On Autoregressive Measurement Errors in a Two-Factor Model

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Abstract In this study, we consider the extended two-factor model, originally introduced by Schwartz and Smith (2000), which has been commonly used for the pricing of commodity derivatives. In this model setup, we assume the latent short and long-term factors represent correlated mean-reverting processes. We develop a Kalman filter for jointly estimating the state variables and unknown parameters. In the measurement equation system, we assume that the residuals are serially correlated and inter-dependent. We derive the Kalman filter under the assumption that these residuals are AR(p) processes. Then the Kalman filter is used for obtaining the marginalised likelihood estimators for the state variables and the model parameters. We provide an extensive, reproducible simulation study for examining the convergence of the estimators in our proposed model. The `MATLAB` code for this study is available via the link to GitHub[1].

1 Introduction

Commodity markets are a crucial part of the global economy, as they enable market participants and industries to manage price risk and ensure efficient allocation of available resources. Challenges arise in the pricing of commodity derivatives due

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[1] https://github.com/Junee1992/EUA-Futures-Pricing/tree/main/MATRIX

to the complex nature of the market, including supply and demand fluctuations, geopolitical risks, and climate risks. Over the years various pricing models have been developed for capturing the complexities of the market dynamics.

The factor models provided a convenient framework for pricing derivatives written on commodities with unobservable spot prices. Study in [4] proposed a reduced form model, a single factor model that models the spot price of a commodity using the geometric Brownian motion, with the convenience yield being a deterministic function of the commodity price and a constant interest rate. The model was extended in [10] and [14], allowing stochasticity in the convenience yield and incorporating volatilities of historical returns on futures contracts of different maturities. Comparative analysis was conducted in [25], assessing reduced-form models that are comprised of one to three factors, starting from the spot price following a mean-reverting process, and subsequently adding stochastic convenience yields and interest rates.

The Schwartz-Smith two-factor model was introduced in [24] and since then is widely used in the pricing of commodities. In fact, the authors acknowledged that their approach is formally equivalent to their pioneering pricing framework in [14], where the stochastic yield factor was considered. The spot price of a commodity in [24] is decomposed into two distinct unobservable factors, representing short-term deviations and a long-term equilibrium price level. The short-term factor is assumed to converge to zero over time, reflecting the temporary effects of fluctuations in supply, demand, and market conditions. Although the long-term factor was originally assumed to follow a Brownian motion with drift, often it is also assumed to follow a mean-reverting process, see [8] and the references therein.

In this paper, we will study the joint estimation problem of the two latent variables, as well as model parameters, by establishing the likelihood estimation procedure using the Kalman filter in the extended setting. In previous works the two variants of Schwartz-Smith (2000) model have been calibrated to various futures markets such as the copper market (see [15], [23]), the crude oil market (see [7], [3], [8]), and also the carbon emission allowance market (see [1], [16]).

As supply and demand for certain commodities are influenced by the time of the year, the deterministic functions representing seasonal components were introduced in the model framework. Some common functions include a step function to model seasonal effects for different calendar months, and the sum of sinusoids (see studies in [18], [21], [27]). Stochastic jumps were incorporated in a short-term factor in [29] and [26] for modelling sudden changes in the market in the data-driven setting. The generalisation of the Schwartz-Smith Two-Factor model has been designed in [18] and [9], allowing for more than two factors that follow joint mean-reverting processes as well as introducing time-varying economic factors in the pricing model. Study in [6] developed a pricing model for long-dated commodity derivatives, allowing for both stochastic volatilities and interest rates to be correlated. In [30]. In this study, the risk premium followed a stochastic process, and its performance was empirically analysed using the European Union Allowance futures (EUA) and options prices.

The joint estimation of state variables and model parameters has several limitations. The study in [13] pointed out the identifiability problem due to the invertibility issue of the observed Fisher information matrix of model parameters. An additional constraint was proposed in [3] for rectifying the parameter identification problem within the maximum likelihood estimation procedure using Kalman filter approach. The filtering process itself was also limited to a set of standard assumptions on measurement errors, e.g. Gaussian white noise. The study in [17] relaxed the assumption of independence of the state and measurement error processes, and allowed the measurement errors to follow a GARCH(1,1) process to incorporate the heteroscedastic feature of commodity spot prices. For relaxing the assumption of independence between different futures contracts, [27] has taken an approach of estimating covariances, by defining each element as a parametrised function of time-to-maturity, but no substantial improvement was made in parameter estimation. The study in [11] has demonstrated through numerical simulation, that augmentation methods do not work well for the estimation of the covariance matrix of measurement errors. In [12], the effects on parameter estimation were studied in the framework when the option prices were used instead of futures prices; the extended Kalman filter was deployed for the estimation of state variables in a nonlinear system. Other than financial application, estimation of dependence component-wise is of high importance in geoscience, and the ensemble Kalman filter is usually adopted for developing various methods of estimation of the covariance matrix of measurement errors, (see [20], [28]).

In [25], the correlations of log prices of commodity contracts were modelled through the correlated bivariate latent state variables. The empirical studies have shown that often the observed measurement equation residuals exhibit significant correlations, necessitating a closer investigation of their sources. In a recent study, [16] proposed a two-step approach to joint estimation of model parameters and inter-correlations of the errors in measurement equations for the contracts with different maturities. For the incorporation of inter-correlated measurement errors and AR(1) serial correlation in each error series, a two-step approach was used, first estimating the model parameters via maximising the marginalised likelihood process and subsequently fitting an autoregressive model to the model residuals.

In difference from [16], here we propose a unified estimation procedure for the estimation of serially correlated series in the form of AR(p) of any order $p \geq 1$. The derivation of Kalman filter for AR(1) measurement errors can be found in [5] and [22]. The method suggested in this paper, can be used for improving out-of-sample commodity price forecasts over the method with any other reduced form of the setting of the measurement equations. In this paper, using a simulation study, we illustrate the convergence of the estimates of parameters in the Schwartz-Smith two-factor model by taking different numbers of contracts and sample sizes.

This paper is outlined as follows. In Section 2, we formulate the main model with a measurement equations system accommodating inter-dependent and serially correlated error terms. Under these assumptions on the error processes, we derive the Kalman filter. In Section 3, we present the results of the simulation study illustrating the convergence of the estimates to true values of model parameters for different practical scenarios. Section 4 concludes with a brief discussion of the results.

