that the claim is true for shellable combinatorial n-balls of dimension less than r. If we denote by $K_1 := \mathrm{sh}_t K$ the result of the first elementary shelling in a sequence $K \xrightarrow{\mathrm{sh}} D_n$, where $s * t$ is an n-simplex of K that fulfills $s \cap \partial K = \partial s$ and $t * \partial s \subseteq \partial K$. We also denote by k the dimension of t. By construction, K_1 has $r-1$ simplices of dimension n, so it follows from the induction hypothesis that $K_1 \approx_{\mathrm{bst}} \mathrm{Cone}(\partial K_1)$. We use this to rewrite

$$K = K_1 \cup s * t \approx_{\mathrm{bst}} \mathrm{Cone}(\partial K_1) \cup s * t.$$

Now, we have the following claim:

Claim. The link of s in this new combinatorial manifold is given by

$$\mathrm{Lk}\left(\mathrm{Cone}(\partial K_1) \cup s * t, s\right) = \partial \mathrm{Cone}(t),$$

and it is a non-trivial k-sphere in $\mathrm{Cone}(\partial K_1) \cup s * t$.

Proof. The second claim follows immediately from the first one since $\mathrm{Cone}(t) \cong_{\mathrm{si}} D_{k+1}$. For the first claim, we perform the following calculation:

$$\partial \mathrm{Cone}(t) = t \cup \mathrm{Cone}(\partial t) = t \cup \mathrm{Cone}\big(\mathrm{Lk}(\partial K_1, s)\big) = \mathrm{Lk}\left(\mathrm{Cone}(\partial K_1) \cup s * t, s\right)$$

The reader is invited to draw some pictures in order to verify that chain of equations.

Thanks to the claim, we can now perform[34]

$$\begin{aligned}
\tau_s\big(\mathrm{Cone}(\partial K_1) \cup s * t\big) &= \big(\mathrm{Cone}(\partial K_1) \cup s * t\big) \setminus \mathring{\mathrm{St}}\big(\mathrm{Cone}(\partial K_1) \cup s * t, s\big) \cup \partial s * \mathrm{Cone}(t) = \\
&= \big(\mathrm{Cone}(\partial K_1)\big) \setminus \mathring{\mathrm{St}}\big(\mathrm{Cone}(\partial K_1), \mathrm{Cone}(s)\big) \cup \mathrm{Cone}(t * \partial s) = \\
&= \mathrm{Cone}\big(\partial K_1 \setminus \mathring{\mathrm{St}}(\partial K_1, s) \cup t * \partial s\big).
\end{aligned}$$

[33] Recall that combinatorial balls are combinatorial manifolds by Lemma 6.

[34] The dual simplex of s in $\mathrm{Cone}(\partial K_1) \cup s * t$ is given by $\mathrm{Cone}(t)$.

In the last step, we used commutativity of the join and the fact that $s * t$ is entirely part of $\mathrm{St}\big(\mathrm{Cone}(\partial K_1) \cup s * t, s\big)$. Now, we use Lemma 17 to calculate the boundary of K_1 as follows:

$$\partial K_1 = \tau_t \partial K = \partial K \setminus \mathring{\mathrm{St}}(\partial K, t * \partial s) \cup s * \partial t.$$

This allows us to simplify the above expression.

$$\begin{aligned}
K &\approx_{\mathrm{bst}} \mathrm{Cone}(\partial K_1) \cup s * t \approx_{\mathrm{bst}} \\
&\approx_{\mathrm{bst}} \mathrm{Cone}\big(\partial K_1 \setminus \mathring{\mathrm{St}}(\partial K_1, s)\big) \cup \mathrm{Cone}(t * \partial s) = \\
&= \mathrm{Cone}\Big(\big(\partial K \setminus \mathring{\mathrm{St}}(\partial K, t * \partial s) \cup s * \partial t\big) \setminus \mathring{\mathrm{St}}(\partial K, s * \partial t)\Big) \cup \mathrm{Cone}(t * \partial s) = \\
&= \mathrm{Cone}\big(\partial K \setminus \mathring{\mathrm{St}}(\partial K, t * \partial s)\big) \cup \mathrm{Cone}(t * \partial s) = \\
&= \mathrm{Cone}\big(\partial K \setminus \mathring{\mathrm{St}}(\partial K, t * \partial s) \cup (t * \partial s)\big) = \\
&= \mathrm{Cone}(\partial K).
\end{aligned}$$

This concludes the induction step and the proof.

