On Dupire Formula and Diffusion with Given Marginals

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Abstract This paper presents relations between the drift and the diffusion coefficients of a diffusion in terms of the prices of a European call option, giving an extension to the Dupire formula. Due to the correspondence between the call option prices and the marginal distributions of the underlying process, a necessary condition for a diffusion with given marginal distributions is obtained. Some specific examples, including the the cases of Normal and Lognormal marginals, are considered. In particular, we construct fake Brownian motion diffusions, a family of diffusions, which all are Gaussian processes with Brownian marginals $N(0,t)$ but not a Brownian motion unless $\sigma^2 = 1$.

1 Introduction

Motivated by Hamza and Klebaner, [9], where a family of processes with Normal marginal distributions was constructed, Mudakkar [15] considered diffusion processes with Normal marginals and obtained a necessary relation between the drift and the diffusion coefficients. In this paper, we generalised Mudakkar's result via a Dupire-like formula. We derive a general version of Dupire formula. Under the assumption of Normal marginal distributions, we recover Mudakkar's result. We can also extend the result to other types of distributions, such as the Lognormal distribution.

The topic of constructing processes with given marginal distributions, also known as mimicking, has been of interest of many in the last two decades. For example,

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[1, 2, 3, 9, 16] gave explicit constructions to mimic Brownian motion, i.e. construct processes of marginals $N(0,t)$. Generalisation from [9] to mimicking self-similar processes was carried out in [7]. For Ito processes, studies include [4, 8, 12]. In the monograph [11] by Hirsch, Profeta, Roynette and Yor on peacocks (processes that are increasing in the convex order), a few methods for constructing processes with given marginals are also discussed.

We give the main results in Section 2 and postpone the proofs to Section 3. In Section 4, we will consider some examples.

2 A general version of Dupire formula

Consider a local martingale $(X_t)_{t\geq 0}$ satisfying the stochastic differential equation $dX_t = \sigma(X_t,t)dB_t$. Let $C(x,t) = \mathbb{E}[(X_t-x)^+]$, the price of a call option with strike x and maturity t. Then, the relation between the functions σ and C is given by the well-known Dupire formula [6]

$$\sigma^2(x,t) = \frac{2\frac{\partial C}{\partial t}(x,t)}{\frac{\partial^2 C}{\partial x^2}(x,t)}, \tag{1}$$

or $\frac{\partial C}{\partial t}(x,t) = \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 C}{\partial x^2}(\cdot,t)$. Note that the probability density function of X_t, if exists, coincides with $\frac{\partial^2 C}{\partial x^2}$. Thus, (1) gives the unique diffusion with given marginal distributions. (In this paper, we assume that the distribution of X_t is absolutely continuous with respect to the Lebesgue measure, i.e. the probability density function exists, and that $\mathbb{E}[(X_t-x)^+]$ is finite.)

This can be generalised to semimartingales as follows.

Theorem 1. *Let $(X_t)_{t\geq 0}$ be a diffusion process satisfying*

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dB_t, \tag{2}$$

with $\int_0^t \sigma^2(X_s,s)ds < \infty$. And, let $C(x,t) = \mathbb{E}[(X_t-x)^+]$. Then,

$$\frac{\partial C}{\partial t}(x,t) = \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 C}{\partial x^2}(x,t) + \int_x^\infty \mu(y,t)\frac{\partial^2 C}{\partial x^2}(y,t)dy, \tag{3}$$

equivalently,

$$\frac{\partial C}{\partial t}(x,t) = \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 C}{\partial x^2}(x,t) + \mathbb{E}[\mu(X_t,t)1_{X_t>x}]. \tag{4}$$

From the put-call parity, we can establish a relation similar to (4) but with put option prices $P(x,t) = \mathbb{E}\left[(x-X_t)^+\right]$. Note that

$$\frac{\partial P}{\partial x}(x,t) \ = \ \int_{-\infty}^{x} p_t(y)dy \ = \ 1+\frac{\partial C}{\partial x}(x,t) \tag{5}$$

and

$$\frac{\partial^2 P}{\partial x^2}(x,t) \ = \ p_t(x) \ = \ \frac{\partial^2 C}{\partial x^2}(x,t). \tag{6}$$

