On some asymptotic expansions of skew diffusions

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Abstract Relying on a perturbation technique of Markov semigroup, also known as the parametrix method, we establish some asymptotic expansion for the Markov semigroup generated by a skew diffusion process with respect to a small parameter. Though the approach developed here is general, we focus on the case of short time and small skew heat kernel expansions.

1 Introduction

The parametrix method introduced by Levi [18] in 1907 has been extensively used to establish semi-explicit expansions of the fundamental solution of a variety of partial differential equations. For a review on these subjects, we refer the reader to [12]. It has also recently been revisited in a series of works [1, 7, 4] to provide some probabilistic representation and integration by parts formula of the marginal law of various processes as an alternative to standard Malliavin calculus techniques or to establish the well-posedness in the weak sense of some degenerate stochastic differential equations (SDEs for short) [8, 9].

The purpose of the present paper is to introduce the inexperienced reader to an application of this method to some asymptotic expansions of skew diffusions. For a general introduction to skew Brownian motion and its applications, we refer the reader to [17]. For an application of the parametrix technique to skew diffusion for the study of its transition density or a time discretization scheme, we refer the reader to [15] and [6] respectively.

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While there have been several important works on the problem of density expansions for diffusions with regular coefficients using different methods, see e.g. [21], [2] or [3], to the best of our knowledge, asymptotic expansions for the transition density of a skew diffusion have not been considered so far. For explicit asymptotic expansions using this method in the case of regular diffusions, we refer the reader to [14].

To motivate our results, we also present a financial application with a model representing regime switching which has been recently discussed in [10]. The parameters that are considered to be small in this model are not the same as in our main results but we will show the reader that the same technique applies. Clearly, these techniques may be useful in other models as well and should have broad applications in pricing and risk management.

Our approach in this article is more didactical than general so that we have chosen some simplified models in order for the reader to grasp the ideas quickly. In this sense, some technical details are deliberately omitted by referring the reader to the corresponding article.

2 Asymptotic expansions of skew diffusions

In this section we study a one-dimensional Markov semigroup with singular coefficients for which we derive an asymptotic expansion with respect to a small parameter ε. We consider a skew diffusion process, that is the unique strong solution of the following one dimensional SDE with dynamics

$$X_t^{\varepsilon} = x + \int_0^t b_{\varepsilon}(X_s^{\varepsilon})ds + \int_0^t \sigma_{\varepsilon}(X_s^{\varepsilon})dW_s + \alpha\varepsilon^{\ell} L_t^0(X^{\varepsilon}), \quad \ell \in \mathbb{R}_+, \tag{1}$$

where $(\alpha,\varepsilon) \in (-1,1) \times [0,1]$, $W = (W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion and $L_t^0(X^{\varepsilon})$ is the symmetric local time at the origin accumulated by X^{ε} up to time t.

Note that for $\sigma \equiv 1$, $b \equiv 0$, $\varepsilon = 1$ and $\ell = 0$ the process X^{ε} is known as the skew Brownian motion. We refer the reader to [17] for a survey on this process (see also Section 5 for some basic information). When $b \equiv 0$ and $\sigma \equiv 1$, [20] constructed a semigroup, closely related to the process (1), for which the associated infinitesimal generator has a Dirac mass as drift part. Additionally, he proved that $P_t^{\varepsilon} f(x) = \mathbb{E}[f(X_t^{\varepsilon})]$ generates a Feller semigroup. Actually, his approach is more general since he considers a multi-dimensional framework and the singular drift is the Dirac mass of a smooth surface.

In order to use the parametrix method for the expansion, we need to first gather some regularity properties for the law of X_t^{ε}.

Regularity properties: Let $T > 0$. Assuming that for any $\varepsilon \in [0,1]$, b_ε is continuous[1] and σ_ε is bounded, uniformly elliptic and Hölder continuous, the random variable X^ε_t given by the unique weak solution at time $t > 0$ of the SDE (1) admits a density $p^\varepsilon_t(x,y)$ satisfying some Gaussian upper bounds as well as the Markov property[2]. We refer the reader to Kohatsu-Higa *et al.* [15]. Notably, the authors proved that for all $t \in (0,T]$, $y \mapsto p^\varepsilon_t(x,y)$ is not continuous at 0 but admits left and right limits (as it is the case for the standard skew Brownian motion) and that there exist $C > 1$ and $c > 1$ that may depend on ε such that

$$\forall (x,y) \in \mathbb{R} \times \mathbb{R}\backslash\{0\},\ p^\varepsilon_t(x,y) \le C g_{c\varepsilon^2 t}(y-x), \tag{2}$$

where $g_c(y) := 1/(2\pi c)^{1/2} \exp(-y^2/(2c))$.

In order to carry out the expansion, we introduce the time-homogeneous semigroup $(P^\varepsilon_t)_{t\ge 0}$ induced by the SDE (1) defined for any bounded and measurable function f by $P^\varepsilon_t f(x) = \mathbb{E}[f(X^\varepsilon_t)]$. We will frequently use the following boundary condition which is called the "transmission condition" due to its meaning as interaction at the boundary. Given a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0+)$ and $f'(0-)$ exist, we say that f satisfies the transmission condition at 0 if

$$(1+\alpha\varepsilon^\ell) f'(0+) = (1-\alpha\varepsilon^\ell) f'(0-). \tag{3}$$

We are interested in obtaining the asymptotic expansion of $p^\varepsilon_t(x,y)$ in powers of ε. Importantly, by a weak uniqueness argument, we remark that the case $b_\varepsilon(x) = \varepsilon^2 b(x)$, $\sigma_\varepsilon(x) = \varepsilon\sigma(x)$ and $\ell = 0$ corresponds to the short time asymptotics of the density of X_t given by the unique weak solution taken at time t of the following SDE

$$X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s + \alpha L^0_t(X). \tag{4}$$

Indeed, it is sufficient to remark that by weak uniqueness $\{X_{\varepsilon^2 t}; t \ge 0\}$ is equal in law to $\{X^\varepsilon_t; t \ge 0\}$. Hence, the case $\ell > 0$ corresponds to a skew parameter α close to zero and will be referred as to the small skew case.

