Treewidth, Circle Graphs and Circular Drawings

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Abstract A circle graph is an intersection graph of a set of chords of a circle. We describe the unavoidable induced subgraphs of circle graphs with large treewidth. This includes examples that are far from the 'usual suspects'. Our results imply that treewidth and Hadwiger number are linearly tied on the class of circle graphs, and that the unavoidable induced subgraphs of a vertex-minor-closed class with large treewidth are the usual suspects if and only if the class has bounded rank-width. Using the same tools, we also study the treewidth of graphs G that have a circular drawing whose crossing graph is well-behaved in some way. In this setting, we show that if the crossing graph is K_t-minor-free, then G has treewidth at most $12t-23$ and has no $K_{2,4t}$-topological minor. On the other hand, we show that there are graphs with arbitrarily large Hadwiger number that have circular drawings whose crossing graphs are 2-degenerate.

1 Introduction

This paper studies the treewidth of graphs that are defined by circular drawings. Treewidth is the standard measure of how similar a graph is to a tree, and is of fundamental importance in structural and algorithmic graph theory. The motivation for

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this study is two-fold. See Section 2 for definitions omitted from this introduction. In this extended abstract, most proofs are omitted; see [28] for all the details.

1.1 Theme #1: Circle Graphs

A *circle graph* is the intersection graph of a set of chords of a circle. Circle graphs form a widely studied graph class [15, 17, 23, 25, 26, 31, 33] and there have been several recent breakthroughs concerning them. In the study of graph colourings, Davies and McCarty [17] showed that circle graphs are quadratically χ-bounded improving upon a previous longstanding exponential upper bound. Davies [15] further improved this bound to $\chi(G) \in O(\omega(G)\log\omega(G))$, which is best possible. Circle graphs are also fundamental to the study of vertex-minors and are conjectured to lie at the heart of a global structure theorem for vertex-minor-closed graph classes (see [36]). To this end, Geelen, Kwon, McCarty, and Wollan [26] recently proved an analogous result to the excluded grid minor theorem for vertex-minors using circle graphs. In particular, they showed that a vertex-minor-closed graph class has bounded rankwidth if and only if it excludes a circle graph as a vertex-minor. For further motivation and background on circle graphs, see [16, 36].

Our first contribution determines when a circle graph has large treewidth.

Theorem 1. *Let $t \in \mathbb{N}$ and let G be a circle graph with treewidth at least $12t+2$. Then G contains an induced subgraph H that consists of t vertex-disjoint cycles $(C_1,\dots,C_t)$ such that for all $i < j$ every vertex of C_i has at least two neighbours in C_j. Moreover, every vertex of G has at most four neighbours in any C_i $(1 \leqslant i \leqslant t)$.*

Observe that in Theorem 1 the subgraph H has a K_t-minor obtained by contracting each of the cycles C_i to a single vertex, implying that H has treewidth at least $t-1$. Moreover, since circle graphs are closed under taking induced subgraphs, H is also a circle graph. We now highlight several consequences of Theorem 1.

First, Theorem 1 describes the unavoidable induced subgraphs of circle graphs with large treewidth. Recently, there has been significant interest in understanding the induced subgraphs of graphs with large treewidth [2, 3, 4, 5, 6, 7, 8, 11, 34, 41, 46]. To date, most of the results in this area have focused on graph classes where the unavoidable induced subgraphs are the following graphs, the *usual suspects*: a complete graph K_t, a complete bipartite graph $K_{t,t}$, a subdivision of the $(t\times t)$-wall, or the line graph of a subdivision of the $(t\times t)$-wall (see [46] for definitions). Circle graphs do not contain subdivisions of large walls nor the line graphs of subdivisions of large walls and there are circle graphs of large treewidth that do not contain large complete graphs nor large complete bipartite graphs (see Theorem 17). To the best of our knowledge this is the first result to describe the unavoidable induced subgraphs of the large treewidth graphs in a natural hereditary class when they are not the usual suspects. Later we show that the unavoidable induced subgraphs of graphs with large treewidth in a vertex-minor-closed class $\mathscr{G}$ are the usual suspects if and only if $\mathscr{G}$ has bounded rankwidth (see Theorem 18).

Second, the subgraph H in Theorem 1 is an explicit witness to the large treewidth of G (with only a multiplicative loss). Circle graphs being χ-bounded says that circle graphs with large chromatic number must contain a large clique witnessing this. Theorem 1 can therefore be considered to be a treewidth analogue to the χ-boundedness of circle graphs.

Third, since the subgraph H has a K_t-minor, it follows that every circle graph contains a complete minor whose order is at least one twelfth of its treewidth. This is in stark contrast to the general setting where there are K_5-minor-free graphs with arbitrarily large treewidth (for example, grids). Theorem 1 also implies the following relationship between the treewidth, Hadwiger number and Hajós number of circle graphs (see Section 5)[1].

Theorem 2. *For the class of circle graphs, the treewidth and Hadwiger number are linearly tied. Moreover, the Hajós number is quadratically tied to both of them. Both 'linear' and 'quadratic' are best possible.*

1.2 Theme #2: Graph Drawing

The second thread of this paper aims to understand the relationship between circular drawings of graphs and their crossing graphs. A *circular drawing* (also called *convex* drawing) of a graph places the vertices on a circle with edges drawn as straight line segments. Circular drawings are well-studied by the graph drawing community. The *crossing graph* of a drawing D of a graph G has vertex-set $E(G)$ where two vertices are adjacent if the corresponding edges cross. Circle graphs are precisely the crossing graphs of circular drawings. If a graph has a circular drawing with a well-behaved crossing graph, must the graph itself also be well-behaved? Graphs that have a circular drawing with no crossings are exactly the outerplanar graphs, which have treewidth at most 2. Put another way, outerplanar graphs are those that have a circular drawing whose crossing graph is K_2-minor-free. Our next result extends this fact, relaxing 'K_2-minor-free' to 'K_t-minor-free'.

