On ruin probabilities in a Sparre Andersen type model in the presence of risky investments and random switching

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Abstract We study the asymptotic behavior of ruin probabilities, as the initial reserve goes to infinity, for a reserve process model where claims arrive according to a renewal process, while between the claim times the process has the dynamics of geometric Brownian motion-type Itô processes with time-dependent random coefficients. These coefficients are "reset" after each claim time, switching to new values independent of the past history of the process. We use the implicit renewal theory to obtain power-function bounds for the eventual ruin probability. In the special case when the random drift and diffusion coefficients of the investment returns process remain unchanged between consecutive claim arrivals, we obtain conditions for the validity of the power function decay behaviour (as the initial reserve tends to infinity) for the ruin probability for our model.

1 Introduction and the main result

In the classical Cramér–Lundberg collective risk model (going back to a 1903 F. Lundberg's work), the insurance company reserve process X is assumed to have dynamics of the form

$$X(t) = u + ct - \sum_{j=1}^{N(t)} \xi_j, \qquad t \geq 0, \tag{1}$$

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where c is a constant premium payment rate, N is a Poisson process of claim epochs T_j, $j \geq 1$, and ξ_j, $j \geq 1$, are positive i.i.d. random variables modeling claim sizes made at the respective claim times, their sequence being independent of N.

The main question posed in the context of this model was on the behavior of the ultimate ruin probability

$$\psi(u) := \mathbf{P}\Big(\inf_{t>0} X(t) < 0\Big)$$

as the initial reserve u tends to infinity. Clearly, in model (1), the ruin (X turning negative) can only occur at a claim time. Hence one deals here with a question on the asymptotic behavior of the distribution tail of the global maximum of a random walk with jumps of the form $Z_j := \xi_j - c\tau_j$, where $\tau_j := T_j - T_{j-1}$, $j \geq 1$ (setting $T_0 := 0$). Hence $\psi(u) < 1$ for $u > 0$ and $\psi(u) \to 0$ as $u \to \infty$ once the safety loading condition

$$\mathbf{E}Z \equiv \mathbf{E}\xi - c\mathbf{E}\tau < 0 \tag{2}$$

is met (here and in what follows, we use the convention that $Z \stackrel{d}{=} Z_1$, $\xi \stackrel{d}{=} \xi_1$, *etc.*). But what can one say about the rate at which $\psi(u)$ vanishes at infinity? The most famous result in this classical setting is the celebrated Cramér–Lundberg approximation that holds in the case of exponentially light tails and can be stated as follows.

For a random variable V, denote by

$$\phi_V(q) := \mathbf{E}e^{qV}, \quad q \in \mathbb{R}, \qquad q_V := \sup\{q \in \mathbb{R} : \phi_V(q) < \infty\} \tag{3}$$

its moment generating function and the right end-point of the interval on which the latter is finite, respectively. If $\phi_Z(q_Z) \geq 1$ then, under condition (2), there exists a unique solution $\gamma > 0$ to the equation $\phi_Z(q) = 1$ and if, in addition, $\phi_Z(q_Z) > 1$ or $\phi_Z(q_Z) = 1$ and $\phi_Z'(q_Z-) < \infty$ then

$$\psi(u) = Ce^{-\gamma u}(1 + o(1)) \quad \text{as} \quad u \to \infty, \tag{4}$$

where the constant C admits a closed-form expression (see e.g. Section 22 in [3] or Section I.4d in [2]). Moreover, it turns out that approximation (4) is rather sharp: there is an $\varepsilon > 0$ such that the remainder term $o(1)$ in it can be replaced with $o(e^{-\varepsilon u})$. Furthermore, under the same moment assumptions on the distribution of Z, approximation (4) also holds for the Sparre Andresen model that differs from (1) only in that the process N is just a renewal process (so that the inter-claim times τ_j are general positive i.i.d. random variables). In this case, the remainder term will be decaying exponentially fast under the additional assumption that the distribution of Z contains an absolutely continuous component (p. 129 in [3]).

Note that in the case where $\phi_Z(q_Z) = 1$ and $\phi_Z'(q_Z-) = \infty$, the problem on the asymptotics of $\psi(u)$ is more difficult and the asymptotic behavior of this probability as $u \to \infty$ can have a different form, see e.g. p. 136 in [3] and Section 6.5 in [4].

Of course, the Cramér–Lundeberg model (1) and its Sparre Andersen extension are oversimplifications of real-life situations. These models assume that all the reserves of the insurance company are kept in a safe bank account. Over the last two decades, several authors turned their attention to more realistic models in which the reserve capital can be invested in a risky financial asset (considering a single risky asset is reasonable due to the common practice of investing in a market portfolio or an index). Models with surplus generating process and investments in risky asset modelled by Lévy processes were discussed, e.g., in [20, 21, 23]. In particular, it was noted in [23] that the ultimate ruin probability and the Laplace transform of the ruin time are solutions to suitable boundary value problems for the respective integro-differential equations.

A discrete time model with stochastic interest rates and returns was considered in [18], the main results (obtained using the "crude" large deviation theory) included power asymptotic behavior of the ruin probability as a function of the initial reserve. A power function ruin probability asymptotics behavior was also obtained in [22] for the Lévy processes-based models under suitable conditions, basing on the results from [21]. Assuming (1) (and also allowing a more general Lévy process model) and that the risky investment returns follow an independent geometric Lévy process, power function bounds for $\psi(u)$ were obtained in [15]. A power function asymptotic behavior was obtained in [9] for a modification of the classical model (1) with investments in a risky asset with price following an independent geometric Brownian motion (BM) process with mean return $\mu \in \mathbb{R}$ and volatility $\sigma > 0$ (as in (6) below, but with a constant $c(s) \equiv c$); the first steps in that direction were announced in [8]). Assuming that the claim sizes are exponentially distributed and setting $\beta := 2\mu/\sigma^2 - 1$, it was shown in [9] that

$$\psi(u) = Cu^{-\beta}(1 + o(1)) \quad \text{as} \quad u \to \infty \tag{5}$$

for some constant $C > 0$ when $\beta > 0$ (and that $\psi(u) \equiv 1$ when $\beta < 0$). For claims with a general distribution such that $\mathbf{E}\xi_1^\beta < \infty$, were obtained upper and lover power bounds with the right-hand sides of the form $Cu^{-\beta}$ for come constants C.