Our main results are established in Sections 2.1 and 4. The first one is Theorem 2.1 where we establish an asymptotic expansion of the transition density $y \mapsto p^\varepsilon_t(x,y)$ with respect to the skew parameter. Not only such expansion appears to be new and of independent interest but also it is the first step towards our second main result, developed in Section 4, where the first order expansion of $y \mapsto p^\varepsilon_t(x,y)$ (in the case $b_\varepsilon \equiv \varepsilon^2 b$, $\sigma_\varepsilon = \varepsilon\sigma$), which notably includes the short time asymptotic, is established.

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[1] In fact, a deeper analysis of the proofs in [15] results in the conclusion that this assumption is only used in Proposition 3.2 which is not required in order to obtain Proposition 5.3. In fact, this result can be obtained using an equivalent proof which uses the Itô-Tanaka formula instead of the generator L (see the proof of Lemma 1). Therefore one may assume that b is a measurable and bounded function. For a quick review of the argument, we refer to the Appendix.

[2] The proof of this fact in [15] can also be applied under the assumption that b is bounded and measurable.

2.1 Small skew expansion: a representation formula for the transition density

The first step consists in defining an approximation process that does not involve the local time component of X^{ε}. For some prescribed finite time horizon $T>0$, we want to expand $(P_t^{\varepsilon})_{t\in[0,T]}$ around the semigroup $(\bar{P}_t^{\varepsilon})_{t\in[0,T]}$ associated to the unique weak solution of the following SDE with dynamics

$$\bar{X}_t^{\varepsilon} = x + \int_0^t b_{\varepsilon}(\bar{X}_s^{\varepsilon})ds + \int_0^t \sigma_{\varepsilon}(\bar{X}_s^{\varepsilon})dW_s. \tag{5}$$

We will work under the following mild assumptions on the coefficients:

(R) For any $\varepsilon \in [0,1]$, the measurable functions $b_{\varepsilon}:\mathbb{R}\to\mathbb{R}$ and $\sigma_{\varepsilon}:\mathbb{R}\to\mathbb{R}$ are bounded uniformly in ε. Moreover, $a_{\varepsilon}=\sigma_{\varepsilon}^2$ is η-Hölder continuous for some $\eta\in(0,1]$ uniformly in ε.

(UE) For any $\varepsilon\in[0,1]$, $\sigma_{\varepsilon}(x)=\sigma_0+\varepsilon\sigma(x)$ for some constant $\sigma_0\geq 0$ and where σ is a positive function such that there exist positive constants $\underline{a},\overline{a}$ satisfying $\underline{a}\leq a(x)=\sigma^2(x)\leq\overline{a}$ for any $x\in\mathbb{R}$.

Since σ_{ε} is continuous and uniformly elliptic, without loss of generality, we assume that σ_{ε} is positive. Under **(R)** and **(UE)**, it is known that the transition density associated to $(\bar{P}_t^{\varepsilon})_{t\geq 0}$ exists for $\varepsilon\in[0,1]$. We denote it by $(t,x,y)\mapsto \bar{p}_t^{\varepsilon}(x,y)$. According to classical results using the parametrix method (see e.g. Chapter 1, Section 6 in [5][3]), under **(R)** and **(UE)**, it is known that $\bar{p}^{\varepsilon}$ satisfies the following properties: for any $(t,x,y)\in(0,T)\times\mathbb{R}\times\mathbb{R}$,

$$\bar{p}_t^{\varepsilon}(x,y) = \bar{p}_t^{\varepsilon,1}(x,y)+\bar{p}_t^{\varepsilon,2}(x,y),$$

where

$$\bar{p}_t^{\varepsilon,1}(x,y) = \frac{1}{\sqrt{2\pi t}\,\sigma_{\varepsilon}(y)}\exp\left(-\frac{(y-b_{\varepsilon}(y)t-x)^2}{2\sigma_{\varepsilon}^2(y)t}\right), \tag{6}$$

and $\bar{p}_t^{\varepsilon,2}(x,y)$ satisfies the following Gaussian type estimates for $r=0$ and $r=1$

$$|\partial_x^r \bar{p}_t^{\varepsilon,2}(x,y)| \leq K_{\varepsilon} t^{-\frac{(1+r-\eta)}{2}}\exp\left(-C_{\varepsilon}\frac{(y-x)^2}{\varepsilon^2 t}\right), \tag{7}$$

for some positive constants C_{ε} and K_{ε}. Note that $\bar{p}_t^{\varepsilon,1}(x,y)$ and $\bar{p}_t^{\varepsilon,2}(x,y)$ are not density functions. Moreover, it holds

$$|\partial_x \bar{p}_t^{\varepsilon}(x,y)| \leq K_{\varepsilon} t^{-1}\exp\left(-C_{\varepsilon}\frac{(y-x)^2}{\varepsilon^2 t}\right), \tag{8}$$

[3] Although in this reference, the author assumes that the drift coefficient b is Hölder continuous, a careful analysis of the arguments reveals that this can be replaced by the weaker condition of being bounded and measurable. For a sketch of the argument in a particular case, see the Appendix.

and from (6) and (7), we easily obtain for $(y,t) \in \mathbb{R} \times (0,T]$,

$$|\partial_x \bar{p}_t^{\varepsilon,1}(y,y)| + |\partial_x \bar{p}_t^{\varepsilon,2}(y,y)| \leq K_\varepsilon t^{-1+\frac{\eta}{2}}. \tag{9}$$

Note also that if $\sigma_0 > 0$ then there exist $M > 1$ such that for any $\varepsilon \in [0,1]$,

$$M^{-1} \leq K_\varepsilon \vee \frac{C_\varepsilon}{\varepsilon^2} \leq M. \tag{10}$$

An important point has to be noted at this step. Since $x \mapsto \bar{p}_t^\varepsilon(x,y)$ is continuously differentiable under **(R)** and **(UE)**, $\bar{P}_t^\varepsilon f$ does not satisfy the transmission condition (3). This is due to the fact that the dynamics (5) does not contain a local time term.