The induction step actually has a somewhat nice picture to back it up, see below. We remove a simplex, turn the remaining complex into a cone on its boundary, and are just one bistellar move away from a cone on the boundary of the original complex.

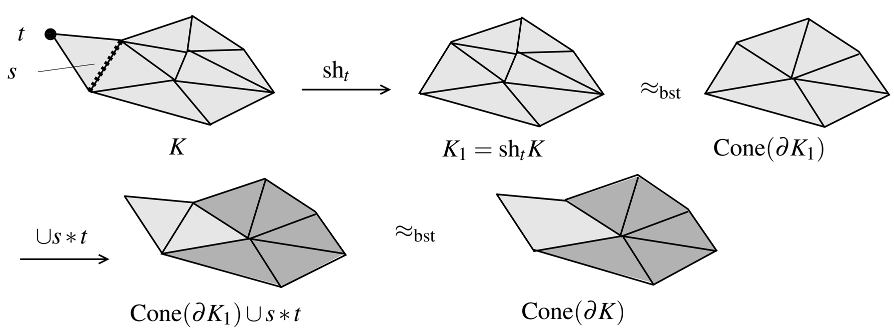


Fig. 20: The induction step in the proof of Lemma 20

The next corollary is beautiful not only because it is the first time we are able to connect stellar moves, bistellar moves, and shellability in a meaningful yet concise statement, but also because it will play a crucial role in proving Pachner's theorem.

Corollary 4. *Let $K = (V, S)$ be a combinatorial n-manifold, and let $s \in S$ be a k-simplex with $s \notin \partial K$. If* $\mathrm{Lk}(K, s)$ *is shellable, then $K \approx_{\mathrm{bst}} \sigma_s K$.*

Both the statement and the proof of Corollary 4 can be found in [Lic99, Corollary 5.8], but as per usual, we have attempted to fill in the details.

Proof. Since $s \notin \partial K$, its link $\mathrm{Lk}(K,s)$ is a shellable combinatorial $(n-k-1)$-sphere. We can apply Lemma 18 $(k+1)$-times to conclude that

$$\mathrm{St}(K,s) = s * \mathrm{Lk}(K,s) \cong_{\mathrm{si}} \underbrace{D_0 * \cdots * D_0}_{(k+1)\text{-times}} * \mathrm{Lk}(K,s)$$

is a shellable combinatorial n-ball. Thus, it follows from Lemma 20 that

$$\mathrm{St}(K,s) \approx_{\mathrm{bst}} \mathrm{Cone}\big(\partial \mathrm{St}(K,s)\big).$$

Since s is a combinatorial k-ball and $\mathrm{Lk}(K,s)$ is a combinatorial $(n-k-1)$-sphere, it follows from Proposition 2 (2) that we have $\partial \mathrm{St}(K,s) = \partial s * \mathrm{Lk}(K,s)$. Using this, we calculate

$$\begin{aligned}
\sigma_s K &= K \setminus \mathring{\mathrm{St}}(K,s) \cup \underline{s} * \partial s * \mathrm{Lk}(K,s) \cong_{\mathrm{si}} \\
&\cong_{\mathrm{si}} K \setminus \mathring{\mathrm{St}}(K,s) \cup \mathrm{Cone}\big(\partial s * \mathrm{Lk}(K,s)\big) = \\
&= K \setminus \mathring{\mathrm{St}}(K,s) \cup \mathrm{Cone}\big(\partial \mathrm{St}(K,s)\big) \approx_{\mathrm{bst}} \\
&\approx_{\mathrm{bst}} K \setminus \mathring{\mathrm{St}}(K,s) \cup \mathrm{St}(K,s) = \\
&= K.
\end{aligned}$$

Since any simplicial isomorphism is also a bistellar equivalence, this finishes the proof.

Apart from the fact that they will prove to be extremely useful in proving Pachner's theorem, reading and proving the statements in this chapter might have had the added benefit of making the reader grow used to bistellar moves.

7 Proof of Pachner's theorem

This penultimate chapter usues a lot of the results from the previous chapters in order to prove Pachner's theorem. For the reader's convenience, we shall quickly restate it in all its glory.