Note that the presence of the moment condition on ξ_1 (here and in our Theorem 1 below) is quite natural as for heavy-tailed claim distributions, the asymptotics of the ruin will be governed by the distribution tail of the "integrated tail law" for ξ_1 when that tail dominates $u^{-\beta}$ (cf. [1] and Chapter X in [2]).

These results were extended in [24] to a modification of the above model with a variable premium payment rate $c(t)$ yielding the following dynamics:

$$X(t) = u + \int_0^t c(s)\,ds + \int_0^t \mu X(s)\,ds + \int_0^t \sigma X(s)\,dW(s) - \sum_{j=1}^{N(t)} \xi_j, \tag{6}$$

where W is a BM process independent of N and $\{\xi_j\}$, the coefficients μ and σ are constant, and $c(t) = c(t,X) \in [0,\overline{c}]$ (with a constant $\overline{c} < \infty$) is a bounded adapted function such that there exists a unique strong solution to the above equation. Upper and lover bounds with the right-hand sides of the form $Cu^{-\beta}$ were obtained under appropriate moment conditions on ξ_1, whereas exact asymptotics of the form (5) were established for generally distributed ξ_1 (satisfying $\mathbf{E}\xi_1^{\beta+\delta} < \infty$ for some $\delta > 0$) in the special case where $c(t) = c_1 e^{\gamma t}$ for some $\gamma \le 0$. The toolbox used in that paper, as in some other previous work as well, was based on the implicit renewal theory.

It may seem paradoxical at the first glance that, in all these papers establishing power asymptotics of the form (5), the distributions of the "main source of risk" — the claims made against the insurer — could have a finite exponential moment, as in the case leading to the much faster exponential decay (4). This means that investing in a risky asset (even with significant mean positive returns) dramatically increases the riskiness of the insurance business. In Remark 5 below we will provide an intuitive explanation of the emergence of the power behavior at infinity for ψ. Roughly speaking, it is due to the closeness of the dynamics of an embedded discrete time process (the values of the risk process $u - X$ at the claims times) to those of the exponential of a random walk with i.i.d. jumps and negative trend. The ruin occurs when the global supremum of that walk is "large", of the order of magnitude of $\ln u$, and the probability of this has the form of the right-hand side of (4), with u replaced by $\ln u$

Over the last few years, several authors turned their attention to versions of model (6) with random switching. In [25], the authors considered a Markov-modulated model, governed by an "external environment process" $\{J(t)\}_{t\ge 0}$, assumed to be a homogeneous, irreducible and recurrent continuous time Markov process, with a finite state space E. All the components of the reserve process depend on the state of $J(t)$: given that $J(t) = i \in E$, arrivals occur according to a Poisson process with rate λ_i, the (absolutely continuous) claim distribution depends on i, premiums are received at a rate $c_i > 0$, and the drift and diffusion coefficients of the risky return process also depend on i. Under these assumptions, a system of integro-differential equations was derived for the vector of ruin probabilities (indexed by the initial state of the modulating Markov chain). Switching to the corresponding Laplace transforms and using Karamata Tauberian theorems, the authors then derive the asymptotic behaviour of the ruin probailities, but only in the case when both the drift and diffusions coefficients for the risky asset price process do not depend on the state of the modulating process J.

In [7], it was assumed that the geometric BM process modelling the dynamics of the risky asset has stochastic drift and volatility coefficients: $\mu = \mu_{\theta(t)}$, $\sigma = \sigma_{\theta(t)}$, where $\{\theta(t)\}_{t\ge 0}$ is a time-homogeneous (hidden) Markov chain with state space $\{0,1\}$ independent of all the other stochastic ingredients of the model. Using implicit renewal theory, the authors derived two-sided power function bounds of the form

$$0 < \liminf_{u\to\infty} u^\beta \psi(u) \le \limsup_{u\to\infty} u^\beta \psi(u) < \infty \qquad (7)$$

for the ruin probability. These results were extended in [12] to the case where $\{\theta(t)\}_{t\ge 0}$ has an arbitrary finite state space.

In [6] a Sparre Andersen type model was considered, where the dynamics of the risky asset used for investment was given by a general Lévy process $\{R(t)\}_{t\ge 0}$ (with the assumption that its jumps are always greater than -1):

$$X(t) = u + \int_0^t X(s-) dR(s) - \sum_{j=1}^{N(t)} \xi_j,$$

where $\{N(t)\}_{t\geq 0}$ is now a renewal process (all the components of the model were, as usual, assumed to be independent). Using recent results from the theory of distributional equations, the authors derived for this model two-sided power function bounds of the form (7). That paper was complemented by note [13], where the authors used semi-Markov processes techniques to extend the results to the case of annuities and models with two-sided jumps.

One can also mention here paper [14] that deals with the ruin problem when an insurance company that has two business branches, life insurance and non-life insurance (so that the reserve process would have both positive and negative jumps), and invests its reserve into a risky asset with the price dynamics given by a geometric Brownian motion. For the case of exponentially distributed jumps, it was shown that the non-ruin probability is a solution of an ordinary differential equation of the fourth order. Asymptotic analysis of the latter led to the conclusion that the ruin probability vanishes the same way as in the already studied cases of models with one-side jumps.