This important observation will characterize the first order expansion of P^ε around $\bar{P}^\varepsilon$ that represents a first step towards our main results. In order to simplify the writing of our next result, we introduce the following operators

$$\Delta_m^{\pm} f = \frac{f(m+) \pm f(m-)}{2}.$$

Lemma 1. *Assume that* **(UE)** *and* **(R)** *are satisfied. For all* $f \in \mathscr{C}_c^\infty(\mathbb{R})$ *and all* $(t,x) \in [0,T] \times \mathbb{R}$*, it holds*

$$P_t^\varepsilon f(x) = \bar{P}_t^\varepsilon f(x) + \alpha a_\varepsilon(0)\varepsilon^\ell \int_0^t \int_{\mathbb{R}} \Delta_0^+ p_{s_1}^\varepsilon(x,\cdot)\partial_x \bar{p}_{t-s_1}^\varepsilon(0,y) f(y)\, dy\, ds_1.$$

By a monotone class argument, one also has the following representation for $(t,x,y) \in (0,T] \times \mathbb{R} \times (\mathbb{R}\backslash\{0\})$*,*

$$p_t^\varepsilon(x,y) = \bar{p}_t^\varepsilon(x,y) + \alpha a_\varepsilon(0)\varepsilon^\ell \int_0^t \Delta_0^+ p_{s_1}^\varepsilon(x,\cdot)\partial_x \bar{p}_{t-s_1}^\varepsilon(0,y)\, ds_1. \tag{11}$$

Proof. We apply the Itô-Tanaka formula to the process $(X_s^\varepsilon)_{s\in[0,t]}$ with the map $[0,t] \times \mathbb{R} \ni (s,x) \mapsto \bar{P}_{t-s}^\varepsilon f(x)$ (see e.g. [11]) and take expectations to obtain:

$$\mathbb{E}[f(X_t^\varepsilon)] = \bar{P}_t^\varepsilon f(x) + \frac{1}{2}\alpha\varepsilon^\ell \mathbb{E}\left[\int_0^t \partial_x \bar{P}_{t-s}^\varepsilon f(0) dL_s^0(X^\varepsilon)\right].$$

Therefore the result follows from the space time formula for local times and the properties stated for the transition density of X^ε. For more details, we apply formula (7.3) in Section 3.7 of [13] and remark that the process $2(\Lambda_t(0,\omega) + \Lambda_t(0-,\omega))$ therein corresponds to $L_t^0(X^\varepsilon)$ in our situation.

For the density representation, one has to use the fact that the formula obtained can be extended to measurable and bounded functions whose support does not contain 0 and from here the density representation follows. □

In the sense of operators which generate the corresponding diffusions we see that the following expansion is satisfied in a *generalized sense*:

$$\begin{aligned}\mathscr{L}^{\varepsilon} =& \bar{\mathscr{L}}^{\varepsilon} + \varepsilon^{\ell}\mathscr{L}^{1,\varepsilon},\\ \mathscr{L}^{1,\varepsilon} =& \alpha a_{\varepsilon}(0)\delta_0\partial_x.\end{aligned}$$

where $\mathscr{L}^{\varepsilon}$ and $\bar{\mathscr{L}}^{\varepsilon}$ stand for the infinitesimal generators associated to P^{ε} and $\bar{P}^{\varepsilon}$.

Our next goal in order to obtain an expansion for p^{ε}_t is to iterate the above identity. This gives our first main result.

Theorem 2.1 *Let $T>0$. Under* **(UE)** *and* **(R)***, for all $(\varepsilon,\alpha)\in[0,1]\times(-1,1)$, the transition density associated to the unique weak solution to the SDE* (1) *admits the following representation for any $(t,x,y)\in(0,T]\times\mathbb{R}\times(\mathbb{R}\backslash\{0\})$,*

$$p^{\varepsilon}_t(x,y) = \bar{p}^{\varepsilon}_t(x,y) + \sum_{p\geq 1}(\alpha a_{\varepsilon}(0)\varepsilon^{\ell})^p I^{\varepsilon}_p(t,x,y), \tag{12}$$

with

$$I^{\varepsilon}_p(t,x,y) := \int_{\Delta_p(t)} \prod_{k=1}^{p-1}\partial_x\bar{p}^{\varepsilon}_{s_k-s_{k+1}}(0,0)\bar{p}^{\varepsilon}_{s_p}(x,0)\partial_x\bar{p}^{\varepsilon}_{t-s_1}(0,y)\,d\mathbf{s}_p, \quad p\geq 1,$$

and the convention $s_0=t$ with $\Delta_p(t) := \{(s_1,....,s_p); 0<s_p<...<s_1<t\}$ and $d\mathbf{s}_p = ds_1...ds_p$.

Proof. In order to iterate formula (11), one needs to derive a similar formula for $\Delta^+_0 p^{\varepsilon}_{s_1}(x,\cdot)$. As we want to take limits in (11), we need to prove the uniform integrability of the integrands. We note here that the estimate in (9) can not be directly used here because formula (11) is not valid for $y=0$.

In order to obtain the uniform integrability, we remark that (11) implies that for any $h\in(0,1]$

$$\begin{aligned}\frac{p^{\varepsilon}_t(x,h)+p^{\varepsilon}_t(x,-h)}{2} =& \frac{\bar{p}^{\varepsilon}_t(x,h)+p^{0}_t(x,-h)}{2}\\ &+\alpha a_{\varepsilon}(0)\varepsilon^{\ell}\int_0^t \Delta^+_0 p^{\varepsilon}_{s_1}(x,\cdot)\frac{\partial_x\bar{p}^{\varepsilon}_{t-s_1}(0,h)\ +\partial_x\bar{p}^{\varepsilon}_{t-s_1}(0,-h)}{2}\,ds_1.\end{aligned}$$

Moreover, from (6) and (7), one has

$$\begin{aligned}&\frac{\partial_x\bar{p}^{\varepsilon}_{t-s_1}(0,h)\ +\partial_x\bar{p}^{\varepsilon}_{t-s_1}(0,-h)}{2}\\ =&\frac{1}{2}\left\{\frac{h-b_{\varepsilon}(h)(t-s_1)}{\sigma^2_{\varepsilon}(h)(t-s_1)} - \frac{h+b_{\varepsilon}(-h)(t-s_1)}{\sigma^2_{\varepsilon}(-h)(t-s_1)}\right\}\bar{p}^{\varepsilon,1}_{t-s_1}(0,h)\\ &+\frac{h+b_{\varepsilon}(-h)(t-s_1)}{2\sigma^2_{\varepsilon}(-h)(t-s_1)}(\bar{p}^{\varepsilon,1}_{t-s_1}(0,h)-\bar{p}^{\varepsilon,1}_{t-s_1}(0,-h))\\ &+\frac{\partial_x\bar{p}^{\varepsilon,2}_{t-s_1}(0,h)\ +\partial_x\bar{p}^{\varepsilon,2}_{t-s_1}(0,-h)}{2}\\ =:&A_1+A_2+A_3.\end{aligned}$$