Theorem 7. (Pachner's Theorem) *Let $K = (V,S)$ and $K' = (V',S')$ be closed combinatorial n-manifolds. Then*

$$K \approx_{\mathrm{bst}} K' \iff K \approx_{\mathrm{st}} K'.$$

The proof of Pachner's theorem closely follows the one given in [Lic99, Theorem 5.9], which itself follows the original proof by Pachner (cf. [Pac91, Theorem 5.5]). However, Lickorish and Pachner make use of a concept called "stellar exchanges" that is essentially a generalisation of both stellar and bistellar moves. We stick to the (bi)stellar moves that we are familiar with in order to minimise confusion, but the interested reader can find the proof using stellar exchanges in the

aforementioned papers.
Another thing to note is that the proof uses a lot of notation and looks rather involved, but in the end boils down to the effects of stellar moves, Corollary 4, Lemma 15, and a glorified exercise in mathematical induction. For the reader just trying to skim over the proof, there are diagrams scattered throughout that attempt to capture a lot of what happens.

Proof. Let $K=(V,S)$ and $K'=(V',S')$ be closed combinatorial n-manifolds.
We start with the significantly easier direction. Assume that $K \approx_{\mathrm{bst}} K'$. Recall that by Lemma 16 (5), any bistellar move in a sequence $K \approx_{\mathrm{bst}} K'$ is (potentially up to a simplicial isomorphism) the composition of two stellar moves. Therefore, we have $K \approx_{\mathrm{st}} K'$, and applying the Alexander-Newman Theorem concludes the proof of this implication.
For the converse, assume that $|K| \cong_{\mathrm{PL}} |K'|$, or equivalently that $K \approx_{\mathrm{st}} K'$. Due to the transitivity of bistellar equivalence, it suffices to prove the claim in the case that

$$K' = \sigma_s K = K \setminus \mathring{\mathrm{St}}(K,s) \cup \underline{s} * \partial s * \mathrm{Lk}(K,s)$$

for some k-simplex $s \in S$. For the remainder of the proof, we also use the notation $L := \mathrm{Lk}(K,s)$. As a first step, we split up L into

$$L = L' * \Sigma, \text{ where } \Sigma \cong_{\mathrm{si}} S_{m_1} * \cdots * S_{m_p}, \text{ for suitable values of } p, m_1, \ldots, m_p \in \mathbb{N}_0.$$

Essentially, we are splitting off a part that is simplicially isomorphic to a join of spheres, but we do allow for Σ to be empty (in that case, $p=0$). Note that since K is closed, the link L is a combinatorial $(n-k-1)$-sphere by Proposition 1 (1).
By applying Lemma 15 p-times, we obtain that L' is also a combinatorial sphere, the dimension of which we denote by[35]

$$m := \dim L' \in \mathbb{N}_0 \cup \{-1\}.$$

Furthermore, since L' is a combinatorial sphere, we can denote the minimum number of stellar moves in a sequence $L' \approx_{\mathrm{st}} S_m$ by $r \in \mathbb{N}_0$.
We perform induction on m. The case $m=-1$ is non-trivial already: If $\dim L' = -1$, then L' is the empty abstract simplicial complex, and

$$L = \Sigma \cong_{\mathrm{si}} S_{m_1} * \cdots * S_{m_p}.$$

Since the sphere S_{m_1} is a shellable combinatorial sphere, it follows from $(p-1)$-fold application of Lemma 19 that L is a shellable combinatorial $(n-k-1)$-sphere. Therefore, since $\partial s \cong_{\mathrm{si}} S_{k-1}$, we obtain from applying Lemma 19 again that $\mathrm{Lk}(K', \underline{s}) = \partial s * \mathrm{Lk}(K,s)$ is a shellable combinatorial $(n-1)$-sphere. Since K is closed, we can apply Corollary 4 in order to conclude that

$$K \approx_{\mathrm{bst}} \sigma_s K = K'.$$

---

[35] Recall that we had defined the dimension of the empty abstract simplicial complex as -1.