Proposition 1. *Let $(X_t)_{t\geq 0}$ satisfies SDE* (2)*, and let $P(x,t)=\mathbb{E}\left[(x-X_t)^+\right]$. Then,*

$$\frac{\partial P}{\partial t}(x,t)=\frac{1}{2}\sigma^2(x,t)\frac{\partial^2 P}{\partial x^2}(x,t)-\mathbb{E}\left[\mu(X_t,t)1_{X_t<x}\right]. \tag{7}$$

Comparing (4) and (7), we see that

$$\frac{\partial C}{\partial t}(x,t)-\frac{\partial P}{\partial t}(x,t)=\mathbb{E}[\mu(X_t,t)].$$

Remark 1. To get an expression relating $\mu(x,t)$ and $\sigma(x,t)$ without integral, we differentiate (3) with respect to x, resulting in

$$\frac{\partial^2 C}{\partial x \partial t}(x,t)=\frac{1}{2}\sigma^2(x,t)\frac{\partial^3 C}{\partial x^3}(x,t)+\frac{1}{2}\frac{\partial}{\partial x}\left\{\sigma^2(x,t)\right\}\frac{\partial^2 C}{\partial x^2}(x,t)-\mu(x,t)\frac{\partial^2 C}{\partial x^2}(x,t). \tag{8}$$

The "put" version follows from (5) and (6),

$$\frac{\partial^2 P}{\partial x \partial t}(x,t)=\frac{1}{2}\sigma^2(x,t)\frac{\partial^3 P}{\partial x^3}(x,t)+\frac{1}{2}\frac{\partial}{\partial x}\left\{\sigma^2(x,t)\right\}\frac{\partial^2 P}{\partial x^2}(x,t)-\mu(x,t)\frac{\partial^2 P}{\partial x^2}(x,t),$$

which can also be obtained from (7).

One can easily see that, when $\mu(y,t)=0$, (3) reduces to the classical Dupire formula (1), and when $\mu(y,t)=\mu(t)$, (3) becomes

$$\frac{\partial C}{\partial t}(x,t)=\frac{1}{2}\sigma^2(x,t)\frac{\partial^2 C}{\partial x^2}(x,t)-\mu(t)\frac{\partial C}{\partial x}(x,t),$$

which is also a known result (see for example [5]). Thus, (3) can be considered as a general version of the Dupire formula.

3 Proofs

There are a few ways to derive (3) and prove Theorem 1. The usual approach is through the forward Kolmogorov equation. Alternatively, it can be obtained through the Meyer-Tanaka formula and local times, like in [10] and [13]. Detailed derivation by the two approaches will be given in Sections 3.1 and 3.2.

3.1 Proof of Theorem 1 by forward Kolmogorov equation

This proof requires further assumptions and is given as a motivation.

Let $(X_t)_{t\geq 0}$ be a process satisfying the SDE (2). Denote by $p(y,t)$ the density function of X_t at point y. Suppose that $\mu(x,t)$ and $\sigma(x,t)$ are bounded and continuous, with $\sigma^2(x,t)$ bounded away from 0; that $\mu(x,t)$ and $\sigma^2(x,t)$ are Holder continuous with respect to x and t; and that $\mu(x,t)$ and $\sigma(x,t)$ have two partial derivatives with respect to x, which are bounded and Holder continuous with respect to x. (See [14, Theorem 5.15], for example.) Then, we have the forward Kolmogorov equation

$$\frac{\partial p}{\partial t}(y,t) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\{\sigma^2(y,t)p(y,t)\} - \frac{\partial}{\partial y}\{\mu(y,t)p(y,t)\}. \tag{9}$$