From (7), it is clear that $|A_3| \le K_\varepsilon (t-s_1)^{-1+\frac{\eta}{2}}$ for some $K_\varepsilon > 0$ independent of h. Now, from **(UE)**, **(R)** and standard computations that we omit, on the set $\left\{s_1 \in [0,t] : h < (t-s_1)^\theta\right\}$, with θ chosen so that $\frac{1}{1+\eta} < 2\theta < 1$, we obtain

$$|A_1| + |A_2| \le K_\varepsilon \left\{ (t-s_1)^{-\frac{1}{2}} + (t-s_1)^{-\frac{3}{2}+\theta(1+\eta)} \right\}.$$

On the set $\left\{s_1 \in [0,t] : h \ge (t-s_1)^\theta\right\}$, from (8) and $\theta < 1/2$, one gets

$$\begin{aligned}|\partial_x \bar{p}^\varepsilon_{t-s_1}(0,h)| + |\partial_x \bar{p}^\varepsilon_{t-s_1}(0,-h)| \le & K_\varepsilon \frac{1}{(t-s_1)} \exp\left(-C_\varepsilon \frac{h^2}{\varepsilon^2 (t-s_1)}\right) \\ \le & K_\varepsilon \frac{1}{(t-s_1)} \exp\left(-C_\varepsilon \frac{1}{\varepsilon^2 (t-s_1)^{1-2\theta}}\right).\end{aligned}$$

Letting $h \downarrow 0$, it then follows from the dominated convergence and the continuity at 0 of $y \mapsto \bar{p}^\varepsilon_t(0,y)$ and $y \mapsto \partial_x \bar{p}^\varepsilon_t(0,y)$ that

$$\Delta_0^+ p^\varepsilon_t(x,.) = \bar{p}^\varepsilon_t(x,0) + \alpha a_\varepsilon(0) \varepsilon^\ell \int_0^t \Delta_0^+ p^\varepsilon_{s_1}(x,.) \partial_x \bar{p}^\varepsilon_{t-s_1}(0,0)\, ds_1. \qquad (13)$$

Moreover, it follows from estimates (2) and (9) that

$$s_1 \mapsto \Delta_0^+ p^\varepsilon_{s_1}(x,.) \partial_x \bar{p}^\varepsilon_{t-s_1}(0,0) \in L^1([0,t]).$$

Hence, one may iterate (13) so that

$$\Delta_0^+ p^\varepsilon_t(x,.) = \bar{p}^\varepsilon_t(x,0) + \sum_{p \ge 1} (\alpha a_\varepsilon(0) \varepsilon^\ell)^p I^\varepsilon_p(t,x,0). \qquad (14)$$

Indeed, it follows from (8) and (9) that there exists K_ε such that for any positive integer p

$$\begin{aligned}&(\alpha a_\varepsilon(0) \varepsilon^\ell)^p \left| I^\varepsilon_p(t,x,0) \right| \\ \le & K_\varepsilon (\alpha a_\varepsilon(0) \varepsilon^\ell K_\varepsilon)^p \int_{\Delta_p(t)} \prod_{k=1}^{p-1} (s_k - s_{k+1})^{-1+\frac{\eta}{2}} s_p^{-\frac{1}{2}} (t-s_1)^{-1+\frac{\eta}{2}}\, d\mathbf{s}_p \\ =: & \int_{\Delta_p(t)} G_p(\mathbf{s}_p)\, d\mathbf{s}_p.\end{aligned}$$

The function $G_p(\mathbf{s}_p)$ defined above satisfies for $C^\varepsilon_1 := \alpha \varepsilon^\ell a_\varepsilon(0) K_\varepsilon t^{\frac{\eta}{2}}$ and any $r > 0$

$$
\begin{aligned}
&\sum_{p\geq 1}\int_{\Delta_p(t)} d\mathbf{s}_p G_p(\mathbf{s}_p) \qquad (15)\\
=&K_\varepsilon t^{-\frac{1+\eta}{2}}\sum_{p\geq 1}(C_1^\varepsilon)^p\prod_{k=0}^{p-1}B\left(\frac{1+k\eta}{2},\frac{\eta}{2}\right)\\
\leq &K_\varepsilon t^{-\frac{1+\eta}{2}}\sum_{p\geq 1}\left(C_1^\varepsilon\Gamma(\frac{\eta}{2})\right)^p\frac{1}{\Gamma(\frac{1+p\eta}{2})}<\infty.
\end{aligned}
$$

where we have used the inequality $x^r e^{-x}\leq C_r$, $x>0$, for the last inequality. Here, B and Γ denote the Beta and Gamma functions respectively. From this last estimate, we deduce that the expansion (14) is satisfied. We then plug (14) into (13) and use Fubini's theorem in order to derive the small skew expansion (12). □

As an illustration of the theorem, we consider the case of the skew Brownian motion that is we set $\sigma_0=0$, $\sigma\equiv 1$, $b\equiv 0$. Hence, $\bar{p}^\varepsilon$ is the fundamental solution associated to the operator $\frac{1}{2}\varepsilon^2\partial_x^2$ and is given by $\bar{p}_t^\varepsilon(x,y)=g_{\varepsilon^2 t}(y-x)$ so that Theorem 2.1 asserts that for all $(t,x,y)\in(0,T]\times\mathbb{R}\times(\mathbb{R}\backslash\{0\})$

$$
p_t^\varepsilon(x,y)=\bar{p}_t^\varepsilon(x,y)+\alpha\varepsilon^{2+\ell}\int_0^t ds\,\bar{p}_s^\varepsilon(x,0)\partial_x\bar{p}_{t-s}^\varepsilon(0,y). \qquad (16)
$$

This is one possible expression for the transition density of the skew Brownian motion, see e.g. Lejay [17]. Hence, Theorem 2.1 appears as an extension of the representation formula (16) to the more general class of skew diffusions.

Another application consists of measuring the distance between models.

Corollary 1. *For any $t\in(0,T]$, the total variation distance between the laws of X_t^ε and $\bar{X}_t^\varepsilon$ can be bounded by*

$$
\int|p_t^\varepsilon(x,y)-\bar{p}_t^\varepsilon(x,y)|dy\leq K_\varepsilon\alpha a_\varepsilon(0)\varepsilon^\ell,
$$

for some constant $K_\varepsilon:=K_\varepsilon(T)$, uniformly bounded in $\varepsilon\in[0,1]$ such that $T\mapsto K_\varepsilon(T)$ is non-decreasing.