Theorem 3. *For every integer $t \geqslant 3$, if a graph G has a circular drawing where the crossing graph has no K_t-minor, then G has treewidth at most $12t-23$.*

Theorem 3 says that G having large treewidth is sufficient to force a complicated crossing graph in every circular drawing of G. A topological $K_{2,4t}$-minor also suffices.

Theorem 4. *If a graph G has a circular drawing where the crossing graph has no K_t-minor, then G contains no $K_{2,4t}$ as a topological minor.*

Outerplanar graphs are exactly those graphs that have treewidth at most 2 and exclude a topological $K_{2,3}$-minor. As such, Theorems 3 and 4 extends these struc-

[1] For a graph class $\mathscr{G}$, two graph parameters α and β are *tied on* $\mathscr{G}$ if there exists a function f such that $\alpha(G) \leqslant f(\beta(G))$ and $\beta(G) \leqslant f(\alpha(G))$ for every graph $G \in \mathscr{G}$. Moreover, α and β are *quadratically/linearly tied on* $\mathscr{G}$ if f may be taken to be quadratic/linear.

tural properties of outerplanar graphs to graphs with circular drawings whose crossing graphs are K_t-minor-free. We also prove a product structure theorem for such graphs, showing that every graph that has a circular drawing whose crossing graph has no K_t-minor is isomorphic to a subgraph of $H \boxtimes K_{O(t^3)}$ where $\text{tw}(H) \leqslant 2$ (see Corollary 11).

In the other direction, we consider sufficient conditions for a graph G to have a circular drawing whose crossing graph has no K_t-minor. By Theorems 3 and 4, G must have bounded treewidth and no $K_{2,4t}$-topological minor. While these conditions are necessary, we show that they are not sufficient, but that bounded treewidth with bounded maximum degree is; see Lemma 14 and Proposition 15 in Section 4.2 for details.

In addition, we show that the assumption in Theorem 3 that the crossing graph has bounded Hadwiger number cannot be weakened to bounded degeneracy. In particular, we construct graphs with arbitrarily large complete graph minors that have a circular drawing whose crossing graph is 2-degenerate (Theorem 16). This result has applications to the study of general (non-circular) graph drawings, and in particular, leads to the solution of an open problem asked by Hickingbotham and Wood [29].

Our proofs of Theorems 1, 2 and 3 are all based on the same core lemmas in Section 3. The results about circle graphs are in Section 5, while the results about graph drawings are in Section 4.

2 Preliminaries

2.1 Graph Basics

We use standard graph-theoretic definitions and notation; see [18].

For a tree T, a *T-decomposition* of a graph G is a collection $\mathscr{W} = (W_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of T such that (i) for every edge $vw \in E(G)$, there exists a node $x \in V(T)$ with $v, w \in W_x$; and (ii) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in W_x\}$ induces a (connected) subtree of T. Each set W_x in $\mathscr{W}$ is called a *bag*. The *width* of $\mathscr{W}$ is $\max\{|W_x| : x \in V(T)\} - 1$. A *tree-decomposition* is a T-decomposition for any tree T. The *treewidth* $\text{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G. When T is a path, $\mathscr{W}$ is a *path-decomposition*. The *pathwidth*, $\text{pw}(G)$, of G is the minimum width of a path-decomposition of G.

A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. The *Hadwiger number*, $h(G)$, of a graph G is the maximum integer t such that K_t is a minor of G.

A graph $\tilde{G}$ is a *subdivision* of a graph G if $\tilde{G}$ can be obtained from G by replacing each edge vw by a path P_{vw} with endpoints v and w (internally disjoint from the rest of $\tilde{G}$). A graph H is a *topological minor* of G if a subgraph of G is isomorphic to a subdivision of H. The *Hajós number*, $h'(G)$, of G is the maximum integer t such that K_t is a topological minor of G. A graph G is *H-topological minor-free* if H is not a topological minor of G.

It is well-known that for every graph G, $h'(G) \leqslant h(G) \leqslant \text{tw}(G)+1$.

A *graph class* is a collection of graphs closed under isomorphism. A *graph parameter* is a real-valued function α defined on all graphs such that $\alpha(G_1) = \alpha(G_2)$ whenever G_1 and G_2 are isomorphic.

2.2 Drawings of Graphs

A *drawing* of a graph G is a function ϕ that maps each vertex $v \in V(G)$ to a point $\phi(v) \in \mathbb{R}^2$ and maps each edge $e = vw \in E(G)$ to a non-self-intersecting curve $\phi(e)$ in $\mathbb{R}^2$ with endpoints $\phi(v)$ and $\phi(w)$, such that:

- $\phi(v) \neq \phi(w)$ for all distinct vertices v and w;
- $\phi(x) \notin \phi(e)$ for each edge $e = vw$ and each vertex $x \in V(G) \setminus \{v,w\}$;
- each pair of edges intersect at a finite number of points: $\phi(e) \cap \phi(f)$ is finite for all distinct edge e, f; and
- no three edges internally intersect at a common point: for distinct edges e, f, g the only possible element of $\phi(e) \cap \phi(f) \cap \phi(g)$ is $\phi(v)$ where v is a vertex incident to all of e, f, g.

A *crossing* of distinct edges $e = uv$ and $f = xy$ is a point in $(\phi(e) \cap \phi(f)) \setminus \{\phi(u), \phi(v), \phi(x), \phi(y)\}$; that is, an internal intersection point. A *plane graph* is a graph G equipped with a drawing of G with no crossings.