2 The Model

Consider the risk-neutral probability space denoted by $(\Omega, \Im, \mathbb{Q})$. We define the risk-neutral dynamics of the spot price S_t as $\ln S_t = \chi_t + \xi_t$, where χ_t and ξ_t are latent variables that are short-term and long-term factors, respectively, given through following stochastic differential equations

$$d\chi_t = (-\kappa\chi_t - \lambda_\chi)dt + \sigma_\chi dW_t^\chi, \tag{1}$$

$$d\xi_t = (\mu_\xi - \lambda_\xi - \gamma\xi_t)dt + \sigma_\xi dW_t^\xi, \tag{2}$$

$$\mathbb{E}_\mathbb{Q}[dW_t^\chi dW_t^\xi] = \rho_{\chi\xi} dt, \tag{3}$$

where $\kappa, \gamma \in \mathbb{R}^+$ are the rate of the mean-reversion for χ_t and ξ_t, respectively, λ_χ, $\lambda_\xi \in \mathbb{R}$ are deterministic unknown risk premia, σ_χ, $\sigma_\xi \in \mathbb{R}^+$ are the volatility parameters, and $\mu_\xi \in \mathbb{R}$ is the mean of the long-term factor. W_t^χ, W_t^ξ are correlated standard Brownian processes under $\mathbb{Q}$, where $\rho_{\chi\xi}$ is the correlation coefficient of the two processes. The futures price at time t, maturing in T years, is $F_{t,T} = \mathbb{E}[S_T|\Im_t]$.

In discrete time, we consider the following linear state space model

$$\mathbf{x}_t = \mathbf{c} + \mathbf{G}\mathbf{x}_{t-1} + \mathbf{w}_t, \ \ \mathbf{w}_t \sim \mathcal{N}(0, \mathbf{W}), \tag{4}$$

$$\mathbf{y}_t = \mathbf{d}_t + \mathbf{B}_t\mathbf{x}_t + \mathbf{v}_t, \tag{5}$$

$$\mathbf{v}_t = \Phi_p(L)\mathbf{v}_t + \varepsilon_t, \ \ \varepsilon_t \sim \mathcal{N}(0, \mathbf{V}_\varepsilon), \tag{6}$$

where, for $t = 1, 2, \cdots, n$, $\mathbf{x}_t$ and $\mathbf{y}_t$ are the vectors of latent and observable variables, respectively. Further, we assume that N futures contracts have the maturities $T_1, \cdots, T_N$, where $T_1 < T_2 < \cdots < T_N$, and

$$\mathbf{x}_t = \begin{pmatrix} \chi_t \\ \xi_t \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} 0 \\ \frac{\mu_\xi(1-e^{-\gamma\Delta t})}{\gamma} \end{pmatrix}, \ \mathbf{G} = \begin{pmatrix} e^{-\kappa\Delta t} & 0 \\ 0 & e^{-\gamma\Delta t} \end{pmatrix}, \tag{7}$$

$$\mathbf{y}_t = (\ln F_{t,T_1}, \ln F_{t,T_2}, \cdots, \ln F_{t,T_N})', \tag{8}$$

$$\mathbf{d}_t = (A(T_1 - t), A(T_2 - t), \cdots, A(T_N - t))', \tag{9}$$

$$\mathbf{B}_t = \begin{pmatrix} e^{-\kappa(T_1-t)} & \dots & e^{-\kappa(T_N-t)} \\ e^{-\gamma(T_1-t)} & \dots & e^{-\gamma(T_N-t)} \end{pmatrix}', \tag{10}$$

with $\Phi_p(L) = \sum_{r=1}^p \phi_r L^r$ being a function of the lag operator, Δt is the time difference in years between discretised time instances $t-1$ and t, and the function $A(T-t)$ is defined as

$$A(T-t) = -\frac{\lambda_\chi}{\kappa}\left(1-e^{-\kappa(T-t)}\right) + \frac{\mu_\xi - \lambda_\xi}{\gamma}\left(1-e^{-\gamma(T-t)}\right) + \frac{1}{2}\left(\frac{1-e^{-2\kappa(T-t)}}{2\kappa}\sigma_\chi^2 + \frac{2(1-e^{-(\kappa+\gamma)(T-t)})}{\kappa+\gamma}\sigma_\chi\sigma_\xi\rho_{\chi\xi} + \frac{1-e^{-2\gamma(T-t)}}{2\gamma}\sigma_\xi^2\right). \tag{11}$$

The covariance matrices of error terms $\mathbf{w}_t$ and ε_t are defined as

$$\mathbf{W} = \begin{pmatrix} \frac{1-e^{-2\kappa\Delta t}}{2\kappa}\sigma_\chi^2 & \frac{1-e^{-(\kappa+\gamma)\Delta t}}{\kappa+\gamma}\sigma_\chi\sigma_\xi\rho_{\chi\xi} \\ \frac{1-e^{-(\kappa+\gamma)\Delta t}}{\kappa+\gamma}\sigma_\chi\sigma_\xi\rho_{\chi\xi} & \frac{1-e^{-2\gamma\Delta t}}{2\gamma}\sigma_\xi^2 \end{pmatrix}, \quad \mathbf{V}_\varepsilon = \begin{pmatrix} s_{11}^2 & s_{12} & \cdots & s_{1N} \\ s_{12} & s_{22}^2 & \cdots & s_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1N} & s_{2N} & \cdots & s_{NN}^2 \end{pmatrix}. \tag{12}$$

Applying the lag polynomial $I - \Phi_p(L)$ to (5), where I is the identity matrix, the measurement equation is expressed as

$$\mathbf{y}_t = \mathbf{d}_t^*(p) + \mathbf{B}_t^*(p)\mathbf{x}_t + \mathbf{v}_t^*(p), \tag{13}$$

where

$$\mathbf{d}_t^*(p) = \sum_{r=1}^{p}\phi_r\mathbf{y}_{t-r} + \mathbf{d}_t - \sum_{r=1}^{p}\phi_r\mathbf{d}_{t-r} + \sum_{r=1}^{p}\phi_r\mathbf{B}_{t-r}\left[\sum_{k=1}^{r}(\mathbf{G}^{-1})^k\right]\mathbf{c}, \tag{14}$$

$$\mathbf{B}_t^*(p) = \mathbf{B}_t - \sum_{r=1}^{p}\phi_r\mathbf{B}_{t-r}(\mathbf{G}^{-1})^r, \tag{15}$$

$$\mathbf{v}_t^*(p) = \sum_{r=1}^{p}\phi_r\mathbf{B}_{t-r}\left[\sum_{k=1}^{r}(\mathbf{G}^{-1})^{r-k+1}\mathbf{w}_{t-k+1}\right] + \varepsilon_t. \tag{16}$$