The case $m = 0$ is also of importance: If L' is a combinatorial 0-sphere, then it is simplicially isomorphic to S_0 since stellar moves along 0-simplices are just simplicial isomorphisms. But then L' can be absorbed into Σ, allowing us to reduce this case to the case that $m = -1$.
Now let $m \in \mathbb{N}$, and assume the claim is true for stellar moves along simplices whose links have a part "outside of Σ" of dimension less than m.
We shall now also perform induction on r. The case $r = 0$ is rather straightforward: If we have $L' \cong_{\text{si}} S_m$, then L' can again be absorbed into Σ, which also puts us back to the case that $m = -1$.
Now let $r \in \mathbb{N}$, and assume the claim is true for stellar moves along simplices whose links have a part "outside of Σ" that are stellar equivalent to S_m in less than r stellar moves.
Idea for the proof: *Cleverly subdivide both K and K' and use the induction hypotheses in order to find bistellar equivalent combinatorial manifolds.*
As we have done before, we distinguish two cases:

Case 1: The first of the r moves in $L' \approx_{\text{st}} S_m$ is a stellar subdivision σ_t for some simplex t of L'. We denote its link in L' by $L'' := \mathrm{Lk}(L', t)$. We consider the following combinatorial n-manifolds: Since t is a simplex in $\mathrm{Lk}(K, s)$, the simplex $s * t$ lies in K, and we can define

$$K_1 := \sigma_{s*t} K.$$

Furthermore, we use the fact that s is a simplex of

$$K_1 = K \setminus \mathring{\mathrm{St}}(K, s*t) \cup \underline{s*t} * \underbrace{\partial(s*t)}_{=\partial s * t \cup s * \partial t} * \mathrm{Lk}(K, s*t)$$

in order to be able to define

$$K_2 := \sigma_s K_1 = \sigma_s(\sigma_{s*t} K).$$

We are now interested in certain links that allow us to actually use the induction hypotheses. First, we calculate

$$\mathrm{St}(K, s) = s * L = s * L' * \Sigma = s * t * L'' * \Sigma,$$

and therefore have $\mathrm{Lk}(K, s*t) = L'' * \Sigma$. But since

$$m = \dim L' \geq \dim(L'' * t) = \dim L'' + \dim t + 1 > \dim L'',$$

we can conclude from the induction hypothesis on m that $K \approx_{\text{bst}} K_1$.
Now we use the above description of K_1 in order to calculate

$$\mathrm{St}(K_1, s) = \underline{s*t} * s * \partial t * \underbrace{\mathrm{Lk}(K, s*t)}_{=L'' * \Sigma} = s * \Sigma * \underbrace{\underline{s*t} * \partial t * L''}_{=\sigma_t L'} = s * \Sigma * \sigma_t L',$$

which yields us $\mathrm{Lk}(K_1, s) = \Sigma * \sigma_t L'$. However, by construction, $\sigma_t L' \approx_{\text{st}} S_m$ in $r - 1$ stellar moves, thus it follows from the induction hypothesis on r that

$K_1 \approx_{\mathrm{bst}} K_2$.

Note that $\underline{s} * t$ is a simplex of K' because t is a simplex in the link $\mathrm{Lk}(K,s)$. Therefore, it makes sense to consider the following claim:

Claim. We have $\sigma_{\underline{s}*t} K' = K_2$.

Proof. It suffices to calculate the effects of[36] $\sigma_{\underline{s}*t}\sigma_s$ and $\sigma_s\sigma_{s*t}$ on $\mathrm{St}(K, s*t)$ because on $K \setminus \mathring{\mathrm{St}}(K, s*t)$, they certainly produce identical results. Using the results from above, we calculate[37]

$$\begin{aligned}
\sigma_{\underline{s}*t}\big(\sigma_s\big(\mathrm{St}(K,s*t)\big)\big) &= \sigma_{\underline{s}*t}\big(\sigma_s(s*t*\Sigma*L'')\big) = \\
&= \sigma_{\underline{s}*t}\big(\underline{s}*\partial s*t*\Sigma*L''\big) = \\
&= \underline{s*t}*\partial(\underline{s}*t)*\partial s*\Sigma*L'' = \\
&= \underline{s*t}*(t\cup \underline{s}*\partial t)*\partial s*\Sigma*L'',
\end{aligned}$$

and

$$\begin{aligned}
\sigma_s\big(\sigma_{s*t}\big(\mathrm{St}(K,s*t)\big)\big) &= \sigma_s\big(\sigma_{s*t}(s*t*\Sigma*L'')\big) = \\
&= \sigma_s\big(\underline{s*t}*\partial(s*t)*\Sigma*L''\big) = \\
&= \sigma_s\big(\underline{s*t}*(\partial s*t\cup s*\partial t)*\Sigma*L''\big) = \\
&= \underline{s*t}*\partial s*t*\Sigma*L''\cup \underline{s*t}*\underline{s}*\partial s*\partial t*\Sigma*L'' = \\
&= \underline{s*t}*\partial s*\Sigma*L''*(t\cup \underline{s}*\partial t),
\end{aligned}$$

which are identical by commutativity of the join operation.