Proof. Note that

$$
\begin{aligned}
&(\alpha a_\varepsilon(0)\varepsilon^\ell)^p\left|I_p^\varepsilon(t,x,y)\right|\\
\leq &K_\varepsilon(\alpha a_\varepsilon(0)\varepsilon^\ell K_\varepsilon)^p\int_{\Delta_p(t)}\prod_{k=1}^{p-1}(s_k-s_{k+1})^{-1+\frac{\eta}{2}}s_p^{-\frac{1}{2}}(t-s_1)^{-1}\exp\left(-C_\varepsilon\frac{y^2}{\varepsilon^2(t-s_1)}\right)d\mathbf{s}_p
\end{aligned}
$$

so that

$$
\begin{aligned}
&(\alpha a_\varepsilon(0)\varepsilon^\ell)^p\int_{\mathbb{R}}\left|I_p^\varepsilon(t,x,y)\right|dy\\
\leq &\varepsilon C_\varepsilon^{-1/2}(\alpha a_\varepsilon(0)\varepsilon^\ell)^p K_\varepsilon^{p+1}\int_{\Delta_p(t)}d\mathbf{s}_p\prod_{k=1}^{p-1}(s_k-s_{k+1})^{-1+\frac{\eta}{2}}s_p^{-\frac{1}{2}}(t-s_1)^{-1/2}.
\end{aligned}
$$

Now the same line of calculations using the Beta function and (10) yields the result.

3 Application to a financial model

We will consider the following stochastic differential equation as an extension of the Black-Scholes model where the local volatility is given by a simple regime switching in the following sense

$$S_t = S_0 + \int_0^t \sigma(S_s) S_s dB_s, \tag{17}$$

with

$$\sigma(s) = \sigma_1 1_{[0,\mathsf{s}_1)}(s) + \sigma_2 1_{[\mathsf{s}_1,\infty)}(s).$$

For more background on the financial meaning of the model, we refer the reader to [10]. In order to transform this model into a skew diffusion, we let $F : \mathbb{R} \to \mathbb{R}$ be the continuous increasing function defined by

$$\begin{aligned}
F(y) &= \frac{y}{\sigma_1} 1_{(-\infty,\mathsf{s}_1^*)}(y) + \left(\frac{y}{\sigma_2} + c_2 \right) 1_{[\mathsf{s}_1^*,\infty)}(y), \\
c_2 &= -\frac{\mathsf{s}_1^*}{\sigma_2} + \frac{\mathsf{s}_1^*}{\sigma_1}, \\
\mathsf{s}_1^* &= \exp(s_1).
\end{aligned}$$

Using Itô's formula, we have

$$\ln(S_t) = \ln(S_0) + \int_0^t \left(\sigma(S_u) dB_u - \frac{\sigma^2(S_u)}{2} du \right).$$

Furthermore, applying the Itô-Tanaka formula for $X_t = F(\ln(S_t))$, we obtain

$$X_t = X_0 + B_t - \int_0^t \frac{\tilde{\sigma}(X_u)}{2} du + \Delta_{a_1}^-(\tilde{\sigma}^{-1}) L_t^{a_1}(\ln(S)).$$

Here, $X_0 = F(\ln(S_0))$, $a_1 := \mathsf{s}_1^*/\sigma_1$ and

$$\tilde{\sigma}(x) := \sigma(e^{F^{-1}(x)}) = \sigma_1 1_{(-\infty,a_1)}(x) + \sigma_2 1_{[a_1,\infty)}(x).$$

Similarly, we apply the Itô-Tanaka formula to $g(\ln(S_t)) := |F(\ln(S_t)) - a_1|$. This gives for $\Delta_{a_1}^+(\tilde{\sigma}^{-1}) := 2^{-1}(\sigma_2^{-1} + \sigma_1^{-1})$,

$$g(\ln(S_t)) = g(\ln(S_0)) + \int_0^t g'(\ln(S_u)) d\ln(S_u) + \Delta_{a_1}^+(\tilde{\sigma}^{-1}) L_t^{a_1}(\ln(S)). \tag{18}$$

Furthermore,

$$\begin{aligned}\int_0^t \operatorname{sgn}(X_u - a_1) dX_u &= \int_0^t \operatorname{sgn}(F(\ln(S_u)) - a_1) \left(dB_u - \frac{\tilde{\sigma}(X_u)}{2} du \right) \\ &= \int_0^t \operatorname{sgn}(F(\ln(S_u)) - a_1) \sigma^{-1}(X_u) d\ln(S_u) \\ &= \int_0^t g'(\ln(S_u)) d\ln(S_u). \end{aligned} \tag{19}$$

Recall that an application of the Itô-Tanaka formula to $|X_t - a_1|$ gives

$$L_t^{a_1}(X) = |X_t - a_1| - |X_0 - a_1| - \int_0^t \operatorname{sgn}(X_u - a_1) dX_u.$$

Comparing the above results in (18) and (19), we get

$$\Delta_{a_1}^{+}(\tilde{\sigma}^{-1}) L_t^{a_1}(\ln(S)) = L_t^{a_1}(X).$$

This gives that

$$X_t = X_0 + B_t - \int_0^t \frac{\tilde{\sigma}(X_u)}{2} du + \alpha_1 L_t^{a_1}(X),$$

with $\alpha_1 := \frac{\Delta_{a_1}^{-}(\tilde{\sigma}^{-1})}{\Delta_{a_1}^{+}(\tilde{\sigma}^{-1})} \in (-1,1)$ for $\Delta_{a_1}^{-}(\tilde{\sigma}^{-1}) := 2^{-1}(\sigma_2^{-1} - \sigma_1^{-1})$.