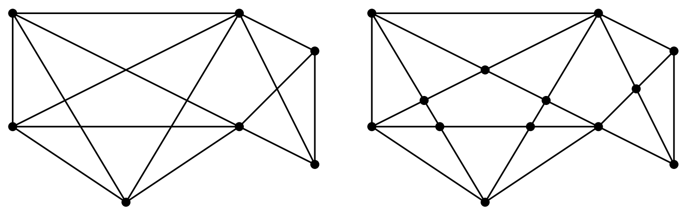
The *crossing graph* of a drawing D of a graph G is the graph X_D with vertex set $E(G)$, where for each crossing between edges e and f in D, there is an edge of X_D between the vertices corresponding to e and f. Note that X_D is actually a multigraph, where the multiplicity of ef equals the number of times e and f cross in D. In most drawings that we consider, each pair of edges cross at most once, in which case X_D has no parallel edges.

Numerous papers have studied graphs that have a drawing whose crossing graph is well-behaved in some way; for example, see [9, 20, 21, 24, 40]. A drawing is *circular* if the vertices are positioned on a circle and the edges are straight line segments. A theme of this paper is to study circular drawings D in which X_D is well-behaved in some way. Many papers have considered properties of X_D in this setting; see for example [19, 22, 45].

3 Tools

In this section, we introduce two auxiliary graphs that are useful tools for proving our main theorems. For a drawing D of a graph G, the *planarisation*, P_D, of D is the plane graph obtained by replacing each crossing with a dummy vertex of degree 4.

Note that P_D depends upon the drawing D (and not just upon G). Figure 1 shows a drawing and its planarisation.



(a) Drawing D of a graph G. (b) Planarisation P_D.

Fig. 1: A drawing and its planarisation.

For a drawing D of a graph G, the *map graph*, M_D, of D is obtained as follows. First let P_D be the planarisation of D. The vertices of M_D are the faces of P_D, where two vertices are adjacent in M_D if the corresponding faces share a vertex. If G is itself a plane graph, then it is already drawn in the plane and so we may talk about the map graph, M_G, of G. Note that all map graphs are connected graphs. Figure 2 shows the map graph M_D for the drawing D in Figure 1.

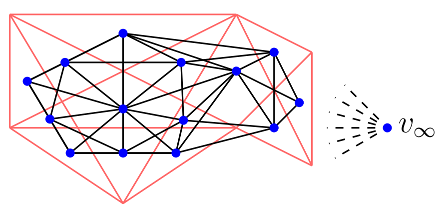


Fig. 2: Map graph M_D. v_∞ is the vertex corresponding to the outer face: it is adjacent to all vertices except the central vertex of degree 10.

The *radius* of a connected graph G, denoted $\text{rad}(G)$, is the minimum non-negative integer r such that for some vertex $v \in V(G)$ and for every vertex $w \in V(G)$ we have $\text{dist}_G(v,w) \leqslant r$.

The following results say that the radius of M_D provides a useful bridge between the treewidth of G, the treewidth of X_D, and the subgraphs of X_D. First, the radius of M_D acts as an upper bound for both the treewidth of G and the treewidth of X_D.

Theorem 5. *For every drawing D of a graph G,*

$$\text{tw}(G) \leqslant 6\,\text{rad}(M_D) + 7 \quad \text{and} \quad \text{tw}(X_D) \leqslant 6\,\text{rad}(M_D) + 7.$$

It is not surprising that treewidth and radius are related for drawings. A classical result of Robertson and Seymour [44, (2.7)] says that $\text{tw}(G) \leqslant 3\,\text{rad}(G) + 1$ for every connected planar graph G. Several authors improved this bound as follows.

Lemma 6 ([10, 21])**.** *For every connected planar graph G,*

$$\text{tw}(G) \leqslant 3\,\text{rad}(G).$$

The next lemma says that if a planar graph G has large treewidth, then the map graph of any plane drawing of G has large radius. A *triangulation* of a plane G is a plane supergraph of G on the same vertex set and where each face is a triangle.

Lemma 7. *Let G be a plane graph with map graph M_G. Then there is a plane triangulation H of G with* $\text{rad}(H) \leqslant \text{rad}(M_G)+1$. *In particular,*

$$\text{tw}(G) \leqslant 3\,\text{rad}(M_G)+3.$$

We use the following lemma about planarisations to extend Lemma 7 from plane drawings to arbitrary drawings.

Lemma 8. *For every drawing D of a graph G, the planarisation P_D of D satisfies*

$$\text{tw}(G) \leqslant 2\,\text{tw}(P_D)+1 \quad \text{and} \quad \text{tw}(X_D) \leqslant 2\,\text{tw}(P_D)+1.$$

To prove Theorem 5, let P_D be the planarisation of D. By definition, $M_D \cong M_{P_D}$. Lemma 7 implies

$$2\,\text{tw}(P_D)+1 \leqslant 2(3\,\text{rad}(M_{P_D})+3)+1 = 6\,\text{rad}(M_D)+7,$$

and Lemma 8 now gives the required result.

The next lemma is a cornerstone of this paper. It shows that if the map graph of a circular drawing has large radius, then the crossing graph contains a useful substructure.

Lemma 9. *Let D be a circular drawing of a graph G. If the map graph M_D has radius at least $2t$, then the crossing graph X_D contains t vertex-disjoint induced cycles $C_1,\dots,C_t$ such that for all $i<j$ every vertex of C_i has at least two neighbours in C_j. Moreover, every vertex of X_D has at most four neighbours in any C_i $(1 \leqslant i \leqslant t)$.*

4 Structural Properties of Circular Drawings

Theorem 5 says that for any drawing D of a graph G, the radius of M_D provides an upper bound for $\text{tw}(G)$ and $\text{tw}(X_D)$. For a general drawing it is impossible to relate $\text{tw}(X_D)$ to $\text{tw}(G)$. Firstly, planar graphs can have arbitrarily large treewidth (for example, the $(n \times n)$-grid has treewidth n) and admit drawings with no crossings. In the other direction, $K_{3,n}$ has treewidth 3 and crossing number $\Omega(n^2)$, as shown by Kleitman [30]. In particular, the crossing graph of any drawing of $K_{3,n}$ has average degree linear in n and thus has arbitrarily large complete minors [35] and so arbitrarily large treewidth.