Proof. We wish to prove that (14) - (16) are true for forming the measurement equation (13) when measurement errors marginally follow AR(p) processes. For $p = 1$, we consider the following linear state-space model

$$\mathbf{x}_t = \mathbf{c} + \mathbf{G}\mathbf{x}_{t-1} + \mathbf{w}_t, \;\; \mathbf{w}_t \sim \mathcal{N}(0, \mathbf{W}), \tag{17}$$

$$\mathbf{y}_t = \mathbf{d}_t + \mathbf{B}_t \mathbf{x}_t + \mathbf{v}_t, \tag{18}$$

$$\mathbf{v}_t = \phi \mathbf{v}_{t-1} + \varepsilon_t, \;\; \varepsilon_t \sim \mathcal{N}(0, \mathbf{V}_\varepsilon). \tag{19}$$

Let $\Phi(L) = \phi_1 L$. By applying $I - \phi_1 L$ to equation (18), and using equation (17), we obtain

$$(I - \Phi(L))\mathbf{y}_t = (I - \Phi(L))\mathbf{d}_t + (I - \Phi(L))\mathbf{B}_t \mathbf{x}_t + (I - \Phi(L))\mathbf{v}_t,$$

which can be extended to

$$\mathbf{y}_t = \phi \mathbf{y}_{t-1} + \mathbf{d}_t - \phi_1 \mathbf{d}_{t-1} + (\mathbf{B}_t - \phi_1 \mathbf{B}_{t-1}\mathbf{G}^{-1})\mathbf{x}_t + \phi_1 \mathbf{B}_{t-1}\mathbf{G}^{-1}(\mathbf{c} + \mathbf{w}_t) + \varepsilon_t.$$

Hence,

$$\mathbf{y}_t = \mathbf{d}_t^*(1) + \mathbf{B}_t^*(1)\mathbf{x}_t + \mathbf{v}_t^*(1),$$

where

$$\begin{aligned}
\mathbf{d}_t^*(1) &= \phi_1 \mathbf{y}_{t-1} + \mathbf{d}_t - \phi_1 \mathbf{d}_{t-1} + \phi_1 \mathbf{B}_{t-1}\mathbf{G}^{-1}\mathbf{c}, \\
\mathbf{B}_t^*(1) &= \mathbf{B}_t - \phi_1 \mathbf{B}_{t-1}\mathbf{G}^{-1}, \\
\mathbf{v}_t^*(1) &= \phi_1 \mathbf{B}_{t-1}\mathbf{G}^{-1}\mathbf{w}_t + \varepsilon_t.
\end{aligned}$$

Therefore, (14) - (16) are obvious for $p = 1$. Now we assume that (13) - (16) holds for $p = m$. Then for $p = m + 1$ we obtain

$$\begin{aligned}
\mathbf{y}_t &= \phi_1 \mathbf{y}_{t-1} + \cdots + \phi_m \mathbf{y}_{t-m} + \phi_{m+1}\mathbf{y}_{t-(m+1)} \\
&\quad + \mathbf{d}_t - \phi_1 \mathbf{d}_{t-1} - \cdots - \phi_m \mathbf{d}_{t-m} - \phi_{m+1}\mathbf{d}_{t-(m+1)} \\
&\quad + \mathbf{B}_t \mathbf{x}_t - \phi_1 \mathbf{B}_{t-1}\mathbf{x}_{t-1} - \cdots - \phi_m \mathbf{B}_{t-m}\mathbf{x}_{t-m} - \phi_{m+1}\mathbf{B}_{t-(m+1)}\mathbf{x}_{t-(m+1)} + \varepsilon_t \\
&= \mathbf{d}_t^*(m) + \phi_{m+1}\mathbf{y}_{t-(m+1)} - \phi_{m+1}\mathbf{d}_{t-(m+1)} + \mathbf{B}_t^*(m)\mathbf{x}_t - \phi_{m+1}\mathbf{B}_{t-(m+1)}\mathbf{x}_{t-(m+1)} \\
&\quad + \mathbf{v}_t^*(m) \\
&= \mathbf{d}_t^*(m+1) + \mathbf{B}_t^*(m+1)\mathbf{x}_t + \mathbf{v}_t^*(m+1).
\end{aligned}$$

Hence, (14) - (16) are proven true for all $p \geq 1$. $\square$

Given the first p observations, the prediction error for $t > p$ is defined as

$$\begin{aligned}
\mathbf{e}_{t|t-1}^* &= \mathbf{y}_t - \mathbb{E}[\mathbf{y}_t | \Im_{t-1}] \\
&= \mathbf{y}_t - \mathbf{d}_t^*(p) - \mathbf{B}_t^*(p)\mathbb{E}[\mathbf{x}_t | \Im_{t-1}] - \mathbb{E}[\mathbf{v}_t^*(p) | \Im_{t-1}],
\end{aligned} \tag{20}$$

and we assume $\mathbf{w}_t$ and ε_t are mutually independent, and serially uncorrelated. For $p = 1$, we have $\mathbb{E}[\mathbf{v}_t^*(1) | \Im_{t-1}] = 0$, since $\mathbb{E}[\mathbf{w}_t | \Im_{t-1}] = 0$. For $p > 1, \mathbb{E}[\mathbf{v}_t^*(p) | \Im_{t-1}]$ is defined as

$$\mathbb{E}[\mathbf{v}_t^*(p)|\Im_{t-1}] = \sum_{r=2}^{p} \phi_r \mathbf{B}_{t-r} \left[\sum_{k=2}^{r} (\mathbf{G}^{-1})^{r-k+1} \mathbb{E}[\mathbf{w}_{t-k+1}|\Im_{t-1}] \right],$$

and the conditional expectation of $\mathbf{w}_{t-k+1}|\Im_{t-1}$ can be estimated as

$$\mathbb{E}[\mathbf{w}_{t-k+1}|\Im_{t-1}] = \mathbb{E}[\mathbf{x}_{t-k+1}|\Im_{t-1}] - \mathbf{c} - \mathbf{G}\mathbb{E}[\mathbf{x}_{t-k}|\Im_{t-1}],$$

with $\mathbb{E}[\mathbf{x}_{t-k}|\Im_{t-1}]$ that can be estimated via the Kalman smoother for $k > 1$. We use the following recursive formulas to obtain estimates of smoother,

$$\begin{aligned}
\mathbb{E}[\mathbf{x}_{t-k}|\Im_{t-1}] &= \mathbb{E}[\mathbf{x}_{t-k}|\Im_{t-k}] + \mathbf{J}_{t-k}(\mathbb{E}[\mathbf{x}_{t-k+1}|\Im_{t-1}] - \mathbb{E}[\mathbf{x}_{t-k+1}|\Im_{t-k}]),\\
Var[\mathbf{x}_{t-k}|\Im_{t-1}] &= Var[\mathbf{x}_{t-k}|\Im_{t-k}] + \mathbf{J}_{t-k}(Var[\mathbf{x}_{t-k+1}|\Im_{t-1}] - Var[\mathbf{x}_{t-k+1}|\Im_{t-k}])\mathbf{J}'_{t-k},\\
\mathbf{J}_{t-k} &= Var[\mathbf{x}_{t-k}|\Im_{t-k}]\mathbf{G}'Var[\mathbf{x}_{t-k+1}|\Im_{t-k}]^{-1}.
\end{aligned}$$