There is an attempt at sketching the star of $s*t$ below, for disappointingly low dimensions of s, t, and Σ, the last of which is just a copy of S_0.

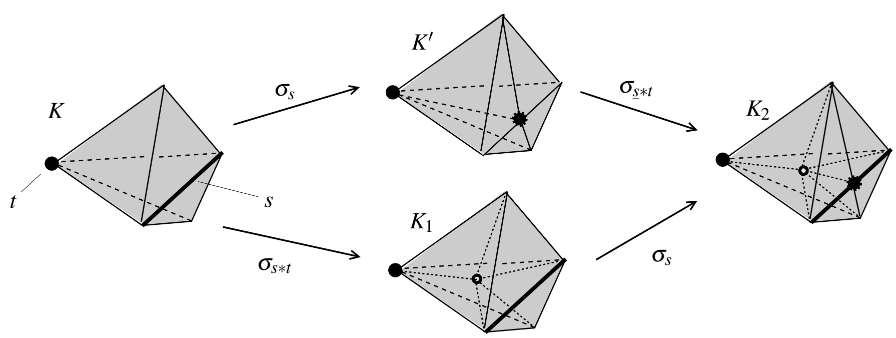


Fig. 21: Two ways to get from K to K_2 using stellar moves

[36] We care about these pairs of stellar subdivisions since they were used to define K' and K_2.

[37] These two calculations essentially utilise the description of stellar moves, and are probably easier to understand by looking at Figure 21.

The following diagram illustrates the situation. Note that the arrows are not maps, but instead indicate stellar subdivisions.

$$\begin{tikzcd} K \arrow[r, "\sigma_{s*t}"] \arrow[d, "\sigma_s"] & K_1 \arrow[d, "\sigma_s"] \\ K' \arrow[r, "\sigma_{\underline{s}*t}"] & K_2. \end{tikzcd}$$

We have already seen that the stellar subdivisions to the top and right induce bistellar equivalences. Our plan is now to show that the bottom horizontal stellar subdivision induces a bistellar equivalence, because then transitivity yields us a bistellar equivalence between K and K'.
Using the definition of $K' = \sigma_s K$, we calculate

$$\mathrm{St}(K', \underline{s} * t) = \underline{s} * \partial s * L = \underline{s} * \partial s * \Sigma * L' = \underline{s} * \partial s * \Sigma * t * L''.$$

Therefore, we have $\mathrm{Lk}(K', \underline{s} * t) = \partial s * \Sigma * L''$, and after absorbing the sphere ∂s into Σ, we can use the claim as well as our hypothesis on m to obtain that

$$K' \approx_{\mathrm{bst}} \sigma_{\underline{s}*t} K' = K_2.$$

But now we have shown the three bistellar equivalences

$$K \approx_{\mathrm{bst}} K_1 \approx_{\mathrm{bst}} K_2 \approx_{\mathrm{bst}} K',$$

so we have $K \approx_{\mathrm{bst}} K'$ by transitivity.

Case 2: The first of the r moves in $L' \approx_{\mathrm{st}} S_m$ is a stellar weld σ_t^{-1} for some simplex t of $\sigma_t^{-1} L'$, and we have[38]

$$\mathrm{Lk}(L', \underline{t}) = \partial t * \mathrm{Lk}(\sigma_t^{-1} L', t).$$

In Case 2, we denote that last link by $L'' := \mathrm{Lk}(\sigma_t^{-1} L', t)$. Because in this case we are dealing with a stellar weld, we have to distinguish another two cases:

Case 2a: The simplex t does not lie in K. We set $K_1 := \sigma_t^{-1} \sigma_{s*\underline{t}} K$ and $K_2 := \sigma_s K_1$. This is possible since $\underline{t}$ is a vertex in L' and hence also in $\mathrm{Lk}(K, s)$. Since for the star of s in K we have

$$\mathrm{St}(K, s) = s * L = s * L' * \Sigma = s * \underline{t} * \mathrm{Lk}(L', \underline{t}) * \Sigma = s * \underline{t} * \partial t * L'' * \Sigma,$$

we obtain $\mathrm{Lk}(K, s * \underline{t}) = \partial t * L'' * \Sigma = L'' * \Sigma'$, for $\Sigma' = \partial t * \Sigma$ still a link of spheres. Furthermore, we again have

$$m = \dim L' \geq \dim(L'' * \underline{t}) = \dim L'' + 0 + 1 > \dim L'',$$

so we can use the induction hypothesis on m in order to obtain that

[38] This is an immediate consequence of the definition of stellar moves.