Before we proceed, we give the derivatives concerning the function

$$C(x,t) = \mathbb{E}[(X_t - x)^+] = \int_x^\infty (y-x)p(y,t)dy.$$

Differentiate with respect to x, we have

$$\frac{\partial C}{\partial x}(x,t) = -\int_x^\infty p(y,t)dy$$

and

$$\frac{\partial^2 C}{\partial x^2}(x,t) = p(x,t).$$

Suppose that for $\frac{\partial p}{\partial t}(y,t)$ exists almost everywhere and that $\int_a^b \int_x^\infty |(y-x)\frac{\partial p}{\partial t}(y,t)|dydt$ is finite for any compact time interval $[a,b]$ and any x. Then, we also have

$$\frac{\partial C}{\partial t}(x,t) = \int_x^\infty (y-x)\frac{\partial p}{\partial t}(y,t)dy.$$

Now, multiply both sides of (9) by $(y-x)^+$ and integrate with respect to y, we obtain

$$\begin{aligned}&\int_x^\infty (y-x)\frac{\partial p}{\partial t}(y,t)dy\\ &= \frac{1}{2}\int_x^\infty (y-x)\frac{\partial^2}{\partial y^2}\{\sigma^2(y,t)p(y,t)\}dy - \int_x^\infty (y-x)\frac{\partial}{\partial y}\{\mu(y,t)p(y,t)\}dy.\end{aligned} \tag{10}$$

By integration by parts, the last integral

$$\begin{aligned}\int_x^\infty (y-x)\frac{\partial}{\partial y}\{\mu(y,t)p(y,t)\}dy &= \Big[(y-x)\mu(y,t)p(y,t)\Big]_x^\infty - \int_x^\infty \mu(y,t)p(y,t)dy\\ &= -\int_x^\infty \mu(y,t)p(y,t)dy,\end{aligned}$$

assuming that $\lim_{y\to\infty}\mu(y,t)p(y,t)=0$. Similarly, the first integral on the right hand side of (10)

$$\begin{aligned}
&\int_x^\infty (y-x)\frac{\partial^2}{\partial y^2}\left\{\sigma^2(y,t)p(y,t)\right\}dy\\
&\qquad=\left[(y-x)\frac{\partial}{\partial y}\left\{\sigma^2(y,t)p(y,t)\right\}\right]_x^\infty-\int_x^\infty\frac{\partial}{\partial y}\left\{\sigma^2(y,t)p(y,t)\right\}dy\\
&\qquad=-\left[\sigma^2(y,t)p(y,t)\right]_x^\infty=\sigma^2(x,t)p(x,t),
\end{aligned}$$

assuming that $\lim_{y\to\infty}\frac{\partial}{\partial y}\left\{\sigma^2(y,t)p(y,t)\right\}=0$ and $\lim_{y\to\infty}\sigma^2(y,t)p(y,t)=0$. Thus, Equation (10) yields the general Dupire formula (3), or (4) equivalently.

Note that in this proof we require $\lim_{y\to\infty}\mu(y,t)p(y,t)$, $\lim_{y\to\infty}\sigma^2(y,t)p(y,t)$ and $\lim_{y\to\infty}\frac{\partial}{\partial y}\left\{\sigma^2(y,t)p(y,t)\right\}$ to be zero. We also assume that $\frac{\partial p}{\partial t}(y,t)$ exists and $\int_a^b\int_x^\infty|(y-x)\frac{\partial p}{\partial t}(y,t)|dydt<\infty$ for any time interval $[a,b]$ and any x. These further assumptions are not required in an alternative proof showing in the next section.

3.2 Proof of Theorem 1 by Meyer-Tanaka formula

Following [10] and [13], we give an alternative derivation of the general Dupire formula (3) through the local time of X. By the Meyer-Tanaka formula, we have

$$(X_t-x)^+=(X_0-x)^++\int_0^t 1_{X_s>x}\,dX_s+\frac{1}{2}L_t^x, \tag{11}$$

where L_t^x denotes the local time of X at point x. Take expectation on both sides and use the SDE of X,

$$\mathbb{E}\left[(X_t-x)^+\right]=\mathbb{E}\left[(X_0-x)^+\right]+\int_0^t\mathbb{E}\left[\mu(X_s,s)1_{X_s>x}\right]ds+\frac{1}{2}\mathbb{E}\left[L_t^x\right]. \tag{12}$$

