A large class of conjecturally stable chromatic symmetric functions

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Abstract The theory of stable and Lorentzian polynomials has recently found a number of successes in a variety of research areas including combinatorics, engineering, and computer science; in particular, they have played a key role in solving long-standing open problems such as the Kadison–Singer problem and Mason's log-concavity conjecture. More recently, the classes of stable polynomials and Lorentzian polynomials have appeared in representation theory, algebraic combinatorics, and even knot theory. We further highlight their ubiquity by introducing a large class of chromatic symmetric functions related to Hessenberg varieties and the Stanley–Stembridge conjecture that are conjecturally Lorentzian and stable.

Graph coloring is a well-studied topic in combinatorics, computer science, and scheduling problems. On the other hand, there has been a recent explosion in the study of stable polynomials: these are a multivariate analogue of real-rooted polynomials that have led the way to the resolution of a number of open problems in fields as diverse as matroid theory [2, 5], knot theory [9], and quantum mechanics, functional analysis, and engineering [11]. This note offers a large class of polynomials constructed via graph coloring that are conjecturally stable.

We write $G = (V, E)$ for an arbitrary graph with finite vertex set V and finite edge set E. The *chromatic symmetric function (CSF)* $X_G(x_1, \ldots, x_m)$ of G in m variables is

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$$X_G(x_1,\ldots,x_m) := \sum_{\substack{f\colon V\to[m] \\ \text{proper coloring}}} \prod_{v\in V} x_{f(v)},$$

where the sum is over all proper (vertex) colorings of G. A proper coloring $f\colon V\to[m]$ is a coloring of the vertices of G such that if $\{v,w\}\in E$, then $f(v)\neq f(w)$; that is, the two vertices of each edge are required to get different colors. This refinement of the chromatic polynomial χ_G of G was introduced by Stanley in [15]; indeed, they are related by the identity $X_G(\underbrace{1,\ldots,1}_{q \text{ times}},0,\ldots,0) = \chi_G(q)$. Note that we may define X_G using any number of variables; see Figure 1 for an example of the CSF of the path graph P_3 in both two and three variables. Much utility can be gained by considering CSFs $X_G(x_1,x_2,\ldots)$ in infinitely-many variables, where they lie in the ring of symmetric functions (see, for example, [7] and [13]); for simplicity, we will not go this route.

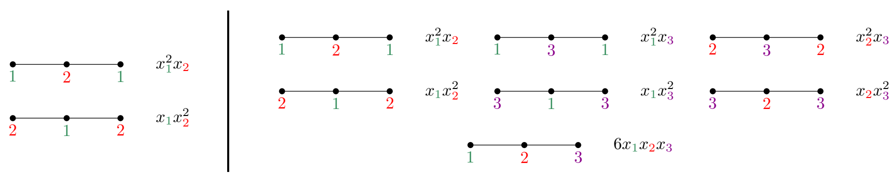

Fig. 1 On the left, the summands in $X_{P_3}(x_1,x_2)$, and on the right, the summands in $X_{P_3}(x_1,x_2,x_3)$.

A polynomial $f\in\mathbb{R}[x_1,\ldots,x_m]$ is called *stable* if it is either identically zero, or is nonvanishing on $\mathscr{H}^m$, where $\mathscr{H}$ is the open upper half-plane in $\mathbb{C}$. See [17, 3, 4] for more about the theory of stable polynomials. The CSF of an arbitrary graph need not be stable: for example, one may check that the CSF $X_{C_4}(x_1,x_2,x_3,x_4)$ in four variables of the four-cycle graph C_4 is not stable. However, the purpose of this note is to point out a large and interesting class where stability conjecturally does hold.

For the remainder of this note, we assume that our graphs G have vertex set $V=[n]=\{1,2,\ldots,n\}$. Define a set of graphs $\mathscr{D}$ by

$$\mathscr{D} = \{\text{graphs } G=([n],E) \mid \text{if } \{i,j\}\in E \text{ and } i\le i' < j' \le j, \text{ then } \{i',j'\}\in E\}.$$

See Figure 2 for both an example of a graph in $\mathscr{D}$ and a non-example of a graph in $\mathscr{D}$ (for the case $n=4$).

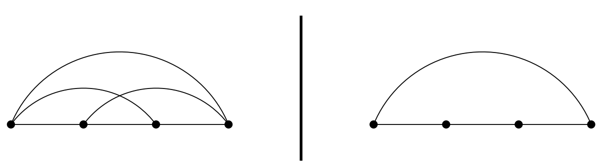

Fig. 2 The graph K_4 on the left is in $\mathscr{D}$, and the graph C_4 on the right is not in $\mathscr{D}$.

Graphs in the class $\mathscr{D}$ are enumerated by the n-th Catalan number. The graphs in $\mathscr{D}$ are also known as indifference graphs of Dyck paths, as unit interval orders, or as incomparability graphs of (2+2)- and (3+1)-free posets. The elusive Stanley–Stembridge conjecture [16] that has guided many developments in algebraic combinatorics involves exactly this class of graphs (due to a reduction in [8]); and on the other hand, the class $\mathscr{D}$ also naturally occurs in the study of Hessenberg varieties [14].

Conjecture 1 ([12, Conjecture 6.5]). If $G \in \mathscr{D}$, then $X_G(x_1,\ldots,x_m)$ is stable for every m.

Checking that a polynomial is stable generally involves verifying that an infinite number of univariate specializations are real-rooted [17, Lemma 2.3]. The Lorentzian property introduced in [6], and independently in [1], is a weaker notion than stability (e.g. Conjecture 1 implies Conjecture 2 below [6, Proposition 2.2]), but only involves a finite number of checks and has had much success uniting discrete and continuous log-concavity phenomena.

Conjecture 2 ([12, Conjecture 6.3]). If $G \in \mathscr{D}$, then $X_G(x_1,\ldots,x_m)$ is Lorentzian for every m.

Conjecture 2 has been checked for $n \leq 7$ and $m \leq 8$; and in the special case when $G \in \mathscr{D}$ has a bipartite complement graph, Conjecture 2 has been settled in [12, Theorem 6.8]. (Normalized) Schur polynomials were recently shown to be Lorentzian [10], and it is expected that many related polynomials in algebraic combinatorics should also have this property [10, Section 3]. On the other hand, (normalized) Schur polynomials are not stable [10, Example 9]. It is interesting to ponder where the line is: is there a satisfying reason why (normalized) Schur polynomials are not stable, but CSFs of graphs in $\mathscr{D}$ conjecturally are?

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