Happily, this is not so for circular drawings. Using the tools in Section 3 we show that if a graph G has large treewidth, then the crossing graph of any circular drawing of G has large treewidth. In fact, the crossing graph must contain a large (topological) complete graph minor (see Theorems 3 and 10). In particular, if X_D is K_t-minor-free, then G has small treewidth. We further show that if X_D is K_t-minor-free, then G does not contain a subdivision of $K_{2,4t}$ (Theorem 4). Using these results, we deduce a product structure theorem for G (Corollary 11).

In the other direction, we ask what properties of a graph G guarantee that it has a circular drawing D where X_D has no K_t-minor. Certainly G must have small treewidth. Adding the constraint that G does not contain a subdivision of $K_{2,f(t)}$ is not sufficient (see Lemma 14) but a bounded maximum degree constraint is: we show that if G has bounded maximum degree and bounded treewidth, then G has a circular drawing where the crossing graph has bounded treewidth (Proposition 15).

4.1 Necessary Conditions for K_t-Minor-Free Crossing Graphs

This subsection studies the structure of graphs that have circular drawings whose crossing graph is (topological) K_t-minor-free. Much of our understanding of the structure of these graphs is summarised by Theorems 3 and 4 and the next two results.

Theorem 10. *If a graph G has a circular drawing where the crossing graph has no topological K_t-minor, then G has treewidth at most* $6t^2+6t+1$.

We may deduce a product structure theorem for graphs that have a circular drawing whose crossing graph is K_t-minor-free. For two graphs G and H, the *strong product* $G \boxtimes H$ is the graph with vertex-set $V(G) \times V(H)$, and with an edge between two vertices (v,w) and (v',w') if and only if $v=v'$ and $ww' \in E(H)$, or $w=w'$ and $vv' \in E(G)$, or $vv' \in E(G)$ and $ww' \in E(H)$. Campbell et al. [13, Prop. 55] showed that if a graph G is $K_{2,t}$-topological minor-free and has treewidth at most k, then G is isomorphic to a subgraph of $H \boxtimes K_{O(t^2k)}$ where $\operatorname{tw}(H) \leqslant 2$. Thus Theorems 3 and 4 imply the following product structure result.

Corollary 11. *If a graph G has a circular drawing where the crossing graph has no K_t-minor, then G is isomorphic to a subgraph of $H \boxtimes K_{O(t^3)}$ where* $\operatorname{tw}(H) \leqslant 2$.

En route to proving these results, we use the cycle structure built by Lemma 9 to find (topological) complete minors in the crossing graph of circular drawings. We first show that the treewidth and Hadwiger number of X_D as well as the radius of M_D are all linearly tied.

Lemma 12. *For every circular drawing D,*

$$\operatorname{tw}(X_D) \leqslant 6\operatorname{rad}(M_D)+7 \leqslant 12\,h(X_D)-11 \leqslant 12\operatorname{tw}(X_D)+1.$$

Proof. The first inequality is exactly Theorem 5, while the final one is the well-known fact that $h(G) \leqslant \operatorname{tw}(G)+1$ for every graph G. To prove the middle inequality we need to show that for any circular drawing D,

$$\operatorname{rad}(M_D) \leqslant 2\,h(X_D)-3. \tag{1}$$

Let $t := h(X_D)$ and suppose, for a contradiction, that $\mathrm{rad}(M_D) \geqslant 2t-2$. By Lemma 9, X_D contains $t-1$ vertex disjoint cycles $C_1,\dots,C_{t-1}$ such that for all $i<j$ every vertex of C_i has a neighbour in C_j. Contracting C_1 to a triangle and each C_i ($i \geqslant 2$) to a vertex gives a K_{t+1}-minor in X_D. This is the required contradiction. □

Clearly the Hajós number of a graph is at most the Hadwiger number. Our next lemma implies that the Hajós number of X_D is quadratically tied to the radius of M_D and to the treewidth and Hadwiger number of X_D.

Lemma 13. *For every circular drawing D,*

$$\mathrm{rad}(M_D) \leqslant h'(X_D)^2 + 3\,h'(X_D) + 1.$$

Proof. Let $t = h'(X_D)+1$ and suppose, for a contradiction, that $\mathrm{rad}(M_D) \geqslant t^2+t$. By Lemma 9, X_D contains $(t^2+t)/2$ vertex disjoint cycles $C_1,\dots,C_{(t^2+t)/2}$ such that for all $i<j$ every vertex of C_i has a neighbour in C_j. For each $i \in \{1,\dots,t\}$, let $v_i \in V(C_i)$. We assume that $V(K_t)=\{1,\dots,t\}$ and let $\phi\colon E(K_t) \to \{t+1,\dots,(t^2+t)/2\}$ be a bijection. Then for each $ij \in E(K_t)$, there is a (v_i,v_j)-path P_{ij} in X_D whose internal vertices are contained in $V(C_{\phi(ij)})$. Since ϕ is a bijection, it follows that $(P_{ij}\colon ij \in E(K_t))$ defines a topological K_t-minor in X_D, a contradiction. □

We are now ready to prove Theorems 3 and 10.

Proof of Theorem 3. Let D be a circular drawing of G with $h(X_D) \leqslant t-1$. By (1), $\mathrm{rad}(M_D) \leqslant 2t-5$. Finally, by Theorem 5, $\mathrm{tw}(G) \leqslant 12t-23$. □

Proof of Theorem 10. Let D be a circular drawing of G with $h'(X_D) \leqslant t-1$. By Lemma 13, $\mathrm{rad}(M_D) \leqslant t^2+t-1$. Finally, by Theorem 5, $\mathrm{tw}(G) \leqslant 6t^2+6t+1$. □

By considering grid graphs, we show that the bound on $\mathrm{tw}(G)$ in Theorem 3 is within a constant factor of being optimal.