Let $\mathbf{P}_{t|t-1} = Var[\mathbf{x}_t|\Im_{t-1}]$ be the variance of state prediction error for the next state. The variance of the prediction error is

$$\begin{aligned}
Var[\mathbf{e}^*_{t|t-1}] &= Var[\mathbf{y}_t|\Im_{t-1}] = \mathbf{L}^*_{t|t-1} = Var[\mathbf{B}_t^*(p)\mathbf{x}_t + \mathbf{v}_t^*(p)|\Im_{t-1}]\\
&= \mathbf{B}_t^*(p)\mathbf{P}_{t|t-1}[\mathbf{B}_t^*(p)]' + \mathbf{V}^*_{t|t-1}(p)\\
&+ \mathbf{B}_t^*(p)\mathbf{W}\left[\sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}(\mathbf{G}^{-1})^r\right] + \left[\sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}(\mathbf{G}^{-1})^r\right]\mathbf{W}'\mathbf{B}_t^*(p)\\
&= \mathbf{B}_t^*(p)\mathbf{P}_{t|t-1}[\mathbf{B}_t^*(p)]' + \mathbf{V}^*_{t|t-1}(p) + \mathbf{B}_t^*(p)\mathbf{C}_t^*(p) + [\mathbf{C}_t^*(p)]'[\mathbf{B}_t^*(p)]'.
\end{aligned} \tag{21}$$

where $\mathbf{C}_t^*(p)$ is the covariance of the two error processes $\mathbf{w}_t, \mathbf{v}_t^*(p)$, as $\mathbf{w}_t$ is a function of $\mathbf{v}_t^*(p)$, and is defined as

$$\mathbf{C}_t^*(p) = \mathbf{W}\left(\sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}(\mathbf{G}^{-1})^r\right)' \tag{22}$$

The conditional variance of $\mathbf{v}_t^*(p)$ is

$$Var[\mathbf{v}_t^*(p)|\Im_{t-1}] = \mathbf{V}^*_{t|t-1}(p) = \left[\sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}(\mathbf{G}^{-1})^r\right]\mathbf{W}\left[\sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}(\mathbf{G}^{-1})^r\right]' + \mathbf{V}_\varepsilon. \tag{23}$$

Proof. Assuming $\mathbf{w}_t, \varepsilon_t$ are mutually independent and serially uncorrelated, the variance of the prediction error is

$$\begin{aligned}
\mathbf{L}^*_{t|t-1} &= Var[\mathbf{e}^*_{t|t-1}] = Var[\mathbf{B}^*_t(p)\mathbf{x}_t + \mathbf{v}^*_t(p)|\Im_{t-1}] \\
&= \mathbb{E}[(\mathbf{B}^*_t(p)\mathbf{x}_t + \mathbf{v}^*_t(p))(\mathbf{B}^*_t(p)\mathbf{x}_t + \mathbf{v}^*_t(p))'|\Im_{t-1}] \\
&- \mathbb{E}[\mathbf{B}^*_t(p)\mathbf{x}_t + \mathbf{v}^*_t(p)|\Im_{t-1}]\mathbb{E}[(\mathbf{B}^*_t(p)\mathbf{x}_t + \mathbf{v}^*_t(p))'|\Im_{t-1}] \\
&= \mathbf{B}^*_t(p)\mathbf{P}_{t|t-1}\mathbf{B}^{*\prime}_t(p) + \mathbf{V}^*_{t|t-1}(p) + \mathbf{B}^*_t(\mathbb{E}[\mathbf{x}_t[\mathbf{v}^*_t(p)]'|\Im_{t-1}] \\
&- \mathbb{E}[\mathbf{x}_t|\Im_{t-1}]\mathbb{E}[\mathbf{v}^*_t(p)|\Im_{t-1}]) + (\mathbb{E}[\mathbf{v}^*_t(p)\mathbf{x}_t{}'|\Im_{t-1}] \\
&- \mathbb{E}[\mathbf{v}^*_t(p)|\Im_{t-1}]\mathbb{E}[\mathbf{x}_t|\Im_{t-1}])[\mathbf{B}^*_t(p)]'|\Im_{t-1}] \\
&= \mathbf{B}^*_t(p)\mathbf{P}_{t|t-1}\mathbf{B}^{*\prime}_t(p) + \mathbf{V}^*_{t|t-1}(p) + \mathbf{B}^*_t(p)Cov(\mathbf{x}_t, \mathbf{v}^*_t(p)|\Im_{t-1}) \\
&+ Cov(\mathbf{v}^*_t(p), \mathbf{x}^{*\prime}_t|\Im_{t-1})'[\mathbf{B}^*_t(p)]',
\end{aligned}$$

where $\mathbf{P}_{t|t-1} = Var[\mathbf{x}_t|\Im_{t-1}]$. Since the new measurement error $\mathbf{v}^*_t(p)$ is a function of $\mathbf{w}_t$,

$$\begin{aligned}
\mathbf{C}^*_t(p) = Cov(\mathbf{x}_t, \mathbf{v}^*_t(p)|\Im_{t-1}) &= Cov(\mathbf{w}_t, \sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}(\mathbf{G}^{-1})^r \mathbf{w}_t|\Im_{t-1}) \\
&= \mathbf{W}\left(\sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}(\mathbf{G}^{-1})^r\right)',
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{V}^*_{t|t-1}(p) &= Var[\mathbf{v}^*_t(p)|\Im_{t-1}] \\
&= Var\left[\sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}\left[\sum_{k=1}^{r} (\mathbf{G}^{-1})^{r-k+1}\mathbf{w}_{t-k+1}\right] + \varepsilon_t \middle| \Im_{t-1}\right] \\
&= \sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}\left[\sum_{k=1}^{r} (\mathbf{G}^{-1})^k \mathbf{W}[(\mathbf{G}^{-1})^k]'\right]\mathbf{B}'_{t-r}\phi'_r + \mathbf{V}_\varepsilon \\
&= \left[\sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}(\mathbf{G}^{-1})^r\right]\mathbf{W}\left[\sum_{r=1}^{p} \phi_r \mathbf{B}_{t-r}(\mathbf{G}^{-1})^r\right]' + \mathbf{V}_\varepsilon,
\end{aligned}$$