Taking expectation of the occupation times formula, we have, for any positive measurable function g,

$$\begin{aligned}
\int g(x)\mathbb{E}\left[L_t^x\right]dx&=\mathbb{E}\left[\int_0^t g(X_s)d\langle X\rangle_s\right]=\int_0^t\mathbb{E}\left[g(X_s)\frac{d\langle X\rangle_s}{ds}\right]ds\\
&=\int_0^t\mathbb{E}\left[g(X_s)\sigma^2(X_s,s)\right]ds\;=\int_0^t\int g(x)\sigma^2(x,s)p_s(x)dxds\\
&=\int g(x)\int_0^t\sigma^2(x,s)p_s(x)dsdx.
\end{aligned}$$

Thus, $\mathbb{E}\left[L_t^x\right]=\int_0^t\sigma^2(x,s)p_s(x)ds$. Substituting this into (12), and taking derivative with respect to t, we have

$$\frac{\partial}{\partial t}\mathbb{E}\big[(X_t-x)^+\big]=\mathbb{E}\big[\mu(X_t,t)1_{X_t>x}\big]+\frac{1}{2}\sigma^2(x,t)p_t(x),$$

which is the general Dupire formula (3).

3.3 Proof of Proposition 1

As $(x-X_t)^+=(X_t-x)^+-(X_t-x)$, using (11) we have

$$\begin{aligned}(x-X_t)^+&=(X_0-x)^++\int_0^t 1_{X_s>x}\,dX_s+\frac{1}{2}L_t^x-(X_t-x)\\&=(x-X_0)^++\int_0^t(1_{X_s>x}-1)dX_s+\frac{1}{2}L_t^x\\&=(x-X_0)^+-\int_0^t 1_{X_s<x}dX_s+\frac{1}{2}L_t^x.\end{aligned}$$

Taking expectation, we have

$$\mathbb{E}\big[(x-X_t)^+\big]=\mathbb{E}\big[(x-X_0)^+\big]-\int_0^t\mathbb{E}\big[\mu(X_s,s)1_{X_s<x}\big]ds+\frac{1}{2}\int_0^t\sigma^2(x,s)p_s(x)ds,$$

where $p_s(x)$ is the probability density function of X_s. Differentiate with respect to t, we obtain (7), the "put" version of (3).

4 Some examples on diffusion with given marginals

As the function C uniquely determines the marginal distributions of the process X, (3) and (8) serve as necessary conditions for the diffusion X to match a given marginal distributions. In this section, we derive from Theorem 1 a necessary relation between the drift and diffusion coefficients of X for some specific marginals. For the Normal marginals, the result coincides with Mudakkar [15]. We will also consider the Lognormal marginals and marginals that satisfy selfsimilarity. Some specific examples are also given in this section.

In the following, we will write $\phi(\cdot)$ for the probability density function of $N(0,1)$ and $\Phi(\cdot)$ for the cumulative distribution function of $N(0,1)$.

4.1 Diffusion with Normal marginals

Suppose that $X_t\sim N(a_t,\gamma_t)$, $\gamma_t>0$, then writing Z for a $N(0,1)$ random variable, we have

$$C(x,t) = \mathbb{E}[(X_t - x)^+] = \sqrt{\gamma_t}\,\phi\Big(\frac{x-a_t}{\sqrt{\gamma_t}}\Big) + (a_t - x)\Big(1 - \Phi\Big(\frac{x-a_t}{\sqrt{\gamma_t}}\Big)\Big).$$

Proposition 2. *Assume that $(X_t)_{t\geq 0}$ has marginals $N(a_t, \gamma_t)$ and satisfies the SDE* (2). *Then the coefficients satisfy*

$$\mu(x,t) = a_t' + \frac{1}{2}\frac{\partial}{\partial x}\big\{\sigma^2(x,t)\big\} + \frac{x-a_t}{2\gamma_t}\big(\gamma_t' - \sigma^2(x,t)\big). \tag{13}$$