The proof of Theorem 4 is more involved—see [28] for details.

4.2 Sufficient Conditions for K_t-Minor-Free Crossing Graphs

It is natural to consider whether the converse of Theorems 3 and 4 holds. That is, does there exist a function f such that if a $K_{2,t}$-topological minor-free graph G has treewidth at most k, then there is a circular drawing of G whose crossing graph is $K_{f(t,k)}$-minor-free. Our next result shows that this is false in general. A *t-rainbow* in a circular drawing of a graph is a non-crossing matching consisting of t edges between two disjoint arcs in the circle.

Lemma 14. *For every $t \in \mathbb{N}$, there exists a $K_{2,4}$-topological minor-free graph G with* $\mathrm{tw}(G)=2$ *such that, for every circular drawing D of G, the crossing graph X_D contains a K_t-minor.*

Proof. Let T be any tree with maximum degree 3 and sufficiently large pathwidth (as a function of t). Such a tree exists as the complete binary tree of height $2h$ has pathwidth h. Let G be obtained from T by adding a dominant vertex v, so G has treewidth 2. Since $G-v$ has maximum degree 3, it follows that G is $K_{2,4}$-topological minor-free.

Let D be a circular drawing of G and let D_T be the induced circular drawing of T. Since T has sufficiently large pathwidth, a result of Pupyrev [43, Thm. 2] implies that X_D has large chromatic number or a $4t$-rainbow[2]. Since the class of circle graphs is χ-bounded [27], it follows that if X_D has large chromatic number, then it contains a large clique and we are done. So we may assume that D_T contains a $4t$-rainbow. By the pigeonhole principle, there is a subset $\{a_1b_1,\dots,a_{2t}b_{2t}\}$ of the rainbow edges such that a_ib_i topologically separates v from a_j and b_j whenever $i<j$. As such, a_ib_i crosses the edges va_j and vb_j in D whenever $i<j$. Therefore X_D contains a $K_{t,2t}$ subgraph with bipartition $(\{a_1b_1,\dots,a_tb_t\},\{va_{t+1},vb_{t+1},\dots,va_{2t},vb_{2t}\})$ and this contains a K_t-minor. □

While $K_{2,4}$-topological minor-free and bounded treewidth is not sufficient to imply that a graph has a circular drawing whose crossing graph is K_t-minor-free, bounded degree and bounded treewidth is sufficient.

Proposition 15. *For $k,\Delta\in\mathbb{N}$, every graph G with treewidth less than k and maximum degree at most Δ has a circular drawing in which the crossing graph X_D has treewidth at most* $(6\Delta+1)(18k\Delta)^2-1$.

4.3 Circular Drawings and Degeneracy

Theorems 3 and 10 say that if a graph G has a circular drawing D where the crossing graph X_D excludes a fixed (topological) minor, then G has bounded treewidth. Graphs excluding a fixed (topological) minor have bounded average degree and degeneracy [35]. Despite this, we now show that X_D having bounded degeneracy is not sufficient to bound the treewidth of G. In fact, it is not even sufficient to bound the Hadwidger number of G.

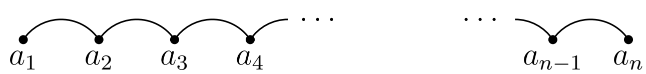
Theorem 16. *For every $t\in\mathbb{N}$, there is a graph G_t and a circular drawing D of G_t such that:*

- *G_t contains a K_t-minor,*
- *G_t has maximum degree 3, and*
- *X_D is 2-degenerate.*

Proof. We draw G_t with vertices placed on the x-axis (x-coordinate between 1 and t) and edges drawn on or above the x-axis. This can then be wrapped to give a circular drawing of G_t.

[2] The result of Pupyrev [43] is in terms of stacks and queues but is equivalent to our statement.

For real numbers $a_1 < a_2 < \cdots < a_n$, we say a path P is drawn as a *monotone path* with vertices $a_1, \ldots, a_n$ if it is drawn as follows where each vertex has x-coordinate equal to its label:



In all our monotone paths, $a_1, a_2, \ldots, a_n$ will be an arithmetic progression. We construct our drawing of G_t as follows (see Figure 3 for the construction with $t = 4$).

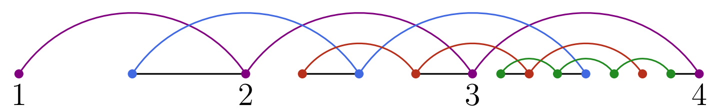


Fig. 3: G_4 where P_0 is purple, P_1 is blue, P_2 is red, P_3 is green, and the $e_{r,s}$ are black.

First let P_0 be the monotone path with vertices $1, 2, \ldots, t$. For $s \in \{1, 2, \ldots, t-1\}$, let P_s be the monotone path with vertices

$$s + 2^{-s}, s + 3 \cdot 2^{-s}, s + 5 \cdot 2^{-s}, \ldots, t - 2^{-s}.$$

Observe that these paths are vertex-disjoint. For $0 \leqslant r < s \leqslant t-1$, let $I_{r,s}$ be the interval

$$[s + 2^{-r} - 2^{-s}, s + 2^{-r}].$$

Note that the lower end-point of $I_{r,s}$ is a vertex in P_s and the upper end-point is a vertex in P_r. Also note that no vertex of any P_i lies in the interior of $I_{r,s}$. Hence for all $r < s$ we may draw a horizontal edge $e_{r,s}$ between the end-points of $I_{r,s}$.