Using the technique exposed in the previous section, we can now study an expansion of this model with $\varepsilon = 2\frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1} > 0$, $\alpha = -\frac{1}{2}$, $\ell = 1$. In this setting, we can set-up

$$\begin{aligned} b_\varepsilon(x) :=& \tilde{\sigma}(x) = \sigma_1 + \frac{\varepsilon}{2}(\sigma_2 + \sigma_1) 1_{[a_1,\infty)}(x) \\ a_\varepsilon(x) =& \sigma_0 = 1. \end{aligned}$$

There will be a slight difference in the definition of $\bar{X}^\varepsilon$ which will be given by

$$\bar{X}_t = X_0 + B_t - \frac{\sigma_1}{2} t.$$

Besides the above change the hypotheses **(R)** and **(UE)** are clearly satisfied. We now mention the main modifications to the proof of Theorem 2.1. First, in Lemma 1, we obtain instead that for all $f \in \mathscr{C}_c^\infty(\mathbb{R})$, for any $x \in \mathbb{R}$, one has

$$\begin{aligned} P_t^\varepsilon f(x) =& \bar{P}_t f(x) + \int_0^t \int_{\mathbb{R}^2} p_s^\varepsilon(x, x_1)(b_\varepsilon - \sigma_1)(x_1) \partial_x \bar{p}_{t-s}(x_1, y) f(y) dx_1 dy ds \\ &+ \alpha_1 \varepsilon \int_0^t \int_{\mathbb{R}} \Delta_{a_1}^{+} p_s^\varepsilon(x, \cdot) \partial_x \bar{p}_{t-s}(a_1, y) f(y) dy ds. \end{aligned} \tag{20}$$

Here $\bar{p}_t(x,y) = g_t(y - x + \frac{\sigma_1}{2}t)$ is the density of $\bar{X}_t$. The above shows that the argument can be applied using the "error operator " $\mathscr{L}^{1,\varepsilon}$ given by

$$\mathscr{L}^{1,\varepsilon} f(x) = \varepsilon \left(\frac{\sigma_2 + \sigma_1}{2} 1_{[a_1,\infty)}(x) + \alpha_1 \delta_{a_1}(x) \right) f'(x).$$

The analysis of integrability of multiple integrals and the estimates in (6)-(9) are straightforward because the density of $\bar{X}_t$ can be explicitly written and in fact the bounds can be improved in comparison with (7)$\sim$(10). This is due to the fact that the diffusion coefficient is constant and previously it was a general Hölder continuous function.

Continuing with the argument, instead of $I_p^\varepsilon(t,x,y)$ in (12), we obtain

$$\begin{aligned}
&\bar{P}_{s_p} \mathscr{L}^{1,\varepsilon} \bar{P}_{s_{p-1}-s_p} \cdots \bar{P}_{s_1-s_2} \mathscr{L}^{1,\varepsilon} \bar{P}_{t-s_1}(\delta_y(.))(x) \\
&= \int_{\mathbb{R}^p} dx_1 ... dx_p \bar{p}_{s_p}(x,x_p) \left((b_\varepsilon(x_1) - \sigma_1) \partial_x \bar{p}_{t-s_1}(x_1,y) + \alpha_1 \varepsilon \delta_{a_1}(x_1) \partial_x \bar{p}_{t-s_1}(x_1,y) \right) \\
&\times \prod_{k=1}^{p-1} \left((b_\varepsilon(x_{k+1}) - \sigma_1) \partial_x \bar{p}_{s_k - s_{k+1}}(x_{k+1},x_k) + \alpha_1 \varepsilon \delta_{a_1}(x_{k+1}) \partial_x \bar{p}_{s_k - s_{k+1}}(x_{k+1},x_k) \right).
\end{aligned}$$

In the present case the partial derivatives are explicit. In particular,

$$\begin{aligned}
|\partial_x \bar{p}_t(x,y)| &\leq \frac{C}{\sqrt{t}} \bar{p}_t(x,y), \\
|\partial_x \bar{p}_t(x,x)| &\leq \frac{C}{\sqrt{t}}.
\end{aligned}$$

Therefore one can easily conclude an equivalent result to Corollary 1 so that for any bounded payoff function f, it holds

$$\left| \mathbb{E}[f(S_t)] - \mathbb{E}[f(\tilde{S}_t)] \right| \leq C |\sigma_2 - \sigma_1|.$$

Here $\tilde{S}$ corresponds to the Black-Scholes model with volatility σ_1. By inspecting the proof, one can see that with a little bit more effort, one can achieve a similar result for linearly growing test functions f. Furthermore as $\bar{p}$ can be expressed explicitly in the present case higher order expansions can be obtained.

A similar argument could be achieved for a more general class of regime switching models such as the following extension of the Black-Scholes model where the volatility is subjected to $n+1$ regime switching models in the following sense

$$S_t = S_0 + \int_0^t \sigma(S_s) S_s dB_s, \tag{21}$$

where σ is defined by

$$\sigma(s) = \sum_{i=1}^{n+1} \sigma_i 1_{[\mathsf{s}_{i-1},\mathsf{s}_i)}(s).$$

Here $\sigma_1,\ldots,\sigma_{n+1} \in \mathbb{R}_+$ and $0 = \mathsf{s}_0 < \mathsf{s}_1 < \cdots < \mathsf{s}_n < \mathsf{s}_{n+1} = \infty$.

Rather than continuing on this path, we would like to present how to compute the full expansion of the density p^ε. First note that the first term in (20) gives as the lowest order term of the expansion

$$\bar{P}_{s_p}\mathscr{L}^{1,\varepsilon}\bar{P}_{s_{p-1}-s_p}\cdots\bar{P}_{s_1-s_2}\mathscr{L}^{1,\varepsilon}\bar{P}_{t-s_1}(\delta_y(.))(x) = \varepsilon^p A_p(s_p,...,s_1).$$

Here, for $\tilde{b}(x) := \frac{\sigma_1+\sigma_2}{2} 1_{[a_1,\infty)}(x)$

$$\begin{aligned}
&A_p(s_p,...,s_1) := \\
&\int_{\mathbb{R}^p} dx_1...dx_p \bar{p}_{s_p}(x,x_p) \left(\tilde{b}(x_1)\partial_x \bar{p}_{t-s_1}(x_1,y) + \alpha_1 \delta_{a_1}(x_1)\partial_x \bar{p}_{t-s_1}(x_1,y)\right) \\
&\times \prod_{k=1}^{p-1} \left(\tilde{b}(x_{k+1})\partial_x \bar{p}_{s_k-s_{k+1}}(x_{k+1},x_k) + \alpha_1 \delta_{a_1}(x_{k+1})\partial_x \bar{p}_{s_k-s_{k+1}}(x_{k+1},x_k)\right).
\end{aligned}$$

From here one has the expansion

$$p_t^\varepsilon(x,y) = \bar{p}_t(x,y) + \sum_{p=1}^{\infty} \varepsilon^p \int_{\Delta_p(t)} d\mathbf{s}_p A_p(s_p,...,s_1).$$

The full expansion for the density of S_t can be easily achieved by a change of variables formula. In the next section, we will return to our main problem which is slightly more complex as the approximating densities also depend on ε.