$$K \approx_{\mathrm{bst}} \sigma_{s*\underline{t}} K = \sigma_t K_1.$$

But $\mathrm{St}(K_1,t) = \partial(s*\underline{t})*t*\Sigma*L''$, so $\mathrm{Lk}(K_1,t) = \partial(s*\underline{t})*\Sigma*L''$, and we can absorb the sphere $\partial(s*\underline{t})$ into Σ in order to obtain from our induction hypothesis on m that $K_1 \approx_{\mathrm{bst}} \sigma_t K_1$.
Similarly, we consider

$$\begin{aligned}\mathrm{St}(K_1,s) = \sigma_t^{-1}\big(\underline{s*t}*\partial(s*\underline{t})*\mathrm{Lk}(K,s*\underline{t})\big) = \\ = s*\sigma_t^{-1}L'*\Sigma \cup \sigma_t^{-1}s*\sigma_t^{-1}L'*\Sigma,\end{aligned}$$

and therefore

$$\mathrm{Lk}(K_1,s) = \sigma_t^{-1}L'*\Sigma.$$

Note that $\sigma_t^{-1}L' \approx_{\mathrm{st}} S_m$ in $r-1$ stellar moves, hence our induction hypothesis on r yields $K_1 \approx_{\mathrm{bst}} K_2$.
Recall that since $\underline{t}$ is a vertex in $\mathrm{Lk}(K,s) = \mathrm{Lk}(K',\underline{s}*\partial s)$, the 1-simplex $\underline{s}*\underline{t}$ lies in K'. This allows us to make the following claim:

Claim. We have $\sigma_t^{-1}\big(\sigma_{\underline{s}*\underline{t}}K'\big) = K_2$.

Proof. It suffices to calculate the effects of both $\sigma_t^{-1}\sigma_{\underline{s}*\underline{t}}\sigma_s$ and $\sigma_s\sigma_t^{-1}\sigma_{s*\underline{t}}$ on $\mathrm{St}(K,s*\underline{t})$ because on $K \setminus \mathring{\mathrm{St}}(K,s*\underline{t})$, they certainly produce identical results. Using the results from above, we calculate

$$\begin{aligned}\sigma_t^{-1}\sigma_{\underline{s}*\underline{t}}\sigma_s\big(\mathrm{St}(K,s*\underline{t})\big) &= \sigma_t^{-1}\sigma_{\underline{s}*\underline{t}}\sigma_s\big(s*\underline{t}*\partial t*\Sigma*L''\big) = \\ &= \sigma_t^{-1}\sigma_{\underline{s}*\underline{t}}\big(\underline{s}*\partial s*\underline{t}*\partial t*\Sigma*L''\big) = \\ &= \sigma_t^{-1}\big(\underline{s*t}*\partial(\underline{s}*\underline{t})*\partial s*\partial t*\Sigma*L''\big) = \\ &= (\underline{s}\cup\underline{t})*\partial s*t*\Sigma*L'',\end{aligned}$$

and

$$\begin{aligned}\sigma_s\sigma_t^{-1}\sigma_{s*\underline{t}}\big(\mathrm{St}(K,s*\underline{t})\big) &= \sigma_s\sigma_t^{-1}\sigma_{s*\underline{t}}\big(s*\underline{t}*\partial t*\Sigma*L''\big) = \\ &= \sigma_s\sigma_t^{-1}\big(\underline{s*t}*\partial(s*\underline{t})*\partial t*\Sigma*L''\big) = \\ &= \sigma_s\sigma_t^{-1}\big(\underline{s*t}*(\partial s*\underline{t}\cup s)*\partial t*\Sigma*L''\big) = \\ &= \sigma_s\big((\partial s*\underline{t}\cup s)*t*\Sigma*L''\big) = \\ &= (\partial s*\underline{t}\cup\underline{s}*\partial s)*t*\Sigma*L'' = \\ &= (\underline{s}\cup\underline{t})*\partial s*t*\Sigma*L'',\end{aligned}$$

which are identical.