In the present note, we extend (6) to another version of the Sparre Andersen-type model with investment in a risky asset that involves random switching. To formally describe our model, in addition to the i.i.d. sequence $\{\xi_n\}_{n\geq 1}$ of claim sizes (as above), introduce an independent of it i.i.d. sequence of quadruples

$$(\mu_n(\cdot), \sigma_n(\cdot), \tau_n, W_n(\cdot)), \quad n \geq 1, \tag{8}$$

and (independent) filtrations $\{\mathbb{H}_n = \{\mathscr{H}_n(t), t \geq 0\}\}_{n\geq 1}$, where W_n is a standard Wiener process which is a martingale w.r.t. filtration $\mathbb{H}_n$, while the process μ_n is adapted to $\mathbb{H}_n$ and locally integrable a.s., σ_n is progressively measurable (w.r.t. $\mathbb{H}_n$) and locally square-integrable a.s., and $\tau_n > 0$ are stopping times w.r.t. $\mathbb{H}_n$ (in particular, they may be independent of W_n, assuming $\mathbb{H}_n$ large enough). About $c(t)$ we will assume, as in [24], that it is right-continuous and takes values in $[0, \overline{c}]$ with some $0 < \overline{c} < \infty$ and is adapted in an appropriate way (omitting technical details to avoid making exposition too cumbersome) such that there exist unique strong solutions to the equations describing our model.

Our reserve process follows the dynamics of (6), where the drift and diffusion coefficients μ and σ are random processes of the form

$$\mu(t) = \sum_{n=1}^{\infty} \mu_n(t - T_{n-1}) \mathbf{1}_{[T_{n-1}, T_n)}(t), \quad \sigma(t) = \sum_{n=1}^{\infty} \sigma_n(t - T_{n-1}) \mathbf{1}_{[T_{n-1}, T_n)}(t),$$

while $N(t) = \sum_{n=1}^{\infty} \mathbf{1}_{(0,t]}(T_n)$, $T_n := \sum_{i=1}^{n} \tau_i$, is the renewal process generated by the inter-arrival times $\tau_n > 0$. We assume that

$$\mu_n(t) \geq \underline{\mu} > -\infty, \quad 0 < \sigma_n(t) \leq \overline{\sigma} < \infty \quad \text{a.s.}$$

for some constant $\underline{\mu}, \overline{\sigma}$.

Thus, according to the suggested model, our insurance company commences at time $t = 0$ with an initial endowment u, faces a renewal-reward claims process with claim sizes ξ_n and inter-claim times τ_n, and receives premium inflow at a bounded non-negative random rate $c(t)$. During the time period (T_{n-1}, T_n), the company obtains a rate of return following a diffusion process with random time-dependent drift coefficient μ_n and volatility σ_n, which are "switched" to μ_{n+1} and σ_{n+1} at time T_n. The random regime switching for the investment component may be related to changing the investment policy or insurer's economic environment following claim payments. Considering the proposed model is also suggested by the inner logic of the mathematical problem per se.

To state our main results, we first need to introduce some notations. Following the standard approach used, in particular, in [15] and [24], we note that ruin for this model can only occur at one of the claim times T_n. Therefore, for the ruin probability analysis, it suffices to consider the embedded discrete time process $\{S_n := X(T_n)\}_{n\geq 0}$ (setting $T_0 := 0$) since

$$\psi(u) = \mathbf{P}\Big(\inf_{n\geq 1} S_n < 0\Big). \tag{9}$$

The dynamics of (6) inside intervals $[T_{n-1}, T_n)$ are those of solutions to linear stochastic differential equations with the respective initial values S_{n-1}. Using the available in closed form solutions to such problems (see e.g. Chapter 9 in [17]), noting that $S_n = X(T_n-) - \xi_n$, and introducing notations

$$K_n(s) := \int_s^{\tau_n} (\mu_n(u) - \sigma_n^2(u)/2)du, \quad Z_n(s) := \int_s^{\tau_n} \sigma_n(u)dW_n(u), \quad s \in [0, \tau_n],$$

$K_n := K_n(0)$, $Z_n := Z_n(0)$,

$$\nu_n := -K_n - Z_n, \quad \lambda_n := e^{-\nu_n}, \quad \zeta_n := \int_0^{\tau_n} e^{K_n(s)+Z_n(s)} c(T_{n-1} + s)ds - \xi_n, \tag{10}$$

we obtain that

$$S_n = \lambda_n S_{n-1} + \zeta_n, \quad n \geq 1, \quad S_0 = u, \tag{11}$$

Note that, due to our assumptions, $\{(K_n, \nu_n)\}_{n\geq 1}$ is an i.i.d. sequence, whereas $\{\zeta_n\}_{n\geq 1}$ does not need to be so.

Recall that, for sequences of random elements, we agreed to omit for brevity's sake the subscript n in the case where $n = 1$.

Referring to (3), we will use the following lemma to introduce one more notation.

Lemma 1. *If $\phi_\nu(q_\nu) \in (1, \infty]$, $\mathbf{E}\tau < \infty$ and $\mathbf{E}K \in (0, \infty)$ then $q_\nu > 0$ and there exists a $\beta \in (0, q_\nu)$ such that $\phi_\nu(\beta) = 1$.*

We will refer to the coefficient β from Lemma 1 as the *Frolova exponent* for our model, as the power decay behaviour of the form (5) (with $\beta := 2\mu/\sigma^2 - 1$ in the basic Cramér–Lundberg model with risky investments described by a geometric BM process) was apparently first conjectured in [8].

Remark 1. Note that as ϕ_ν is left-continuous on $(0, q_\nu)$, one has $\beta < q_\nu$. Therefore $\mathbf{E}\lambda^{-(\beta+\delta)} \equiv \phi_\nu(\beta+\delta) < \infty$ for any $\delta \in (0, q_\nu - \beta) \neq \varnothing$.

Our main result is stated in the following theorem.