4 Asymptotics of the density $\bar{p}^\varepsilon$

Now we return to the general case studied in Theorem 2.1 and note that we have reduced the problem of the expansion of p^ε to the expansion of the density of $\bar{X}_t^\varepsilon$ which is a one dimensional diffusion. In order to carry out this expansion, we can use classical techniques to expand its density in the case[4] $\sigma_0 = 0$. In fact, we will use the Lamperti transformation as follows:

$$F(x) := \int_0^x \frac{1}{\sigma(y)} dy$$

Note that this function is well defined due to hypothesis **(UE)**. Then a combination of Itô's formula and the Girsanov measure transformation under hypothesis **(R)**, gives the following expression with $Z_t^\varepsilon := F(x) + \varepsilon W_t$:

[4] The case $\sigma_0 > 0$ can also be achieved with further computational effort.

$$\begin{aligned}\mathbb{E}[f(\bar{X}_t^\varepsilon)] =& \mathbb{E}\left[f(F^{-1}(Z_t^\varepsilon))M_t^\varepsilon\right] \\ \bar{b}_\varepsilon(x) :=& -\frac{1}{2}\sigma'(F^{-1}(x)) + \varepsilon^{-2}\frac{b_\varepsilon}{\sigma}(F^{-1}(x)) \\ M_t^\varepsilon :=& \exp\left(\varepsilon\int_0^t \bar{b}_\varepsilon(Z_s^\varepsilon)dW_s - \frac{\varepsilon^2}{2}\int_0^t \bar{b}_\varepsilon^2(Z_s^\varepsilon)ds\right).\end{aligned} \tag{22}$$

Using the explicit density of $F(x)+\varepsilon W_t$ and the fact that $\{W_s; s\in[0,t]\}$ conditioned to $W_t = y$ has the law of a Brownian bridge, we obtain :

$$\bar{p}_t^\varepsilon(x,y) = \frac{g_{\varepsilon^2 t}(F(x),F(y))}{\sigma(y)}\mathbb{E}\left[\exp\left(\varepsilon\int_0^t \bar{b}_\varepsilon(B_s^{x,y})dB_s^{x,y} - \frac{\varepsilon^2}{2}\int_0^t \bar{b}_\varepsilon^2(B_s^{x,y})ds\right)\right].$$

Here $B_s^{x,y}$ represents a Brownian bridge with variance ε^2 at time s, starting at $F(x)$ and arriving at $F(y)$ at time t. From now on, we will assume the following hypothesis:

(R') The measurable functions $\bar{b}_\varepsilon : \mathbb{R}\to\mathbb{R}$ are bounded uniformly in ε.

Therefore in order to obtain an expansion for the above expectation, one may just use the Taylor expansion for the exponential function and the following representation for the Brownian bridge (see Section 5.6.B in [13])

$$\begin{aligned}B_s^{x,y} =& F_{s,t}(x,y) + \varepsilon(t-s)\int_0^s \frac{dW_u}{t-u}, \\ F_{s,t}(x,y) :=& F(x)\left(1-\frac{s}{t}\right) + F(y)\frac{s}{t}.\end{aligned}$$

As the proofs on the order of the residues are cumbersome, from now on we only give heuristic expansions so that the way calculations are performed is clear.

For example, the first term in the expansion requires the computation of

$$\mathbb{E}\left[\varepsilon\int_0^t \bar{b}_\varepsilon(B_s^{x,y})dB_s^{x,y}\right] = \mathbb{E}\left[\varepsilon\int_0^t \bar{b}_\varepsilon(B_s^{x,y})\left(\partial_s F_{s,t}(x,y) - \varepsilon\int_0^s \frac{dW_u}{t-u}\right)ds\right]. \tag{23}$$

Further calculations will require explicit values for the coefficient functions b_ε and σ but it should be clear that the only underlying law needed for this calculation is the one related to the Brownian motion W. For example, assume that one may expand $\bar{b}_\varepsilon(B_s^{x,y})$ as follows:

$$\begin{aligned}\bar{b}_\varepsilon(B_s^{x,y}) \approx& \bar{b}_\varepsilon(F_{s,t}(x,y)) + \bar{b}'_\varepsilon(F_{s,t}(x,y))(t-s)\varepsilon\int_0^s \frac{dW_u}{t-u} \\ &+ \frac{1}{2}\bar{b}''_\varepsilon(F_{s,t}(x,y))(t-s)^2\varepsilon^2\left(\int_0^s \frac{dW_u}{t-u}\right)^2.\end{aligned}$$

Using the above expansion in (23), one obtains:

$$\mathbb{E}\left[\varepsilon\int_0^t \bar{b}_\varepsilon(B_s^{x,y})dB_s^{x,y}\right] \approx \varepsilon\int_0^t \bar{b}_\varepsilon(F_{s,t}(x,y))\partial_s F_{s,t}(x,y)ds - \varepsilon^3\int_0^t \bar{b}'_\varepsilon(F_{s,t}(x,y))\frac{s}{t}ds + \frac{\varepsilon^3}{2}\int_0^t \bar{b}_\varepsilon\bar{b}''_\varepsilon(F_{s,t}(x,y))(t-s)^2\partial_s F_{s,t}(x,y)ds.$$

Putting the above expansions together, one obtains:

$$\bar{p}_t^\varepsilon(x,y) \approx \frac{g_{\varepsilon^2 t}(F(x),F(y))}{\sigma(y)} \tag{24}$$

$$\times\left(1+\varepsilon\int_0^t \bar{b}_\varepsilon(F_{s,t}(x,y))\partial_s F_{s,t}(x,y)ds - \frac{\varepsilon^2}{2}\int_0^t \bar{b}_\varepsilon^2(F_{s,t}(x,y))ds\right)$$

$$=: B_0(\varepsilon). \tag{25}$$

This gives the expansion for the first term of the main result in Theorem 2.1. For the following term, we see that one also needs an expansion for the derivative $\partial_x\bar{p}_t^\varepsilon(x,y)$.