The following diagram illustrates the situation. Note that again, the arrows are not maps, but instead indicate stellar subdivisions.

$$\begin{tikzcd}
K \arrow[r, "\sigma_{s*\underline{t}}"] \arrow[d, "\sigma_s"] & \sigma_t K_1 & K_1 \arrow[l, "\sigma_t"'] \arrow[d, "\sigma_s"] \\
K' \arrow[r, "\sigma_{\underline{s}*\underline{t}}"] & \sigma_t K_2 & K_2. \arrow[l, "\sigma_t"']
\end{tikzcd}$$

We have already seen that the two top horizontal stellar subdivisions, as well as the one to right, all induce bistellar equivalences. Our plan is now to show that the two bottom horizontal stellar subdivisions induce bistellar equivalences, because then transitivity yields us a bistellar equivalence between K and K'.

As we had already seen in the proof of the claim, we have

$$\mathrm{St}(K', \underline{s}*\underline{t}) = \underline{s}*\partial s*\underline{t}*\partial t*\Sigma*L'',$$

and therefore

$$\mathrm{Lk}(K', \underline{s}*\underline{t}) = \partial s*\partial t*\Sigma*L''.$$

After absorbing the spheres ∂s and ∂t into Σ, we can use the claim as well as our hypothesis on m to obtain that

$$K' \approx_{\mathrm{bst}} \left(\sigma_{\underline{s}*\underline{t}} K'\right) = \sigma_t K_2.$$

So it remains to show that $\sigma_t K_2 \approx_{\mathrm{bst}} K_2$. But this follows from the above result that

$$\mathrm{St}(K_2, t) = (\underline{s}\cup\underline{t})*\partial s*t*\Sigma*L'',$$

hence

$$\mathrm{Lk}(K_2, t) = (\underline{s}\cup\underline{t})*\partial s*\Sigma*L''.$$

Again, we absorb the spheres ∂s and $\underline{s}\cup\underline{t}$ into Σ, and can use the induction hypothesis on m to obtain $K_2 \approx_{\mathrm{bst}} \sigma_t K_2$.

Now we have shown the five stellar equivalences

$$K \approx_{\mathrm{bst}} \sigma_t K_1 \approx_{\mathrm{bst}} K_1 \approx_{\mathrm{bst}} K_2 \approx_{\mathrm{bst}} \sigma_t K_2 \approx_{\mathrm{bst}} K',$$

so we have $K \approx_{\mathrm{bst}} K'$ by transitivity.

Case 2b: If the simplex t happens to lie in K, we reduce the situation to Case 2a as follows: If t is a 0-simplex, then we can just use an alternative new vertex v_0 instead, and obtain the result from Case 2a up to a simplicial isomorphism. If $\dim t \geq 1$, we pick a vertex $u \in t$ and write $t' := t \setminus \{u\}$ for its opposite face in t. Now choose a vertex v that is not in K. Let φ be a simplicial isomorphism that sends the vertex $\underline{s*u}$ to v. We now define a series of combinatorial manifolds as follows:

$$\hat{K} := \varphi\sigma_{s*u}K, \qquad \hat{K}' := \sigma_s\hat{K}, \qquad K' := \sigma_s K.$$

Claim. We have $\varphi\sigma_{\underline{s}*u}K' = \hat{K}'$.

Proof. As in the previous cases, we calculate how $\varphi\sigma_{\underline{s}*u}\sigma_s$ and $\sigma_s\varphi\sigma_{s*u}$ affect $\mathrm{St}(K,s*t)$, because both clearly do the same on $K\setminus\mathring{\mathrm{S}}\mathrm{t}(K,s*t)$.

$$\begin{aligned}\varphi\sigma_{\underline{s}*u}\sigma_s\big(\mathrm{St}(K,s*t)\big) &= \varphi\sigma_{\underline{s}*u}\sigma_s\big(s*u*t'*L''*\Sigma\big) = \\ &= \varphi\sigma_{\underline{s}*u}\big(\underline{s}*\partial s*u*t'*L''*\Sigma\big) = \\ &= v*\partial(\underline{s}*u)*\partial s*t'*L''*\Sigma = \\ &= v*(u\cup\underline{s})*\partial s*t'*L''*\Sigma,\end{aligned}$$

and

$$\begin{aligned}\sigma_s\varphi\sigma_{s*u}\big(\mathrm{St}(K,s*t)\big) &= \sigma_s\varphi\sigma_{s*u}\big(s*u*t'*L''*\Sigma\big) = \\ &= \sigma_s\big(v*\partial(s*u)*t'*L''*\Sigma\big) = \\ &= \sigma_s\big(v*(\partial s*u\cup s)*t'*L''*\Sigma\big) = \\ &= v*(\partial s*u\cup\underline{s}*\partial s)*t'*L''*\Sigma = \\ &= v*(u\cup\underline{s})*\partial s*t'*L''*\Sigma,\end{aligned}$$

which are clearly identical.