Proof. Computing the partial derivatives of $C(x,t)$ and using the fact that $\phi'(z) = -z\phi(z)$, we obtain

$$\frac{\partial C}{\partial t} = \frac{\gamma_t'}{2\sqrt{\gamma_t}}\phi\Big(\frac{x-a_t}{\sqrt{\gamma_t}}\Big) + a_t'\Big(1 - \Phi\Big(\frac{x-a_t}{\sqrt{\gamma_t}}\Big)\Big),$$
$$\frac{\partial^2 C}{\partial x^2} = \frac{1}{\sqrt{\gamma_t}}\phi\Big(\frac{x-a_t}{\sqrt{\gamma_t}}\Big).$$

Therefore, Equation (3) gives

$$\frac{\gamma_t'}{2\sqrt{\gamma_t}}\phi\Big(\frac{x-a_t}{\sqrt{\gamma_t}}\Big) + a_t'\Big(1 - \Phi\Big(\frac{x-a_t}{\sqrt{\gamma_t}}\Big)\Big) = \frac{1}{2}\sigma^2(x,t)\frac{1}{\sqrt{\gamma_t}}\phi\Big(\frac{x-a_t}{\sqrt{\gamma_t}}\Big) + \int_x^\infty \mu(y,t)\frac{1}{\sqrt{\gamma_t}}\phi\Big(\frac{y-a_t}{\sqrt{\gamma_t}}\Big)dy. \tag{14}$$

Differentiate with respect to x and simplify, we obtain

$$-\frac{\gamma_t'(x-a_t)}{2\gamma_t} - a_t' = -\frac{1}{2}\sigma^2(x,t)\frac{x-a_t}{\gamma_t} + \frac{1}{2}\frac{\partial}{\partial x}\big\{\sigma^2(x,t)\big\} - \mu(x,t).$$

□

Example 1. Suppose $\mu(x,t) = 0$ and let $X_0 = x_0$, i.e. $X_t = x_0 + \int_0^t \sigma(X_s,s)dB_s$. Then, $a_t = x_0$ and (14) gives

$$\sigma^2(x,t) = \gamma_t'$$

as a necessary condition for X to have Normal marginals. In particular, $\sigma^2(x,t)$ does not depend on x.

Example 2. Suppose $\mu(x,t) = \mu_t x$ and $\sigma(x,t) = \sigma_t x$, where μ_t and σ_t are functions of t only, i.e.

$$dX_t = \mu_t X_t dt + \sigma_t X_t dB_t.$$

Since $dX_t = X_t dR_t$, with $dR_t = \mu_t dt + \sigma_t dB_t$, X is the stochastic exponential of R, so that

$$X_t = X_0 \exp\Big(R_t - \frac{1}{2}[R,R]_t\Big) = X_0 \exp\Big(\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dB_s\Big).$$

We will show that σ_t must be zero for the process to have a Normal marginals, provided X_0 has a normal distribution. Moreover, the mean and variance a_t and γ_t must be of a certain exponential form of μ_t. Proposition 2 gives

$$\mu_t x = a_t' + x\sigma_t^2 + \frac{x-a_t}{2\gamma_t}(\gamma_t' - \sigma_t^2 x^2)$$

for all x and t. Grouping the terms,

$$\frac{\sigma_t^2}{2\gamma_t}x^3 - \frac{a_t\sigma_t^2}{2\gamma_t}x^2 + \Big(\mu_t - \sigma_t^2 - \frac{\gamma_t'}{2\gamma_t}\Big)x + \frac{a_t\gamma_t'}{2\gamma_t} - a_t' = 0,$$

thus, we must have

$$\sigma_t^2 = 0\ , \quad \mu_t = \frac{\gamma_t'}{2\gamma_t}\ , \quad a_t' = \mu_t a_t .$$

The last two equations give

$$\gamma_t = \gamma_0 \exp\Big(2\int_0^t \mu_s ds\Big) \qquad \text{and} \qquad a_t = a_0 \exp\Big(\int_0^t \mu_s ds\Big).$$

Note that the a_t and γ_t are consistent with the property of $X_t = X_0 e^{\int_0^t \mu_s ds}$ when X_0 is Normally distributed.