with $Var[\mathbf{w}_{t-k+1}|\Im_{t-1}] = 0$ for any $k > 1$. □

Lastly, the optimal Kalman gain matrix $\mathbf{K}^*_t$ is

$$\mathbf{K}^*_t = (\mathbf{P}_{t|t-1}[\mathbf{B}^*_t(p)]' + \mathbf{C}^*_t(p))(\mathbf{L}^*_{t|t-1} + \mathbf{B}^*_t(p)\mathbf{C}^*_t(p) + [\mathbf{C}^*_t(p)]'[\mathbf{B}^*_t(p)]')^{-1}, \quad (24)$$

which is used to correct and forecast the distribution of the state vectors for $t > p$.

Proof. Defining $\mathbf{a}_{t|t-1} = \mathbb{E}[\mathbf{x}_t|\Im_{t-1}]$ and $\mathbf{v}^*_{t|t-1}(p) = \mathbb{E}[\mathbf{v}^*_t(p)|\Im_{t-1}]$,

$$\begin{aligned}\mathbf{a}_t &= \mathbb{E}[\mathbf{x}_t|\Im_t] = \mathbf{a}_{t|t-1} + \mathbf{K}^*_t\mathbf{e}^*_{t|t-1}\\ &= \mathbf{a}_{t|t-1} + \mathbf{K}^*_t[\mathbf{B}^*_t(p)\mathbf{x}_t + \mathbf{v}^*_t(p) - \mathbf{B}^*_t(p)\mathbf{a}_{t|t-1} - \mathbf{v}^*_{t|t-1}(p)].\end{aligned}$$

and the state estimation error η_t is

$$\begin{aligned}\eta_t &= \mathbf{x}_t - \mathbf{a}_t\\ &= \mathbf{x}_t - \mathbf{a}_{t|t-1} - \mathbf{K}^*_t[\mathbf{B}^*_t(p)\mathbf{x}_t + \mathbf{v}^*_t(p) - \mathbf{B}^*_t(p)\mathbf{a}_{t|t-1} - \mathbf{v}^*_{t|t-1}(p)]\\ &= (\mathbf{I} - \mathbf{K}^*_t\mathbf{B}^*_t(p))(\mathbf{x}_t - \mathbf{a}_{t|t-1}) - \mathbf{K}^*_t(\mathbf{v}^*_t(p) - \mathbf{v}^*_{t|t-1}(p)).\end{aligned}$$

The covariance matrix of the state estimation error will be

$$\begin{aligned}\mathbf{P}_t &= \mathbb{E}[\eta_t\eta_t{}']\\ &= \mathbb{E}[((\mathbf{I} - \mathbf{K}^*_t\mathbf{B}^*_t(p))(\mathbf{x}_t - \mathbf{a}_{t|t-1})(\mathbf{x}_t - \mathbf{a}_{t|t-1})'(\mathbf{I} - \mathbf{K}^*_t\mathbf{B}^*_t(p))'\\ &- (\mathbf{I} - \mathbf{K}^*_t\mathbf{B}^*_t(p))(\mathbf{x}_t - \mathbf{a}_{t|t-1})(\mathbf{K}^*_t(\mathbf{v}^*_t(p) - \mathbf{v}^*_{t|t-1}(p)))'\\ &- \mathbf{K}^*_t(\mathbf{v}^*_t(p) - \mathbf{v}^*_{t|t-1}(p))(\mathbf{x}_t - \mathbf{a}_{t|t-1})'(\mathbf{I} - \mathbf{K}^*_t\mathbf{B}^*_t(p))'\\ &+ \mathbf{K}^*_t(\mathbf{v}^*_t(p) - \mathbf{v}^*_{t|t-1}(p))(\mathbf{K}^*_t(\mathbf{v}^*_t(p) - \mathbf{v}^*_{t|t-1}(p))'] \\ &= \mathbf{P}_{t|t-1} - \mathbf{K}^*_t\mathbf{B}^*_t(p)\mathbf{P}_{t|t-1} - \mathbf{P}_{t|t-1}[\mathbf{B}^*_t(p)]'\mathbf{K}^{*\prime}_t + \mathbf{K}_t\mathbf{B}^*_t(p)\mathbf{P}_{t|t-1}[\mathbf{B}^*_t(p)]'\mathbf{K}^{*\prime}_t\\ &- (\mathbf{I} - \mathbf{K}^*_t\mathbf{B}^*_t(p))\mathbf{C}^*_t(p)\mathbf{K}^{*\prime}_t - \mathbf{K}^*_t[\mathbf{C}^*_t(p)]'(\mathbf{I} - \mathbf{K}^*_t\mathbf{B}^*_t(p))' + \mathbf{K}^*_t\mathbf{V}^*_{t|t-1}\mathbf{K}^{*\prime}_t\\ &= \mathbf{P}_{t|t-1} - \mathbf{K}^*_t\mathbf{B}^*_t(p)\mathbf{P}_{t|t-1} - \mathbf{P}_{t|t-1}[\mathbf{B}^*_t(p)]'\mathbf{K}^{*\prime}_t\\ &+ \mathbf{K}^*_t(\mathbf{B}^*_t(p)\mathbf{P}_{t|t-1}[\mathbf{B}^*_t(p)]' + \mathbf{V}^*_{t|t-1})\mathbf{K}^{*\prime}_t - \mathbf{C}^*_t(p)\mathbf{K}^{*\prime}_t + \mathbf{K}^*_t\mathbf{B}^*_t(p)\mathbf{C}^*_t(p)\mathbf{K}^{*\prime}_t\\ &- \mathbf{K}^*_t[\mathbf{C}^*_t(p)]' + \mathbf{K}^*_t[\mathbf{C}^*_t(p)]'[\mathbf{B}^*_t(p)]'\mathbf{K}^{*\prime}_t.\end{aligned}$$

Applying trace to the above equation, we minimise the following first derivative in order to find the optimal Kalman gain matrix.