The situation is now as follows:

$$\begin{array}{ccc} K & \xrightarrow{\varphi\sigma_{s*u}} & \hat{K} \\ \Big\downarrow{\scriptstyle \sigma_s} & & \Big\downarrow{\scriptstyle \sigma_s} \\ K' & \xrightarrow{\varphi\sigma_{\underline{s}*u}} & \hat{K}' \end{array}$$

In order to show that there is a bistellar equivalence between the combinatorial manifolds on the left-hand side, we show that the three stellar subdivisions at the bottom, top, and to the right induce bistellar equivalences between the respective combinatorial manifolds. First, we use that the link of s in $\hat{K}$ is given by[39]

$$\mathrm{Lk}(\hat{K},s) = v*t'*L''*\Sigma = \varphi(t)*L''*\Sigma = \varphi(L')*\Sigma,$$

and that $\varphi(L')$ is just another copy of L', but with one vertex renamed. Therefore, we can use the calculation from Case 2a in order to obtain that $\hat{K}\approx_{\mathrm{bst}}\hat{K}'$.

For the top horizontal map, we use that

$$\mathrm{Lk}(K,s*u) = t'*L''*\Sigma = \mathrm{Lk}(L',u)*\Sigma,$$

and since L' is a combinatorial m-sphere, we have

$$\dim\mathrm{Lk}(L',u) = m-1 < m,$$

[39] As before, we can read it off in the third line of the calculation of $\sigma_s\varphi\sigma_{s*u}\big(\mathrm{St}(K,s*t)\big)$.

so it follows from the induction hypothesis on m that $K \approx_{\text{bst}} \hat{K}$.
The argument that the bottom horizontal map induces a bistellar equivalence $K' \approx_{\text{bst}} \hat{K}'$ is essentially the same. Therefore, we can again conclude that

$$K \approx_{\text{bst}} \hat{K} \approx_{\text{bst}} \hat{K}' \approx_{\text{bst}} K'.$$

In all cases, we have now shown the induction step, therefore Pachner's theorem follows by induction.

Remark 8. One of the downsides of the proof is that regrettably, drawing sketches that are both meaningful and non-trivial is pretty much impossible. This can be seen in the fact that at times in the proof, we had to consider joins of up to 5 simplicial complexes, which, according to the dimension formula from Lemma 1, automatically puts us at a dimension of at least 4, and significantly more if we ask for some of the links or simplices to be of higher dimensions. So unless the reader has an easy time visualising those dimensions, the beauty has to be found in the way everything ultimately ends up falling into place, as well as sketches of tiny parts of the complexes involved, similar to the way we did in Figure 21.

For the sake of completeness, we also state another result that could be described as "Pachner for manifolds with boundary". For this, we need a notion of connectedness for abstract simplicial complexes. Luckily, connectedness for abstract simplicial complexes is a rather straightforward concept to define.

Definition 18. Let $K = (V,S)$ be an abstract simplicial complex. We call K connected if for any two vertices $u, v \in V$, there exists a sequence $w_1, \dots, w_k \in V$ with $w_1 = u$, $w_k = v$, and the simplex $\{w_i, w_{i+1}\} \in S$ for all $i \in \{1, \dots, k-1\}$.

Theorem 8. (Pachner for Bounded Manifolds) *Let K and K' be connected combinatorial n-manifolds with non-empty boundary. Then $|K| \cong_{\text{PL}} |K'|$ if and only if they are related by a sequence of elementary shellings, inverse shellings*[40]*, and simplicial isomorphisms.*

Proof. Proofs for this Theorem can be found in [Lic99, Theorem 5.10], or alternatively in [Pac91, Theorem 6.3].

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[40] An *inverse shelling* is the opposite operation of an elementary shelling, that is, if $K' = \text{sh}_t K$ is an elementary shelling, then $K = \text{sh}_t^{-1} K'$ is an inverse shelling.

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