Graph G_t and the drawing D are obtained as a union of the P_s together with all the $e_{r,s}$. The paths P_s are vertex-disjoint and edge $e_{r,s}$ joins P_r to P_s, so G_t contains a K_t-minor. It is simple to check that the $I_{r,s}$ are pairwise disjoint. In particular, any vertex v is the end-point of at most one $e_{r,s}$ and so G_t has maximum degree three.

Each edge $e_{r,s}$ is horizontal and crosses no other edges so has no neighbours in X_D. Next consider an edge aa' of P_s. We have $a' = a + 2 \cdot 2^{-s}$. Exactly one vertex in $V(P_0) \cup V(P_1) \cup \cdots \cup V(P_s)$ lies between a and a': their midpoint, $m = a + 2^{-s}$. Vertex m has at most two non-horizontal edges incident to it and so, in X_D, every $aa' \in E(P_s)$ has at most two neighbours in $E(P_0) \cup E(P_1) \cup \cdots \cup E(P_s)$. Thus X_D is 2-degenerate, as required. □

5 Structural Properties of Circle Graphs

Recall that a circle graph is the intersection graph of a set of chords of a circle. More formally, let C be a circle in $\mathbb{R}^2$. A *chord* of C is a closed line segment with distinct endpoints on C. Two chords of C either cross, are disjoint, or have a common endpoint. Let S be a set of chords of a circle C such that no three chords in S cross at a single point. Let G be the crossing graph of S. Then G is called a *circle graph*. Note that a graph G is a circle graph if and only if $G \cong X_D$ for some circular drawing D of a graph H, and in fact one can take H to be a matching.

We are now ready to prove Theorems 1 and 2. While the treewidth of circle graphs has previously been studied from an algorithmic perspective [31], to the best of our knowledge, these theorems are the first structural results on the treewidth of circle graphs.

Proof of Theorem 1. Let D be a circular drawing of a graph such that $G \cong X_D$. Let M_D be the map graph of D. Since $\text{tw}(X_D) = \text{tw}(G) \geqslant 12t+2$, it follows by Theorem 5 that M_D has radius at least $2t$. The claim then follows from Lemma 9. □

Proof of Theorem 2. Let G be a circle graph and let D be a circular drawing with $G \cong X_D$. By Lemma 12, $\text{tw}(G) \leqslant 6\,\text{rad}(M_D)+7 \leqslant 12\,h(G)-11 \leqslant 12\,\text{tw}(G)+1$. Hence, the Hadwiger number and treewidth are linearly tied for circle graphs. This inequality and Lemma 13 imply

$$h'(G)-1 \leqslant h(G)-1 \leqslant \text{tw}(G) \leqslant 6\,\text{rad}(M_D)+7 \leqslant 6h'(G)^2+18\,h'(G)+13.$$

Hence the Hajós number is quadratically tied to both the treewidth and Hadwiger number for circle graphs. Finally, $K_{t,t}$ is a circle graph which has treewidth t, Hadwiger number $t+1$, and Hajós number $\Theta(\sqrt{t})$. Hence, 'quadratic' is best possible. □

We now discuss several noteworthy consequences of Theorems 1 and 2. Say an hereditary class of graphs $\mathscr{G}$ is *induced*-tw-*bounded* if there is a function f such that for every graph $G \in \mathscr{G}$ with $\text{tw}(G) \geqslant f(t)$, G contains at least one of the usual suspects defined in Section 1. While the class of all graphs is not induced-tw-bounded [3, 11, 14, 42, 46], many natural graph classes are. For example, Aboulker, Adler, Kim, Sintiari, Trotignon [1] showed that every proper minor-closed class is induced-tw-bounded and Korhonen [32] recently showed that the class of graphs with bounded maximum degree is induced-tw-bounded. We now show that the class of circle graphs is not induced-tw-bounded.

Theorem 17. *The class of circle graphs is not induced*-tw-*bounded.*

Proof. We first show that for all $t \geqslant 50$, no circle graph contains a subdivision of the $(t \times t)$-wall or a line graph of a subdivision of the $(t \times t)$-wall as an induced subgraph. As the class of circle graphs is hereditary, it suffices to show that for all

$t \geqslant 50$, these two graphs are not circle graphs. These two graphs are planar (so K_5-minor-free) and have treewidth $t \geqslant 50$. However, Lemma 12 implies that every K_5-minor-free circle graph has treewidth at most 49, which is the required contradiction.

Now consider the family of couples of graphs $((G_t, X_t) : t \in \mathbb{N})$ given by Theorem 16 where X_t is the crossing graph of the drawing of G_t. Then $(X_t : t \in \mathbb{N})$ is a family of circle graphs. Since $(G_t : t \in \mathbb{N})$ has unbounded treewidth, Theorem 3 implies that $(X_t : t \in \mathbb{N})$ also has unbounded treewidth. Moreover, since X_t is 2-degenerate for all $t \in \mathbb{N}$, it excludes K_4 and $K_{3,3}$ as (induced) subgraphs, as required. □

We now discuss applications of Theorem 1 to vertex-minor-closed classes. For a vertex v of a graph G, to *locally complement at v* means to replace the induced subgraph on the neighbourhood of v by its complement. A graph H is a *vertex-minor* of a graph G if H can be obtained from G by a sequence of vertex deletions and local complementations. Vertex-minors were first studied by Bouchet [12] under the guise of isotropic systems. The name 'vertex-minor' is due to Oum [37]. Circle graphs are a key example of a vertex-minor-closed class.

We now show that a vertex-minor-closed graph class is induced-tw-bounded if and only if it has bounded rank-width. Rank-width is a graph parameter introduced by Oum and Seymour [39] that describes whether a graph can be decomposed into a tree-like structure by simple cuts. For a formal definition and a survey on this parameter, see [38]. Oum [37] showed that rank-width is closed under vertex-minors.