Example 3. Suppose $\mu(x,t) = \mu x$ and $\sigma(x,t) = \sigma$, where μ and σ are constant, i.e.

$$dX_t = \mu X_t dt + \sigma dB_t$$

and X is the Ornstein Uhlenbeck process. Under the assumption of Normal marginals, Proposition 2 gives

$$\mu x = a_t' + \frac{x-a_t}{2\gamma_t}\left(\gamma_t' - \sigma^2\right).$$

Thus, we must have

$$\mu - \frac{\gamma_t' - \sigma^2}{2\gamma_t} = 0 \qquad \text{and} \qquad a_t' - \frac{a_t(\gamma_t' - \sigma^2)}{2\gamma_t} = 0.$$

Solving the two equations, we arrive at

$$a_t = a_0 e^{\mu t} \qquad \text{and} \qquad \gamma_t = e^{2\mu t}\Big(\gamma_0 + \frac{\sigma^2}{2\mu}(1 - e^{-2\mu t})\Big).$$

In this case equation (13) gives also sufficient conditions, because

$$X_t = e^{\mu t}\Big(X_0 + \int_0^t \sigma e^{-\mu s} dB_s\Big)$$

is a Gaussian process.

Example 4 (Fake Brownian motion diffusions). Here we give an example of a family of diffusions (indexed by σ) which are all Gaussian processes with Brownian marginals $N(0,t)$ but not a Brownian motion unless $\sigma^2 = 1$.

Let $\sigma > 0$ and $\alpha = \frac{1}{2}(\sigma^2 - 3)$. Define $(X_t)_{t\geq 0}$ by

$$X_t = \sigma t^{-(\alpha+1)} \int_0^t s^{\alpha+1} dB_s, \quad X_0 = 0.$$

Note that the Ito integral is well defined because $\alpha = -\frac{3}{2} + \frac{\sigma^2}{2} > -\frac{3}{2}$. For any $t > 0$, $\mathbb{E}[X_t] = 0$ and

$$Var(X_t) = \sigma^2 t^{-2(\alpha+1)} \int_0^t s^{2(\alpha+1)} ds = t$$

as $2\alpha + 3 = \sigma^2$. It is clear that X is continuous at 0 for $\alpha \leq -1$. From the calculation of variance, it follows that X is continuous at 0 in L^2 also for $\alpha > -1$. Thus, the process $(X_t)_{t\geq 0}$ is continuous at 0 for any $\sigma > 0$.

Note that X is a Gaussian process, with marginals $N(0,t)$. Obtaining the SDE of X, we have

$$dX_t = \frac{1-\sigma^2}{2t} X_t dt + \sigma dB_t.$$

With

$$a_t = 0, \quad \gamma_t = t, \quad \mu(x,t) = \frac{1-\sigma^2}{2t} x, \quad \sigma(x,t) = \sigma,$$

we see that equation (13) is indeed satisfied.

The process X is a Brownian motion only for $\sigma = 1$ (or $\alpha = -1$). For $\sigma \neq 1$, X is a martingale multiplied by a monotone function.

Remark that the covariance function, which also defines the process uniquely, is given by, for $u < t$,

$$Cov(X_u, X_t) = \sigma^2 (ut)^{-(\alpha+1)} \int_0^u s^{2\alpha+2} ds = u^{\alpha+2} t^{-\alpha-1}.$$

When $\sigma^2 = 1$ (or $\alpha = -1$), the above covariance function is that of Brownian motion.