$$\begin{aligned}tr(\mathbf{P}_t) &= tr(\mathbf{P}_{t|t-1}) - 2tr(\mathbf{K}^*_t\mathbf{B}^*_t(p)\mathbf{P}_{t|t-1}) + tr(\mathbf{K}^*_t\mathbf{L}^*_{t|t-1}\mathbf{K}^{*\prime}_t)\\ &+ tr(\mathbf{K}^*_t\mathbf{B}^*_t(p)\mathbf{C}^*_t(p)\mathbf{K}^{*\prime}_t) + tr(\mathbf{K}^*_t[\mathbf{C}^*_t(p)]'[\mathbf{B}^*_t(p)]'\mathbf{K}^{*\prime}_t)\\ &- 2tr(\mathbf{K}^*_t[\mathbf{C}^*_t(p)]'),\\ \frac{d(tr(\mathbf{P}_t))}{d\mathbf{K}^*_t} &= -2(\mathbf{B}^*_t(p)\mathbf{P}_{t|t-1})' + 2\mathbf{K}^*_t\mathbf{L}^*_{t|t-1} + 2\mathbf{K}^*_t\mathbf{B}^*_t(p)\mathbf{C}^*_t(p)\\ &+ 2\mathbf{K}^*_t[\mathbf{C}^*_t(p)]'[\mathbf{B}^*_t(p)]' - 2\mathbf{C}^*_t(p) = 0.\end{aligned}$$

Therefore, the optimal Kalman gain matrix $\mathbf{K}^*_t$ is

$$\mathbf{K}^*_t = (\mathbf{P}_{t|t-1}[\mathbf{B}^*_t(p)]' + \mathbf{C}^*_t(p))(\mathbf{L}^*_{t|t-1} + \mathbf{B}^*(p)\mathbf{C}^*_t(p) + [\mathbf{C}^*_t(p)]'[\mathbf{B}^*_t(p)]')^{-1}. \quad \square$$

Now, assuming the prediction error $\mathbf{e}^*_{t|t-1}$ follows a multivariate normal distribution, we construct the marginalised likelihood based on the prediction errors

$$\ell_t(\psi, \mathbf{V}_\varepsilon, \phi; \mathbf{y}_t) = -\frac{1}{2}\ln(\det(\mathbf{L}^*_{t|t-1})) - \frac{1}{2}(\mathbf{e}^*_{t|t-1})'(\mathbf{L}^*_{t|t-1})^{-1}\mathbf{e}^*_{t|t-1}, \tag{25}$$

where $\psi = (\kappa, \sigma_\chi, \lambda_\chi, \gamma, \mu_\xi, \sigma_\xi, \lambda_\xi, \rho_{\chi\xi})$ is the set of model parameters and $\det(\cdot)$ denotes the determinant of the matrix. The sum of the marginalised likelihoods for $t > p$ is the likelihood function to be maximised for the estimation of the model parameters. That is,

$$\ell(\psi, \mathbf{V}_\varepsilon, \phi; \mathbf{y}_t) = -\frac{1}{2}\sum_{t=p+1}^{n}\left[\ln(\det(\mathbf{L}^*_{t|t-1})) + (\mathbf{e}^*_{t|t-1})'(\mathbf{L}^*_{t|t-1})^{-1}\mathbf{e}^*_{t|t-1}\right], \tag{26}$$

where, (26) is to be maximised subject to the constraint $\kappa \geq \gamma$ from [3]. As discussed in [3], this constraint is used to overcome the parameter identification problem in the state process $\mathbf{x}_t$. In addition, the code specifications and structure of the covariance matrix of the measurement errors, $\mathbf{V}_\varepsilon$, used are similar to that provided in [16].

3 Simulation Study

In this section, we present the results of the simulation study designed to address the convergence of the estimates of the model parameters and the state variables in the two scenarios, where the error terms in the system of measurement equations are inter-correlated and follow:

Model 1: AR(1) process (the two-step approach is used for estimation, [16]);
Model 2: AR(p) process (the unified procedure presented in Section 2, is used for estimation).

In this study, the system of measurement equations is formulated for the logarithms of the prices of futures contracts with varying maturity times. The samples of the futures prices (with the number of the contracts $N = 5, 10$) were simulated using Model 2 scenario with $p = 1$. The futures contracts have maturities ranging from 1 to 5 years. Specifically, for $N = 5$, we simulate annual contracts, while for $N = 10$, each vector of futures contracts matures biannually. To align with real market conditions, we took the model parameters from an empirical study on the EUA futures prices in [1] as the true values in our simulation study. In addition, to initialise the serial and inter-correlations of the measurement errors in Model 2 we use the results from Model 1, [16]. The long-term factor ξ_t is simulated assuming it follows a Brownian motion, equivalent to setting $\gamma = 0$.

Figures 1 - 2 present visualisations illustrating the convergence of the model parameter estimates by plotting the differences between the estimates and their true values. The figures correspond to the cases of $N = 5$ and $N = 10$, respectively. For $N = 5$, we observe that the estimates tend to exhibit fluctuations for sample sizes $n \leq 1000$. However, they start stabilising thereafter. In most cases, Model 2 demonstrates superior performance compared to Model 1, particularly when the sample size is large. Although there are minimal differences between the two models, except for