Theorem 18. *A vertex-minor-closed class $\mathcal{G}$ is induced-*tw*-bounded if and only if it has bounded rankwidth.*

Proof. Suppose $\mathcal{G}$ has bounded rankwidth. By a result of Abrishami, Chudnovsky, Hajebi, and Spirkl [7], there is a function f such that every graph in $\mathcal{G}$ with treewidth at least $f(t)$ contains K_t or $K_{t,t}$ as an induced subgraph. Thus $\mathcal{G}$ is induced-tw-bounded. Now suppose $\mathcal{G}$ has unbounded rank-width. By a result of Gellen, Kwon, McCarty, and Wollan [26], $\mathcal{G}$ contains all circle graphs. It therefore follows by Theorem 17 that $\mathcal{G}$ is not induced-tw-bounded. □

We conclude with the following question:

> Let $\mathcal{G}$ be a vertex-minor-closed class with unbounded rank-width. What are the unavoidable induced subgraphs of graphs in $\mathcal{G}$ with large treewidth?

The cycle structure (or variants thereof) in Theorem 1 must be included in the list of unavoidable induced subgraphs.

Acknowledgements This research was initiated at the *Structural Graph Theory Downunder II* workshop at the Mathematical Research Institute MATRIX (March 2022). Thanks to all the participants, especially Marc Distel, for helpful conversations.

Research of Robert Hickingbotham supported by an Australian Government Research Training Program Scholarship. Research of Freddie Illingworth supported by EPSRC grant EP/V007327/1. Research of Bojan Mohar supported in part by the NSERC Discovery Grant R611450 (Canada). Research of David Wood supported by the Australian Research Council and a Visiting Research Fellowship of Merton College, University of Oxford.

References

1. Aboulker, P., Adler, I., Kim, E.J., Sintiari, N.L.D., Trotignon, N.: On the tree-width of even-hole-free graphs. European J. Combin. **98**, #103,394 (2021). DOI 10.1016/j.ejc.2021.103394
2. Abrishami, T., Alecu, B., Chudnovsky, M., Hajebi, S., Spirkl, S.: Induced subgraphs and tree decompositions VII. Basic obstructions in H-free graphs (2022). URL https://arxiv.org/abs/2207.05538
3. Abrishami, T., Alecu, B., Chudnovsky, M., Hajebi, S., Spirkl, S., Vušković, K.: Induced subgraphs and tree decompositions V. One neighbor in a hole (2022). URL https://arxiv.org/abs/2205.04420
4. Abrishami, T., Chudnovsky, M., Dibek, C., Hajebi, S., Rzążewski, P., Spirkl, S., Vušković, K.: Induced subgraphs and tree decompositions II. Toward walls and their line graphs in graphs of bounded degree (2021). URL https://arxiv.org/abs/2108.01162
5. Abrishami, T., Chudnovsky, M., Hajebi, S., Spirkl, S.: Induced subgraphs and tree decompositions III. Three-path-configurations and logarithmic treewidth. Adv. Comb. **#6** (2022). DOI 10.19086/aic.2022.6
6. Abrishami, T., Chudnovsky, M., Hajebi, S., Spirkl, S.: Induced subgraphs and tree decompositions IV. (Even hole, diamond, pyramid)-free graphs (2022). URL https://arxiv.org/abs/2203.06775
7. Abrishami, T., Chudnovsky, M., Hajebi, S., Spirkl, S.: Induced subgraphs and tree decompositions VI. Graphs with 2-cutsets (2022). URL https://arxiv.org/abs/2207.05538
8. Abrishami, T., Chudnovsky, M., Vušković, K.: Induced subgraphs and tree decompositions I. Even-hole-free graphs of bounded degree. J. Combin. Theory Ser. B **157**, 144–175 (2022). DOI 10.1016/j.jctb.2022.05.009
9. Bae, S.W., Baffier, J., Chun, J., Eades, P., Eickmeyer, K., Grilli, L., Hong, S., Korman, M., Montecchiani, F., Rutter, I., Tóth, C.D.: Gap-planar graphs. Theor. Comput. Sci. **745**, 36–52 (2018). DOI 10.1016/j.tcs.2018.05.029
10. Bodlaender, H.: Planar graphs with bounded treewidth (1988). URL https://dspace.library.uu.nl/handle/1874/16567. Technical Report RUU-CS-88-14, Department of Computer Science, University of Utrecht
11. Bonamy, M., Bonnet, É., Déprés, H., Esperet, L., Geniet, C., Hilaire, C., Thomassé, S., Wesolek, A.: Sparse graphs with bounded induced cycle packing number have logarithmic treewidth (2022). URL https://arxiv.org/abs/2206.00594
12. Bouchet, A.: Isotropic systems. European J. Combin. **8**(3), 231–244 (1987). DOI 10.1016/S0195-6698(87)80027-6
13. Campbell, R., Clinch, K., Distel, M., Gollin, J.P., Hendrey, K., Hickingbotham, R., Huynh, T., Illingworth, F., Tamitegama, Y., Tan, J., Wood, D.R.: Product structure of graph classes with bounded treewidth (2022). URL https://arxiv.org/abs/2206.02395
14. Davies, J.: Counterexample to the conjecture. In: Graph Theory, *Mathematisches Forschungsinstitut Oberwolfach Reports*, vol. 1, pp. 62–63 (2022). DOI 10.14760/OWR-2022-1
15. Davies, J.: Improved bounds for colouring circle graphs. Proc. Amer. Math. Soc. **150**, 5121–5135 (2022). DOI 10.1090/proc/16044
16. Davies, J.: Local properties of graphs with large chromatic number. Ph.D. thesis, University of Waterloo (2022). URL http://hdl.handle.net/10012/18679
17. Davies, J., McCarty, R.: Circle graphs are quadratically χ-bounded. Bull. Lond. Math. Soc. **53**(3), 673–679 (2021). DOI 10.1112/blms.12447
18. Diestel, R.: Graph theory, *Graduate Texts in Mathematics*, vol. 173, 5th edn. Springer (2018)
19. Dujmovic, V., Eppstein, D., Hickingbotham, R., Morin, P., Wood, D.R.: Stack-number is not bounded by queue-number. Combinatorica **42**(2), 151–164 (2022). DOI 10.1007/s00493-021-4585-7
20. Dujmović, V., Eppstein, D., Wood, D.R.: Structure of graphs with locally restricted crossings. SIAM J. Discrete Math. **31**(2), 805–824 (2017). DOI 10.1137/16M1062879