Theorem 1. *Assume that* $\phi_\nu(q_\nu) \in (1, \infty]$, $\mathbf{E}\tau < \infty$, $\mathbf{E}K \in (0, \infty)$, *and* $\mathbf{E}\xi^\beta < \infty$ *for some* $\delta > 0$, *where* β *is the Frolova exponent for our model. Then*

$$\limsup_{u\to\infty} u^\beta \psi(u) \le C_+. \tag{12}$$

If, in addition, $(\mu(\cdot), \sigma(\cdot), \tau)$ *and* $W(\cdot)$ *are independent,* $\mathbf{E}\xi^{\beta+\delta} < \infty$ *for some* $\delta > 0$ *and*

$$q_\tau > \beta^2\overline{\sigma}^2/2 + \beta(\overline{\sigma}^2/2 - \underline{\mu})^+ \tag{13}$$

then

$$\liminf_{u\to\infty} u^\beta \psi(u) \ge C_-. \tag{14}$$

Here $0 < C_- \le C_+ < \infty$ *are some constants.*

Remark 2. The existence of the Frolova exponent $\beta > 0$ is ensured since the conditions of Lemma 1 are clearly met under the assumptions of Theorem 1. Without loss of generality, in what follows we will assume about the δ from the conditions of Theorem 1 that $\delta \in (0, q_\nu - \beta)$ (see Remark 1).

Remark 3. Condition $\mathbf{E}K = \mathbf{E}\int_0^\tau (\mu(u) - \sigma^2(u)/2)du > 0$ means that "volatility" $\sigma(t)$ cannot be "large" in some average sense. Recall that $\sigma^2/2 > \mu$ implies certain ruin in the models with constant μ and σ considered in [9] and [24].

Remark 4. Observe that if $\phi_\tau(q) < \infty$ for any $q > 0$ then condition (13) is clearly superfluous.

The proof of Theorem 1 is given in Section 2.

The existence of the Frolova exponent β is the key factor for establishing the power behaviour of the ruin probability. Given the structure of our random variable ν, verifying the existence of such a β in the general case is a complicated task. In Section 3, we will establish a sufficient condition for the existence of the Frolova exponent in the more tractable special case when

$$\mu_n(t) \equiv \mu_n(0) =: \mu_n \quad \text{and} \quad \sigma_n(t) \equiv \sigma_n(0) =: \sigma_n \tag{15}$$

do not depend on time (so that the random drift and diffusions coefficients for the return on investments process remain unchanged during each of the intervals

$[T_{n-1}, T_n)$, $n \geq 1$). Moreover, we will assume that the components (μ_n, σ_n), τ_n and $W_n(\cdot)$ of our quadruples (8) are jointly independent. The problem admits in this case an elegant solution: it turns out that the answer (given in Theorem 2 stated and proved in Section 3) basically depends on "concentration of probability" in vicinity of a certain straight line tangent to the support of the distribution of the random vector $(\mu, \sigma^2/2)$.

2 Proof of Theorem 1

Proof of Lemma 1. That $q_\nu > 0$ in clear since $\phi_\nu(q_\nu) > 1$. Further,

$$\mathbf{E}\nu = -\mathbf{E}K - \mathbf{E}Z = -\mathbf{E}K < 0 \tag{16}$$

as $\mathbf{E}Z = \mathbf{E}\int_0^\tau \sigma(u)dW(u) = 0$ by the optional stopping theorem (note that $\mathbf{E}|\nu| < \infty$). Since ϕ_ν is a convex function and $\phi_\nu(q_\nu) \in (1, \infty]$, the existence of the claimed β is equivalent to having $\phi_\nu'(0+) < 0$, which is an immediate consequence of (16). □

Proof of Theorem 1. Our line of argument follows the overall logic employed in [24]. Iterating (11) and setting $\Lambda_n := \prod_{k=1}^n \lambda_k$, $n \geq 1$, we get

$$S_n = \Lambda_n u + \Lambda_n \sum_{k=1}^n \Lambda_k^{-1} \zeta_k, \quad n \geq 1. \tag{17}$$

First we will prove the upper bound (12). Clearly,

$$(Q_k, M_k) := \{(\xi_k, 1)/\lambda_k\}_{k \geq 1}$$

is an i.i.d. sequence. Set

$$R_n := \sum_{k=1}^n Q_k \prod_{i=1}^{k-1} M_i, \quad n \geq 1, \tag{18}$$

with the usual convention that $\prod_{i=j}^k = 1$ when $j > k$. Since $Q, M > 0$, the sequence $\{R_n\}_{n \geq 1}$ is clearly increasing so that

$$R_n \uparrow R \quad \text{a.s.} \tag{19}$$

for some (possibly improper) random variable $R \leq \infty$.

In view of (10), one has $\zeta_k \geq -\xi_k$, $k \geq 1$, and hence we obtain from (17) that

$$S_n \geq \Lambda_n u - \Lambda_n \sum_{k=1}^n \Lambda_k^{-1} \xi_k = \Lambda_n (u - R_n) \geq \Lambda_n (u - R), \quad n \geq 1. \tag{20}$$

Hence it follows from (9) that

$$\psi(u) \le \mathbf{P}(R > u), \quad u > 0. \tag{21}$$

Remark 5. One can clarify the emergence of the power decay for ψ as follows. Clearly, $U_n := \sum_{k=1}^n \nu_k$, $n \ge 1$, is a random walk with i.i.d. jumps ν_k with negative trend (see (16)) and $\phi_\nu(\beta) = 1$. Hence by the classical Cramér–Lundberg result (4) for $\overline{U} := \max_{n\ge1} U_n$, one has $\mathbf{P}(\overline{U} > w) \sim Ce^{-\beta w}$ as $w \to \infty$.

Now in view of (17), ruin is equivalent to the event $\left\{ \sup_{n\ge1} \sum_{k=1}^n (-\zeta_k) e^{U_k} > u \right\}$ which actually occurs "due" to a few terms in these sums, with k close to the point n' such that $\overline{U} = U_{n'}$ (cf. the argument in the proof of Theorem 4 in [5]). So one can expect that the probability of ruin behaves like $\mathbf{P}(\overline{U} > \ln u) \sim Ce^{-\beta \ln u} = Cu^{-\beta}$ as $u \to \infty$.