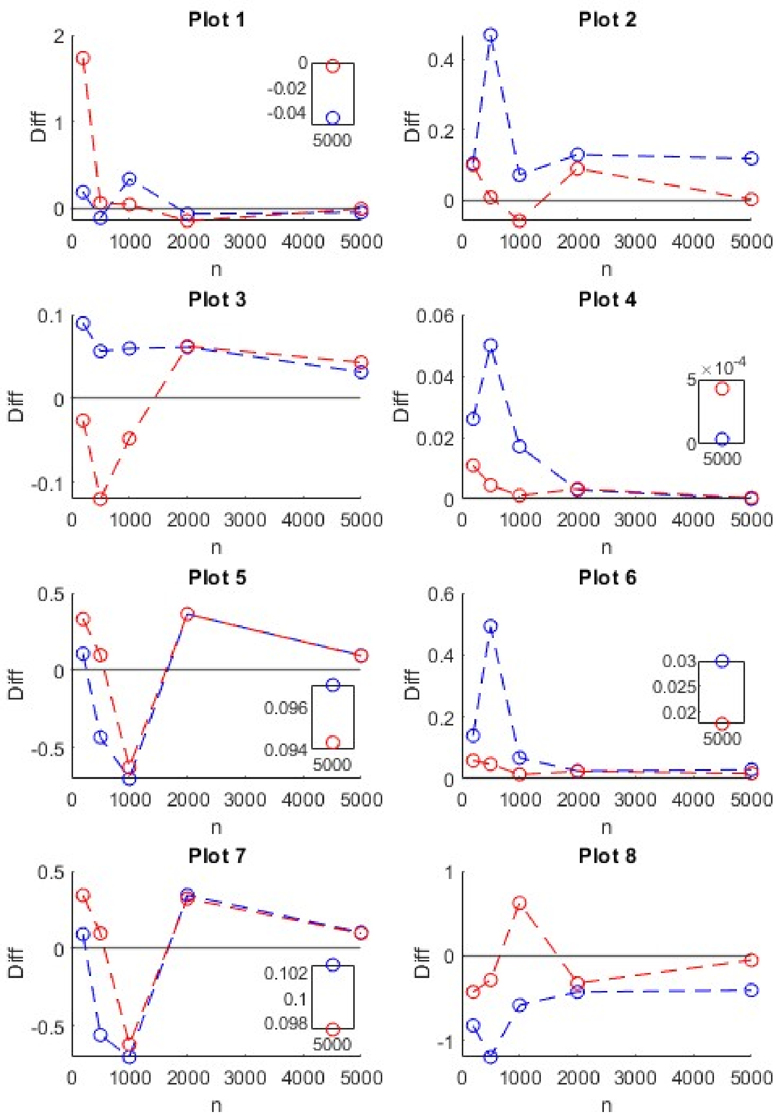

Fig. 1 Plots 1-8 show the difference of the estimated parameter to their true values in the following order - $\kappa, \sigma_\chi, \lambda_\chi, \gamma, \mu_\xi, \sigma_\xi, \lambda_\xi, \rho_{\chi\xi}$, when $N = 5$. The x-axis shows the sample size in the simulated dataset, while the y-axis shows the absolute difference. The blue line represents the result using Model 1, and the red line shows the result using Model 2. Insets are added at $n = 5,000$ for better comparison between two models.

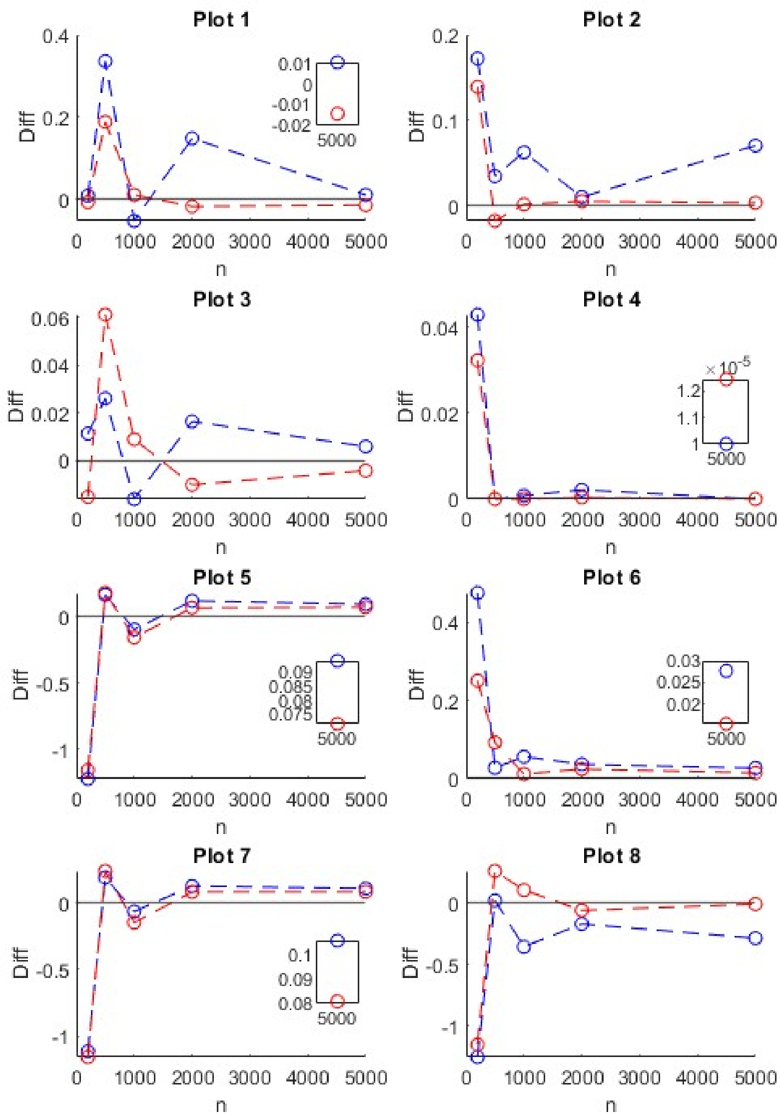

Fig. 2 Plots 1-8 show the difference of the estimated parameter to their true values in the following order - $\kappa, \sigma_\chi, \lambda_\chi, \gamma, \mu_\xi, \sigma_\xi, \lambda_\xi, \rho_{\chi\xi}$, when $N = 10$. The x-axis shows the sample size in the simulated dataset, while the y-axis shows the absolute difference. The blue line represents the result using Model 1, and the red line shows the result using Model 2. Insets are added at $n = 5,000$ for better comparison between two models.

$\hat{\lambda}_\chi$ and $\hat{\gamma}$, Model 2 generally demonstrates better performance. Both models exhibit similar behaviour, except for $\hat{\sigma}_\chi$ and $\hat{\rho}_{\chi\xi}$, where we see a significant difference in estimates at $n = 5,000$.