4.2 Diffusion with Lognormal marginals

Suppose that the marginal distributions of X are $LN(a_t, \gamma_t)$. Let $C(x,t) = \mathbb{E}[(X_t - x)^+]$. The function C and its derivatives can be computed as follows:

$$\begin{aligned} C(x,t) &= e^{a_t+\frac{1}{2}\gamma_t}\Phi\Big(\frac{a_t+\gamma_t-\ln x}{\sqrt{\gamma_t}}\Big) - x\Phi\Big(\frac{a_t-\ln x}{\sqrt{\gamma_t}}\Big), \\ \frac{\partial C}{\partial x}(x,t) &= -\Phi\Big(\frac{a_t-\ln x}{\sqrt{\gamma_t}}\Big), \\ \frac{\partial^2 C}{\partial x^2}(x,t) &= \frac{1}{x\sqrt{\gamma_t}}\phi\Big(\frac{a_t-\ln x}{\sqrt{\gamma_t}}\Big), \\ \frac{\partial^2 C}{\partial x\partial t}(x,t) &= \frac{(a_t-\ln x)\gamma_t' - 2\gamma_t a_t'}{2\gamma_t^{3/2}}\phi\Big(\frac{a_t-\ln x}{\sqrt{\gamma_t}}\Big), \\ \frac{\partial^3 C}{\partial x^3}(x,t) &= \frac{a_t-\gamma_t-\ln x}{x^2\gamma_t^{3/2}}\phi\Big(\frac{a_t-\ln x}{\sqrt{\gamma_t}}\Big). \end{aligned}$$

Proposition 3. *Assume that $(X_t)_{t\geq 0}$ has marginals $LN(a_t,\gamma_t)$ and satisfies the SDE* (2)*. Then we must have*

$$\mu(x,t) = \frac{1}{2}\frac{\partial}{\partial x}\big\{\sigma^2(x,t)\big\} + \frac{a_t-\gamma_t-\ln x}{2x\gamma_t}\sigma^2(x,t) - \frac{x(a_t-\ln x)\gamma_t'}{2\gamma_t} + xa_t'.$$

Proof. This follows from (8). □

Example 5. Suppose that $\mu(x,t)=0$ and $a_t=a_0$. Under the assumption of Lognormal marginals, the diffusion coefficient must satisfy

$$x\gamma_t\frac{\partial}{\partial x}\big\{\sigma^2(x,t)\big\} + (a_0-\gamma_t-\ln x)\sigma^2(x,t) - x^2(a_0-\ln x)\gamma_t' = 0,$$

due to Proposition 3. Suppose $\sigma(0,t)=0$. Solving for $\sigma^2(x,t)$, we obtain

$$\begin{aligned} \sigma^2(x,t) &= \gamma_t' x^{2-\ln x/(2\gamma_t)}\exp\Big(\frac{(\ln x)^2}{2\gamma_t}\Big) \\ &\qquad - \sqrt{2\pi\gamma_t}\gamma_t' x^{1-a_0/\gamma_t}\exp\Big(\frac{(a_0-\gamma_t)^2+(\ln x)^2}{2\gamma_t}\Big)\Phi\Big(\frac{\ln x - a_0-\gamma_t}{\sqrt{\gamma}}\Big) \end{aligned}$$

as a necessary condition for matching the marginal distributions. In particular, if $a_0=0$ and $\gamma_t=t$, we must have

$$\sigma^2(x,t) = x^{2-\ln x/(2t)}\exp\Big(\frac{(\ln x)^2}{2t}\Big) - \sqrt{2\pi t}x\exp\Big(\frac{t^2+(\ln x)^2}{2t}\Big)\Phi\Big(\frac{\ln x - t}{\sqrt{t}}\Big)$$

for the SDE $dX_t=\sigma(X_t,t)dB_t$ to have the $LN(0,t)$ marginal distributions.