21. Dujmović, V., Morin, P., Wood, D.R.: Layered separators in minor-closed graph classes with applications. J. Combin. Theory Ser. B **127**, 111–147 (2017). DOI 10.1016/j.jctb.2017.05.006
22. Dujmović, V., Wood, D.R.: Graph treewidth and geometric thickness parameters. Discrete Comput. Geom. **37**(4), 641–670 (2007). DOI 10.1007/s00454-007-1318-7
23. Durán, G., Grippo, L.N., Safe, M.D.: Structural results on circular-arc graphs and circle graphs: A survey and the main open problems. Discrete Appl. Math. **164**, 427–443 (2014). DOI 10.1016/j.dam.2012.12.021
24. Eppstein, D., Gupta, S.: Crossing patterns in nonplanar road networks. In: Proc. 25th ACM SIGSPATIAL Int'l Conf. on Advances in Geographic Information Systems, pp. 40:1–9 (2017). DOI 10.1145/3139958.3139999
25. de Fraysseix, H.: A characterization of circle graphs. European J. Combin. **5**(3), 223–238 (1984). DOI 10.1016/S0195-6698(84)80005-0
26. Geelen, J., Kwon, O.j., McCarty, R., Wollan, P.: The grid theorem for vertex-minors. J. Combin. Theory, Ser. B **158**, 93–116 (2023). DOI 10.1016/j.jctb.2020.08.004
27. Gyárfás, A.: On the chromatic number of multiple interval graphs and overlap graphs. Discrete Math. **55**(2), 161–166 (1985). DOI 10.1016/0012-365X(85)90044-5
28. Hickingbotham, R., Illingworth, F., Mohar, B., Wood, D.R.: Treewidth, circle graphs and circular drawings (2022). URL https://arxiv.org/abs/2212.11436
29. Hickingbotham, R., Wood, D.R.: Shallow minors, graph products and beyond planar graphs (2021). URL https://arxiv.org/abs/2111.12412
30. Kleitman, D.J.: The crossing number of $K_{5,n}$. J. Combinatorial Theory **9**, 315–323 (1970). DOI 10.1016/S0021-9800(70)80087-4
31. Kloks, T.: Treewidth of circle graphs. Int. J. Found. Comput. Sci. **7**(2), 111–120 (1996). DOI 10.1142/S0129054196000099
32. Korhonen, T.: Grid induced minor theorem for graphs of small degree (2023). DOI 10.1016/j.jctb.2023.01.002
33. Kostochka, A., Kratochvíl, J.: Covering and coloring polygon-circle graphs. Discrete Math. **163**(1–3), 299–305 (1997). DOI 10.1016/S0012-365X(96)00344-5
34. Lozin, V., Razgon, I.: Tree-width dichotomy. European J. Combin. **103**, # 103,517 (2022). DOI 10.1016/j.ejc.2022.103517
35. Mader, W.: Homomorphieeigenschaften und mittlere Kantendichte von Graphen. Math. Ann. **174**, 265–268 (1967). DOI 10.1007/BF01364272
36. McCarty, R.: Local structure for vertex-minors. Ph.D. thesis, University of Waterloo (2021). URL http://hdl.handle.net/10012/17633
37. Oum, S.i.: Rank-width and vertex-minors. J. Combin. Theory Ser. B **95**(1), 79–100 (2005). DOI 10.1016/j.jctb.2005.03.003
38. Oum, S.i.: Rank-width: algorithmic and structural results. Discrete Appl. Math. **231**, 15–24 (2017). DOI 10.1016/j.dam.2016.08.006
39. Oum, S.i., Seymour, P.: Approximating clique-width and branch-width. J. Combin. Theory Ser. B **96**(4), 514–528 (2006). DOI 10.1016/j.jctb.2005.10.006
40. Pach, J., Tóth, G.: Graphs drawn with few crossings per edge. Combinatorica **17**(3), 427–439 (1997). DOI 10.1007/BF01215922
41. Pilipczuk, M., Sintiari, N.L.D., Thomassé, S., Trotignon, N.: (Theta, triangle)-free and (even hole, K_4)-free graphs. Part 2: Bounds on treewidth. J. Graph Theory **97**(4), 624–641 (2021). DOI 10.1002/jgt.22675
42. Pohoata, C.: Unavoidable induced subgraphs of large graphs. Senior thesis, Department of Mathematics, Princeton University (2014). URL http://arks.princeton.edu/ark:/88435/dsp012514nk67q
43. Pupyrev, S.: Book embeddings of graph products (2020). URL https://arxiv.org/abs/2007.15102
44. Robertson, N., Seymour, P.: Graph minors. III. Planar tree-width. J. Combin. Theory Ser. B **36**(1), 49–64 (1984). DOI 10.1016/0095-8956(84)90013-3
45. Schaefer, M.: The graph crossing number and its variants: A survey. Electron. J. Combin. **#DS21** (2022). URL https://www.combinatorics.org/DS21
46. Sintiari, N.L.D., Trotignon, N.: (Theta, triangle)-free and (even hole, K_4)-free graphs—Part 1: Layered wheels. J. Graph Theory **97**(4), 475–509 (2021). DOI 10.1002/jgt.22666