That R is a proper random variable follows immediately from the following lemma, which is a direct consequence of Theorem 1.6 in [26]:

Lemma 2. *Let $\{(A_n, B_n)\}_{n\ge1}$ be an i.i.d. sequence of bivariate random vectors, and*

$$Z_n(x) := x \prod_{j=1}^n A_j + \sum_{k=1}^n B_k \prod_{j=1}^{k-1} A_j, \quad n \ge 1, \quad x \in \mathbb{R}. \tag{22}$$

Assume that $\mathbf{E} \ln |A| < 0$ *and* $\mathbf{E}(\ln |B|)^+ < 0$, *where* $z^+ := \max\{0, z\}$, $z \in \mathbb{R}$. *Then* $Z_n(x) \to Z$ *in distribution as* $n \to \infty$ *for all* $x \in \mathbb{R}$, *where the distribution of the proper random variable* Z *satisfies the random equation*

$$Z \stackrel{d}{=} B + AZ, \tag{23}$$

(A, B) *and* Z *on the right-had side being independent of each other.*

Indeed, our sequence (18) is of the form (22) with $x = 0$ and $(A_n, B_n) = (M_n, Q_n)$, $n \ge 1$, and $\mathbf{E} \ln |A| = \mathbf{E} \ln \lambda^{-1} = \mathbf{E}\nu < 0$ by (16), whereas

$$\mathbf{E}(\ln |B|)^+ = \mathbf{E}(\ln(\xi/\lambda))^+ = \mathbf{E}(\ln \xi + \nu)^+ \le \mathbf{E}(\ln \xi)^+ + \mathbf{E}\nu^+ < \infty$$

as $\mathbf{E}\xi^\beta < \infty$ and $\mathbf{E}|\nu| < \infty$ (cf. Lemma 1).

Hence, by Lemma 2, the sequence $\{R_n\}$ converges as $n \to \infty$ in distribution to a proper random variable, which implies that the a.s. limit R from (19) is proper as well and satisfies the random equation

$$R \stackrel{d}{=} Q + MR, \tag{24}$$

where R and (M, Q) on the right-hand side are independent of each other.

Now to complete the derivation of the desired upper bound using (21) it remains to turn to the implicit renewal theory. We will make use of the following lemma which is a direct consequence of Theorem 4.1 in [10].

Lemma 3. *Assume that the distribution of a bivariate random vector (A, B) with $A \ge 0$ a.s. is such that, for some $\alpha > 0$,*

$$\mathbf{E}A^{\alpha}=1,\quad \mathbf{E}A^{\alpha}(\ln A)^{+}<\infty,\quad \mathbf{E}|B|^{\alpha}<\infty,$$

while the conditional distribution of $\ln A$ *given* $A\neq 0$ *is non-arithmetic. Then solution to* (23) *satisfies*

$$\lim_{u\to\infty}u^{\alpha}\mathbf{P}(Z>u)=C,$$

where $C:=\mathbf{E}\big[((B+AZ)^{+})^{\alpha}-(AZ^{+})^{\alpha}\big]/(\alpha\mathbf{E}A^{\alpha}\ln A)\in(0,\infty)$.

To apply this lemma to our equation (24) with $(A,B)=(M,Q)$ and $\alpha=\beta$, it suffices to note that $\mathbf{E}M^{\beta}=\phi_{\nu}(\beta)=1$, $\mathbf{E}Q^{\beta}=\mathbf{E}\xi^{\beta}M^{\beta}=\mathbf{E}\xi^{\beta}\mathbf{E}M^{\beta}=\mathbf{E}\xi^{\beta}<\infty$ due to independence, and $\mathbf{E}M^{\beta}(\ln M)^{+}=\mathbf{E}e^{\beta\nu}\nu^{+}<\infty$ since $\mathbf{E}e^{(\beta+\delta)\nu}<\infty$ for some $\delta>0$ (see Remark 2). That $\ln M$ given $M\neq 0$ is non-arithmetic is obvious from the definition of $M=e^{\nu}$ and the presence of the Itô integral in ν. This completes the proof of the upper bound (12).

Now we will proceed to proving the lower bound (14). The main tool here is the following assertion from [16] (see also [10] and [19]).

Lemma 4. *Assume that* Y *satisfies the equation*

$$Y\stackrel{d}{=}B+AY^{+},\tag{25}$$

where (A,B) *and* Y *on the right-hand side are independent of each other,* $A>0$ *a.s., and the distribution of* (A,B) *is such that* $\mathbf{P}(A>1,B>0)>0$. *If, for some* $\alpha,\delta>0$,

$$\mathbf{E}A^{\alpha}=1,\quad \mathbf{E}A^{\alpha+\delta}<\infty,\quad \mathbf{E}|B|^{\alpha+\delta}<\infty,$$

and $\ln A$ *is absolutely continuous, then*

$$\lim_{u\to\infty}u^{\alpha}\mathbf{P}(Y>u)=C+o(u^{-h})$$

for some positive constants C *and* h.