Example 6. Suppose $\mu(x,t)=\mu_t x$ and $\sigma(x,t)=\sigma_t x$, where μ_t and σ_t are functions of t only, i.e.

$$dX_t = \mu_t X_t dt + \sigma_t X_t dB_t.$$

Under the Lognormal regime, Proposition 3 gives

$$\mu_t x = x\sigma_t^2 + \frac{a_t - \gamma_t - \ln x}{2\gamma_t} x\sigma_t^2 - \frac{x(a_t - \ln x)\gamma_t'}{2\gamma_t} + xa_t',$$

that is,

$$\mu_t - a_t' - \Big(\frac{1}{2} + \frac{a_t}{2\gamma_t}\Big)\sigma_t^2 + \frac{a_t}{2\gamma_t}\gamma_t' = \Big(\frac{\gamma_t'}{2\gamma_t} - \frac{\sigma_t^2}{2\gamma_t}\Big)\ln x.$$

Therefore, we must have

$$\sigma_t^2 = \gamma_t' \qquad \text{and} \qquad \mu_t - a_t' - \frac{1}{2}\sigma_t^2 = 0. \tag{15}$$

Checking this with the solution to the SDE,

$$X_t = X_0 e^{\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2) ds + \int_0^t \sigma_s dB_s},$$

which has marginal distribution

$$LN\bigg(\ln x_0 + \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\Big) ds, \int_0^t \sigma_s^2 ds\bigg),$$

(15) is indeed satisfied.

Example 7 (Fake Lognormal diffusions). Let $Y_t = e^{X_t}$, where X is the process in Example 4. Then Y has marginal distribution $LN(0,t)$ and

$$\begin{aligned} dY_t &= e^{X_t} dX_t + \frac{1}{2} e^{X_t} (dX_t)^2 \\ &= \sigma Y_t dB_t + \Big(\frac{1}{2}\sigma^2 - (\alpha+1)\frac{1}{t}\ln Y_t\Big) Y_t dt. \end{aligned}$$

That is, $\mu(x,t) = \big(\frac{1}{2}\sigma^2 - (\alpha+1)\frac{1}{t}\ln x\big)x$ and $\sigma(x,t) = \sigma x$. We can see that, as $\alpha + 1 = \frac{\sigma^2 - 1}{2}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial x}\big\{\sigma^2(x,t)\big\} + \frac{a_t - \gamma_t - \ln x}{2x\gamma_t}\sigma^2(x,t) - \frac{x(a_t - \ln x)\gamma_t'}{2\gamma_t} + xa_t' \\ &= \frac{1}{2}\sigma^2 x + (1-\sigma^2)\frac{x\ln x}{2t} \\ &= \mu, \end{aligned}$$

which is consistent with Proposition 3.

4.3 Selfsimilar diffusion

In a more general case, suppose that a diffusion process $(X_t)_{t\geq 0}$, satisfying SDE (2), is selfsimilar. Its marginal densities $(p_t(\cdot))_{t\geq 0}$ satisfies the scaling property $p_t(y) =$

$t^{-\kappa}p_1(yt^{-\kappa})$ for all t, for some $\kappa > 0$. We also have

$$\begin{aligned}\frac{\partial C}{\partial x} &= -\mathbb{P}(X_t \geq x) = \mathbb{P}(X_t < x) - 1 = \mathbb{P}(X_1 < xt^{-\kappa}) - 1 \\ &= \int_{-\infty}^{xt^{-\kappa}} p_1(y)dy = \int_{-\infty}^{t} -\kappa xy^{-\kappa-1} p_1(xy^{-\kappa})dy\end{aligned}$$

and

$$\frac{\partial^2 C}{\partial x \partial t} = -\kappa xt^{-\kappa-1} p_1(xt^{-\kappa}).$$

Then, Equation (8) simplifies to

$$\kappa xt^{-1} p_1(xt^{-\kappa}) + \left(\frac{1}{2}\frac{\partial}{\partial x}\{\sigma^2(x,t)\} - \mu(x,t)\right) p_1(xt^{-\kappa}) + \frac{1}{2}\sigma^2(x,t)t^{-\kappa} p_1'(xt^{-\kappa}) = 0,$$

or,

$$\kappa xt^{-1} p_t(x) + \left(\frac{1}{2}\frac{\partial}{\partial x}\{\sigma^2(x,t)\} - \mu(x,t)\right) p_t(x) + \frac{1}{2}\sigma^2(x,t)\frac{\partial}{\partial x}\{p_t(x)\} = 0.$$

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