Table 1 Simulation Study - Parameter estimates with a full covariance matrix and AR(1) measurement errors, $n = 5,000$, $N = 10$

	Aspinall et al. [2]	Model 1	Model 2	True
$\hat{\kappa}$	0.2746	0.2571	0.2319	0.2462
$\hat{\sigma}_\chi$	0.1171	0.1454	0.0787	0.0748
$\hat{\lambda}_\chi$	-0.0327	0.0122	0.0021	0.0062
$\hat{\gamma}$	1.00×10^{-5}	1.00×10^{-5}	1.25×10^{-5}	0
$\hat{\mu}_\xi$	0.4956	0.5913	0.5692	0.4973
$\hat{\sigma}_\xi$	0.5212	0.5354	0.5231	0.5077
$\hat{\lambda}_\xi$	-0.1748	0.5590	0.5337	0.4527
$\hat{\rho}_{\chi\xi}$	0.2283	0.0094	0.2852	0.2960
$\hat{s}_1$	1.00×10^{-10}	0.0054	0.0046	0.01
$\hat{s}_2$	0.0180	0.0203	0.0046	0.01
$\hat{s}_3$	0.0145	0.0175	0.0047	0.01
$\hat{s}_4$	0.0138	0.0176	0.0043	0.01
$\hat{s}_5$	0.0148	0.0187	0.0042	0.01
$\hat{s}_6$	0.0141	0.0192	0.0047	0.01
$\hat{s}_7$	0.0132	0.0179	0.0047	0.01
$\hat{s}_8$	0.0140	0.0197	0.0048	0.01
$\hat{s}_9$	0.0136	0.0200	0.0045	0.01
$\hat{s}_{10}$	0.0127	0.0010	0.0046	0.01
$\hat{\rho}_1$	-	0.9666	0.3888	0.90
$\hat{\rho}_2$	-	0.5298	0.2730	0.90
$\hat{\rho}_3$	-	0.6047	0.3858	0.90
$\hat{\rho}_4$	-	0.6422	0.1978	0.90
$\hat{\rho}_5$	-	0.6300	0.1509	0.90
$\hat{\rho}_6$	-	0.6983	0.3523	0.90
$\hat{\rho}_7$	-	0.6419	0.3267	0.90
$\hat{\rho}_8$	-	0.6516	0.4005	0.90
$\hat{\rho}_9$	-	0.6737	0.2693	0.90
$\hat{\rho}_{10}$	-	0.3210	0.7079	0.90
$\hat{\phi}_{1,1}$	-	0.8929	0.9467	0.95
$\hat{\phi}_{1,2}$	-	0.9526	0.9558	0.95
$\hat{\phi}_{1,3}$	-	0.9401	0.9451	0.95
$\hat{\phi}_{1,4}$	-	0.9408	0.9456	0.95
$\hat{\phi}_{1,5}$	-	0.9495	0.9592	0.95
$\hat{\phi}_{1,6}$	-	0.9485	0.9446	0.95
$\hat{\phi}_{1,7}$	-	0.9392	0.9485	0.95
$\hat{\phi}_{1,8}$	-	0.9494	0.9498	0.95
$\hat{\phi}_{1,9}$	-	0.9501	0.9500	0.95
$\hat{\phi}_{1,10}$	-	0.9187	0.9468	0.95
AIC	-2.71×10^5	-4.01×10^5	-4.60×10^5	-

The estimation results for model parameters, encompassing volatilities, inter-correlations, and serial correlations, are presented in Table 1 for a sample size of $n = 5,000$ and $N = 10$. Additionally, we present the estimates using the model in [2] in Table 1 for testing on the available `R` package. This work involves a recently published `R` code for the estimation of model parameters under the Schwartz-Smith two-factor model framework, where model parameters are estimated based on the 'Genetic Optimization Using Derivatives' algorithm that does not require setting the values to initiate the estimation procedure, see [19]. However, this model setup assumes measurement errors follow white noise, and are independent component-wise. In contrast, in [16], they use the gradient-based algorithm to minimise the non-linear function with constraints, with the initial values determined based on the grid search method. The result shows the model in [2] is able to closely estimate $\hat{\gamma}$ and $\hat{\mu}_{\xi}$, but other model parameters deviates from their true values.

It is important to note that in Model 1, the estimated parameters $\hat{s}_j$ and $\hat{\rho}_j$ correspond to the volatilities and inter-correlation parameters of the measurement error process $\mathbf{v}_t$. Conversely, in Model 2, they represent parameters related to ε_t. The serial correlations, denoted as $\hat{\phi}$, for Model 1 are estimated by fitting an autoregressive (AR) model to the fitted residuals. Using the Akaike Information Criterion (AIC) Model 2 is selected over Model 1. In [2], $\mathbf{V}$ is assumed to be a diagonal matrix, hence $s_j,\ j = 1,\dots,N$ are to be estimated.

In addition, we evaluate the performance of state estimation for Models 1 and 2. The quality of the fit for the state variables is assessed using the mean squared error (MSE). Table 2 presents the MSE values for each model at various sample sizes when $N = 5, 10$. We observe that for smaller datasets ($n \leq 500$, $N = 5$), estimations of state variables are unstable, whereas when $n \geq 1,000$, we observe, especially with longer-dated contracts, that the fit is better in terms of MSE. Hence, to use this model for empirical applications, at least $n = 1,000$ may be required in order to achieve a better fit of state variables, and hence, closer parameter estimates to their true values.

Table 2 Mean Square Errors of state variables for Models 1 and 2

		Model 1		Model 2	
N	n	MSE(χ)	MSE(ξ)	MSE(χ)	MSE(ξ)
5	200	0.2365	0.2422	0.0691	0.0911
	500	0.7216	0.7230	0.2174	0.2180
	1,000	0.0019	0.0039	0.0343	0.0282
	2,000	0.2432	0.2380	0.1167	0.1118
	5,000	0.1291	0.1313	0.1693	0.1720
10	200	0.0464	0.0528	0.0200	0.0142
	500	0.0134	0.0174	0.0329	0.0418
	1,000	0.0013	0.0022	0.0005	0.0020
	2,000	0.0015	0.0022	0.0015	0.0025
	5,000	0.0016	0.0020	0.0005	0.0011

Our findings, backed by AIC, demonstrate that incorporating the AR(p) structure of the measurement errors in Model 2 and using the unified estimation procedure surpassed the two-step approach employed in Model 1. The noticeable superiority is also evident through MSE criterion in the fitting of the state variables.

In summary, given the assumption of independence of measurement errors cannot be validated, our study provides a new tool which can be used for Schwartz-Smith two factor model calibration and forecasting of futures prices.

4 Conclusion

This paper presents an extended two-factor model from [24] and [8], for which we propose the data-driven parameter estimation through modelling of the autoregressive structure of inter-correlated errors in the measurement equations' system. In this framework we derived the linear Kalman filter, which was used for joint estimation of the state variables and model parameters.

In the simulation study, we carried out a comparative analysis of the convergence of the estimates in Models 1 and 2 with respect to the sample size and number of contracts, as well as accuracy. Model 2 outperformed Model 1 in terms of MSE and AIC for state variables and parameter estimates, respectively. The simulation study offered valuable insights into the performance and suitability of the models, contributing to the advancement of data-driven estimation methodology.

The empirical results presented in Section 3 can be reproduced using `MATLAB` code developed by the first author, the link to the code is provided in the footnote on the title page of the paper.

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