To apply this result, we turn to representation (17) and use the natural upper bound for ζ_n :

$$\zeta_n\le\overline{\zeta}_n:=\overline{c}\int_0^{\tau_n}\exp\{K_n(s)+Z_n(s)\}ds-\xi_n$$

to get the inequality

$$S_n\le\overline{S}_n:=\Lambda_n u+\Lambda_n\sum_{k=1}^{n}\Lambda_k^{-1}\overline{\zeta}_k=\Lambda_n(u-\overline{R}_n),\quad n\ge 1,$$

where

$$\overline{R}_n:=\sum_{k=1}^{n}\overline{Q}_k\prod_{i=1}^{k-1}M_i,\quad \overline{Q}_n:=-\overline{\zeta}_n/\lambda_n,\quad n\ge 1.$$

In view of (9), this implies the bound

$$\psi(u) \geq \mathbf{P}\big(\inf_{n\geq 1} \overline{S}_n < 0\big) \geq \mathbf{P}\big(\overline{R} > u\big), \quad \text{where } \overline{R} := \sup_{n\geq 1} \overline{R}_n.$$

Next we note that, since $\overline{R}_1 = \overline{Q}_1$ and $M_1 > 0$, one has

$$\begin{aligned}\overline{R} &= \overline{Q}_1 \vee \sup_{n\geq 2} \overline{R}_n = \overline{Q}_1 \vee \Big(\overline{Q}_1 + M_1 \sup_{n\geq 2} \sum_{k=2}^{n} \overline{Q}_k \prod_{i=2}^{k-1} M_i\Big) \\ &= \overline{Q}_1 \vee (\overline{Q}_1 + M_1 \overline{R}') = \overline{Q}_1 + M_1 (\overline{R}')^+,\end{aligned}$$

where $\overline{R}' := \sup_{n\geq 2} \sum_{k=2}^{n} \overline{Q}_k \prod_{i=2}^{k-1} M_i \stackrel{d}{=} \overline{R}$ is independent of (M_1, Q_1). Therefore our $\overline{R}$ satisfies the random equation

$$\overline{R} \stackrel{d}{=} \overline{Q} + M(\overline{R})^+,$$

where (M,Q) and $\overline{R}$ on the right-hand side are independent of each other. This relation is exactly of the form (25), and we will now verify whether the conditions of Lemma 4 are met when $(A,B) = (M,\overline{Q})$, $\alpha = \beta$.

First of all, it follows from Proposition 6.1 in [10] that $\overline{R}$ is a proper random variable provided that $\mathbf{E}\ln(1 \vee \overline{Q}) < \infty$. The latter will immediately follow from the condition $\mathbf{E}|\overline{Q}|^{\beta+\delta} < \infty$ of Lemma 4 that we need to verify. To demonstrate the latter relation, note that

$$\begin{aligned}\overline{Q} &= -\frac{\overline{\zeta}}{\lambda} = \frac{\xi}{\lambda} - \overline{c}e^{-K(0)-Z(0)} \int_0^{\tau} e^{K(s)+Z(s)} ds \\ &= \frac{\xi}{\lambda} - \overline{c} \int_0^{\tau} \exp\Big\{ -\int_0^{s} (\mu(u) - \sigma^2(u)/2) du - \int_0^{s} \sigma(u) dW(u) \Big\} ds. \qquad (26)\end{aligned}$$

It is obvious from the elementary inequality $|x+y|^p \leq (1 \vee 2^{p-1})(|x|^p + |y|^p)$, $x, y, p > 0$, that it suffices to show that the absolute moments of the order $\beta + \delta$ are finite for both terms on the right-hand side. By independence, one has

$$\mathbf{E}\left|\frac{\xi}{\lambda}\right|^{\beta+\delta} = \mathbf{E}\xi^{\beta+\delta} \mathbf{E}\lambda^{-(\beta+\delta)} = \mathbf{E}\xi^{\beta+\delta} \phi_\nu(\beta+\delta) < \infty$$

in view of Remark 2.

Next note that, due to our assumption about independence of $(\mu(\cdot), \sigma(\cdot), \tau)$ and $W(\cdot)$, one has $\big\{ -\int_0^s \sigma(u) dW(u) \}_{s\geq 0} \stackrel{d}{=} \{W(\Sigma(s))\big\}_{s\geq 0}$, where we set $\Sigma(s) := \int_0^s \sigma^2(u) du$, $s \geq 0$. Therefore, putting $\overline{W}(t) := \max_{0\leq s\leq t} W(s)$, $t \geq 0$, we get

$$\max_{0\leq s\leq \tau} \left(-\int_0^s \sigma(u) dW(u) \right) \stackrel{d}{=} \max_{0\leq s\leq \tau} W(\Sigma(s)) = \overline{W}(\Sigma(\tau)) \leq \overline{W}(\overline{\sigma}^2 \tau).$$

Now, setting $\kappa(u) := \sigma^2(u)/2 - \mu(u)$ and noting that $\kappa(u) \le \overline{\kappa} := \overline{\sigma}^2/2 - \underline{\mu}$ a.s., we get for the second term on the right-hand side of (26) that

$$\mathbf{E}\left(\int_0^\tau \cdots ds\right)^{\beta+\delta} \le \mathbf{E}\left(e^{\overline{W}(\overline{\sigma}^2\tau)}\int_0^\tau e^{\overline{\kappa}s}ds\right)^{\beta+\delta} \le \mathbf{E}\left(e^{\overline{W}(\overline{\sigma}^2\tau)}\tau e^{\overline{\kappa}^+\tau}\right)^{\beta+\delta}. \quad (27)$$

Due to the reflection principle, for any $a,t>0$,

$$\mathbf{E}e^{a\overline{W}(t)} = 2\mathbf{E}(e^{aW(t)}; W(t)>0) < 2\mathbf{E}e^{aW(t)} = 2e^{a^2t/2}, \quad t>0,$$

so conditioning the last expectation in (27) on τ and using independence, we obtain that it is less than

$$2\mathbf{E}e^{((\beta+\delta)^2\overline{\sigma}^2/2+(\beta+\delta)\overline{\kappa}^+)\tau}\tau^{\beta+\delta} < \infty,$$

using assumption (13) and choosing $\delta > 0$ small enough. Thus we showed that $\mathbf{E}|\overline{Q}|^{\beta+\delta} < \infty$, which implies, in particular, that $\overline{R}$ is proper.

To verify the remaining assumptions of Lemma 4, we observe that condition $\mathbf{E}M^\beta = 1$ is met by Lemma 1 and that $\mathbf{E}M^{\beta+\delta} < \infty$ as explained in Remark 2. That $M = e^\nu$ is absolutely continuous follows from the presence of the Itô integral in ν and independence of W from the other participating random quantities. Thus it only remains to verify that $\mathbf{P}(M>1, \overline{Q}>0) > 0$. Setting

$$V(t) := \int_0^t (\mu(u) - \sigma^2(u)/2)du + \int_0^t \sigma(u)dW(u), \quad t \ge 0,$$

and choosing $a>0$ such that $b := \mathbf{P}(\xi > a\overline{c}) > 0$, the previous probability is clearly equal to

$$\begin{aligned}\mathbf{P}(V(\tau)<0, \overline{\zeta}<0) &\ge \mathbf{P}\left(V(\tau)<0, \overline{c}\int_0^\tau e^{V(\tau)-V(s)}ds < \xi, a\overline{c} < \xi\right)\\ &\ge b\mathbf{P}\left(V(\tau)<0, \int_0^\tau e^{V(\tau)-V(s)}ds < a\right)\\ &\ge b\mathbf{P}\left(V(\tau)<0, \int_0^\tau e^{-V(s)}ds < a\right)\\ &\ge b\mathbf{P}\big(\tau e^{-\underline{V}(\tau)} < a \,|\, V(\tau)<0\big)\mathbf{P}(V(\tau)<0),\end{aligned}$$

where we put $\underline{V}(t) := \inf_{0\le s\le t} V(s)$. Obviously, $\mathbf{P}(V(\tau)<0)>0$, and as $-\underline{V}(\tau)>0$ on the event $\{V(\tau)<0\}$ while a can be chosen arbitrary small, the product in the last line of the displayed formula is positive, establishing that the last condition of Lemma 4 is met as well. This completes the proof of Theorem 1. □

3 Frolova's exponent when coefficients $\mu_n(t)$ and $\sigma_n(t)$ do not depend on time

In this section we assume satisfied condition (15) and also that the components (μ_n, σ_n), τ_n and $W_n(\cdot)$ of our quadruples (8) are jointly independent. Under these assumptions, one has $\nu = -(\mu - \sigma^2/2) - \sigma W(\tau)$. Introducing the random vector $\Theta := (\mu, \sigma^2/2)$, setting

$$u(q) := (-q, q(q+1)), \quad q \in \mathbb{R},$$

and conditioning, we get

$$\begin{aligned}\phi_\nu(q) &= \mathbf{E} e^{q(-(\mu-\sigma^2/2)\tau - \sigma W(\tau))} = \mathbf{E} e^{-q(\mu-\sigma^2/2)\tau + q^2\sigma^2\tau/2} \\ &= \mathbf{E}\phi_\tau(-q(\mu - \sigma^2/2) + q^2\sigma^2/2) = \mathbf{E}\phi_\tau(\langle u(q), \Theta\rangle), \end{aligned} \tag{28}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^2$.

Note that our key condition $\mathbf{E}K > 0$ for the existence of the Frolova exponent is equivalent in the case under consideration to

$$\mathbf{E}(\mu - \sigma^2/2) > 0 \tag{29}$$

(assuming that $\mathbf{E}\tau < \infty$), which is a "mean version" of the condition $2\mu/\sigma^2 - 1 > 0$ under which the asymptotics (5) was established in the case of constant deterministic μ and σ in [24].

Assuming that the above condition is met, the case $q_\tau = \infty$ is trivial: it is clear from Lemma 1 and (28) that β will then always exist. So we will only consider the case where

$$q_\tau < \infty, \quad \phi_\tau(q_\tau) = \infty. \tag{30}$$

Note that the latter is a typical situation when $q_\tau < \infty$; this is so, for instance, for gamma-distributed τ. It turns out that, in this situation, the desired β may or may not exist depending on the distribution of Θ, .

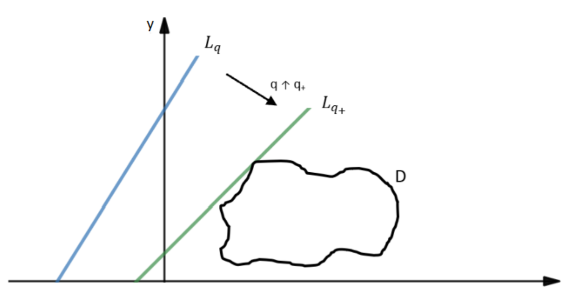

Fig. 1 As $q \uparrow q_+$, the line L_q approaches the set D.

Introduce rays $L_q := \{(x,y) \in \mathbb{R} \times \mathbb{R}^+ : \langle u(q),(x,y)\rangle = q_\tau\},\ q > 0$. Clearly, $(x,y) \in L_q$ iff

$$y = \frac{x}{q+1} + \frac{q_\tau}{q(q+1)}, \quad x \ge -\frac{q_\tau}{q} \tag{31}$$

(the last inequality is equivalent to $y \ge 0$). Denote by $D \subseteq [\underline{\mu},\infty) \times [0,\overline{\sigma}^2/2]$ the support of Θ and put

$$q_+ := \inf\{q > 0 : L_q \cap D \ne \varnothing\}.$$

Note that, as q increases, the ray L_q "moves" to the right and "rotates" in the clockwise direction, and as D is bounded from the left and from the top, q_+ is a finite positive number (see Fig. 1: q_+ is the value of q for which L_q first "touches" D).

Note that if $q > q_+$ then $\mathbf{P}(\langle u(q),\Theta\rangle > q_\tau) > 0$, so that $\phi_\nu(q) = \infty$ by (28). As for $q < q_+$ one clearly has $\mathbf{P}(\langle u(q),\Theta\rangle < q_\tau - \varepsilon) = 1$ for some $\varepsilon > 0$, we get $q < q_\nu$ (again by (28)). We conclude that $q_+ = q_\nu$.

Clearly, $\phi_\nu(q_\nu) = \infty$ is sufficient for the existence of the Frolova exponential under the condition that $\mathbf{E}\nu < 0$. In view of our assumption (30), representation (28) suggests that whether $\phi_\nu(q_\nu)$ is infinite or not depends on how strongly the distribution of Θ is concentrated in vicinity of the ray L_{q_+}. To capture this, we introduce the random variable H by setting, for any $\theta_+ \in L_{q_+}$,

$$H := \langle u(q_+),\theta_+ - \Theta\rangle = q_\tau + q_+\mu - q_+(q_+ + 1)\sigma^2/2,$$

where the second equality was obtained choosing $\theta_+ = \theta_0 := (-q_\tau/q_+,0) \in L_{q_+}$, and denote by F_H its distribution function. We see that $H \ge 0$ a.s. (as the point Θ is below the ray L_{q_+} given by (31)) with $q = q_+$, the value of H being equal to the Euclidean length of the vector $u(q_+)$ times the distance from Θ to L_{q_+}

Now, from (28),

$$\begin{aligned}\phi_\nu(q) &= \mathbf{E}\phi_\tau(\langle u(q) - u(q_+),\Theta\rangle + \langle u(q_+),\Theta - \theta_0\rangle + \langle u(q_+),\theta_0\rangle) \\ &= \mathbf{E}\phi_\tau(q_\tau - H - \langle u(q_+) - u(q),\Theta\rangle) = \mathbf{E}\phi_\tau(q_\tau - H - \varepsilon\langle a(\varepsilon),\Theta\rangle),\end{aligned} \tag{32}$$

where we first noted that $u(q_+) - u(q) = (q_+ - q)(-1,q_+ + q + 1)$ and then put $\varepsilon := q_+ - q$, $a(\varepsilon) := (-1,2q_+ + 1 - \varepsilon) \to (-1,2q_+ + 1)$ as $\varepsilon \downarrow 0$.

There is no monotone dependence on ε in the integrand on the right-hand side on (32), so we need an argument establishing convergence of these expectations as $\varepsilon \downarrow 0$. Let $D'' := \{\theta = (x,y) \in D : y < x/(q_+ + 1)\}$. Clearly, $\langle u(q_+),\theta\rangle = -q_+x + q_+(q_+ + 1)y < 0$ for $\theta \in D''$, so that $\phi_\tau(\langle u(q_+),\theta\rangle) \le 1$ in that domain and hence

$$\mathbf{E}(\phi_\tau(\langle u(q),\Theta\rangle);\Theta \in D'') \to \mathbf{E}(\phi_\tau(q_\tau - H);\Theta \in D''), \quad q \uparrow q_+,$$

by the dominated convergence theorem. Turning to $D' := D \setminus D''$, one can easily verify that there exist $r_\pm \in \mathbb{R}$ such that $r_- \le \langle a(\varepsilon),\theta\rangle \le r_+$ for all $\theta \in D'$, $\varepsilon \in (0,1)$. Since ϕ_τ is an increasing function, we get

$$\begin{aligned}\mathbf{E}(\phi_\tau(q_\tau - H - r_+\varepsilon); \Theta \in D') &\le \mathbf{E}(\phi_\tau(q_\tau - H - \varepsilon\langle a(\varepsilon), \Theta\rangle); \Theta \in D') \\ &\le \mathbf{E}(\phi_\tau(q_\tau - H - r_-\varepsilon); \Theta \in D').\end{aligned}$$

Now we can apply the monotone convergence theorem to both lower and upper bounds in the last displayed formula since the integrands in them have monotone dependence on ε. We conclude that

$$\phi_\nu(q) \to \int_0^\infty \phi_\tau(q_\tau - h)dF_H(h), \quad q \uparrow q_+.$$

Since clearly $\int_\delta^\infty \phi_\tau(q_\tau - h)dF_H(h) \le \phi_\tau(q_\tau - \delta) < \infty$ for any $\delta > 0$, we arrive at the following result.

Theorem 2. *Under the assumptions stated at the beginning of this section, assume that* (29) *and* (30) *hold true. Then $q_\nu = q_+$ and, moreover, $\phi_\nu(q_\nu) = \infty$ iff*

$$\int_0^\delta \phi_\tau(q_\tau - h)dF_H(h) = \infty$$

for some (*and then for any*) $\delta > 0$.

Thus there must be significant presence of probability mass in vicinity of the tangent to D line L_{q_+} to ensure that $\phi_\nu(q_\nu) = \infty$.

It is not hard to get closed-form expressions for ϕ_ν in several tractable examples in the special case of the Poisson arrival process with rate 1, which means that $\phi_\tau(q) = 1/(1-q)$, $q < q_\tau := 1$ (so that (30) is true). In one such example one has $\mathbf{P}(\Theta = (1/j, 1-1/j)) = j^{-p}/\zeta(p)$, $j \ge 1$, for a fixed $p \in \mathbb{N}$, where ζ is the Euler–Riemann zeta function. In this case, $q_+ = (\sqrt{5}-1)/2$ and $D \cap L_1 = \{(0,1)\}$, and it turns out that $\phi_\nu(q_+) = \infty$ iff $p = 2$, in obvious agreement with the claim of Theorem 2. If, further, one assumes that Θ is uniformly distributed in a unit square D with vertices at the points (i,j), $i,j \in \{0,1\}$, then again $q_+ = (\sqrt{5}-1)/2$, $D \cap L_1$ consists of the single point $(0,1)$ (the vertex of our D at which it touches the line L_{q_+}), and one can also derive a closed form expression for ϕ_ν yielding $\phi_\nu(q_+) < \infty$. If, however, we rotate the square in the anticlockwise direction around the vertex $(0,1)$ until its upper edge runs along the line L_{q_+} (with clearly the same value of q_+ as in the previous examples) then one would have $\phi_\nu(q_+) = \infty$, also in agreement with Theorem 2. In the latter case, there is "too much probability" in vicinity of L_{q_+} (the probability mass in the ε-neighbourhood of that line is $\asymp \varepsilon$ as $\varepsilon \downarrow 0$ compared to $\asymp \varepsilon^2$ in the former case).

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