In order to do this, we take $f=\delta_y$ in (22) and differentiate with respect to x. This gives:

$$\partial_x\bar{p}_t^\varepsilon(x,y) = \frac{\sigma(y)}{\sigma(x)}\mathbb{E}\left[\delta'_y(\bar{X}_t^\varepsilon)M_t^\varepsilon\right] + \sigma(x)^{-1}\mathbb{E}\left[\delta_y(\bar{X}_t^\varepsilon)M_t^\varepsilon Y_t^\varepsilon\right] \tag{26}$$

$$Y_t^\varepsilon := \varepsilon\left(\int_0^t \bar{b}'_\varepsilon(F(x)+\varepsilon W_s)dW_s - \varepsilon\int_0^t \bar{b}_\varepsilon\bar{b}'_\varepsilon(F(x)+\varepsilon W_s)ds\right).$$

The second term in (26) can be treated as we did in the expansion of $\bar{p}^\varepsilon$. For the first term, we need to use the integration by parts formula of Malliavin Calculus. Without going into much detail into this theory[5], let us just state the result:

$$\mathbb{E}\left[\delta'_y(\bar{X}_t^\varepsilon)M_t^\varepsilon\right] = (\sigma(y)\varepsilon t)^{-1}\mathbb{E}\left[\delta_y(\bar{X}_t^\varepsilon)M_t^\varepsilon\left(W_t - \varepsilon\int_0^t\left(\bar{b}_\varepsilon(Z_s^\varepsilon)ds + s dY_s^\varepsilon\right)\right)\right]$$

$$=: (\sigma(y)^2\varepsilon t)^{-1}g_{\varepsilon^2 t}(F(x),F(y))A_1(t).$$

The above expression can now be expanded as we did previously using the Brownian bridge $B^{x,y}$. The explicit expansion of $A_1(t)$ is performed in Section 6.

These are the essential mathematical elements required in order to obtain the expansions. The rest of the calculations, which are briefly explained in the Appendix, require careful attention as there are many terms. We refer the reader to Appendix 2 for the explicit but long expressions. Taking into account the calculations in the Appendix 2, we obtain the following result:

Result Let X^ε be the unique solution of (1) with $\ell \geq 1$ and $(t,x,y) \in (0,T]\times\mathbb{R}\times(\mathbb{R}\backslash\{0\})$. Assume the hypotheses **(UE)**, **(R)** and **(R')**. Furthermore, assume that $\bar{b}_\varepsilon \in C^2$ and that $\|\bar{b}_\varepsilon^{(i)}\|_\infty \leq C$, $i=1,2$. Then, it holds

$$p_t^\varepsilon(x,y) \approx B_0(\varepsilon) + \alpha\varepsilon^\ell\left(\sigma(y)^{-1}g_{\varepsilon^2 t}(|F(x)|,|F(y)|) + \varepsilon B_1 + \varepsilon^2 B_2\right).$$

[5] We refer the interested reader to [19] for definitions and notation.

We quote the above statement as a result and not a theorem because proving error rates is a technical and cumbersome argument which we have avoided here. In particular, the error rate seems to be bounded by $\varepsilon^3 g_{\varepsilon^2 t}(F(x),F(y))$. In the case that $\bar{b}_\varepsilon$ is not differentiable but explicit as in Section 3, a careful analysis about the expansion of M_t^ε is needed. We do not develop these issues here.

5 Appendix 1: The parametrix method applied to a skew diffusion

In this section, we give a brief and sketchy argument to show what is claimed in footnote 3. In this sense, it may be easier for a beginner as we will not deal with technicalities.

Our goal is to prove the existence and some regularity estimates for the density of the random variable X_t^ε in (1). In order to simplify notation, we will just explain how to obtain properties of the law of

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \alpha L_t^m(X).$$

Recall that we are assuming that b is bounded and measurable and σ is Hölder continuous and bounded.

The approximation will be based on the skew Brownian motion with $\alpha \in (-1,1)$ and $m \in \mathbb{R}$ which is the unique solution of the equation

$$X_t^\alpha = x + \bar{\sigma} W_t + \alpha L_t^m(X^\alpha).$$

for some well-chosen $\bar{\sigma}$. Its density is given explicitly by

$$\begin{aligned} p_t^\alpha(x,y) :=& g_{t\bar{\sigma}^2}(y-x) + \operatorname{sgn}(y-m)\alpha g_{t\bar{\sigma}^2}(|y-m|+|x-m|) \\ g_t(y) :=& \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right). \end{aligned}$$

Note that this density is in general discontinuous at $y = m$.

We will be using as the parametrix operator, the following linear operator which is well defined for any bounded measurable function f:

$$\begin{aligned} \bar{P}_t f(x) =& \int_{\mathbb{R}} f(y)\bar{p}_t(x,y)dy \\ \bar{p}_t(x,y) :=& g_{t\sigma^2(y)}(y-x) + \operatorname{sgn}(y-m)\alpha g_{t\sigma^2(y)}(|y-m|+|x-m|). \end{aligned}$$

Note that $y \mapsto \bar{p}_t(x,y)$ is not a density but $\bar{P}_t f$ is differentiable w.r.t. x with a second generalized derivative which satisfies the transmission condition (3) with $\varepsilon = 1$ for any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$.

From these properties, we see that the Itô-Tanaka formula can be applied (see e.g. formula 2.5 in [11]) to $\bar{P}_{T-t}f(X_t)$, $t\in[0,T]$. This gives after explicit calculations [6]

$$\begin{aligned}
&\mathbb{E}[f(X_T)]-\bar{P}_T f(x)\\
&=\int_0^T \mathbb{E}\left[\left(\partial_s \bar{P}_{T-s}f\right)(X_s)+b(X_s)\partial_x \bar{P}_{T-s}f(X_s)+\frac{1}{2}\sigma^2(X_s)\partial_x^2\bar{P}_{T-s}f(X_s)1_{(X_s\neq m)}\right]ds\\
&=\int_0^T \mathbb{E}\left[b(X_s)\partial_x \bar{P}_{T-s}f(X_s)+\left(\partial_s \bar{P}_{T-s}f+\frac{1}{2}\sigma^2\partial_x^2\bar{P}_{T-s}f\right)(X_s)1_{(X_s\neq m)}\right]ds.
\end{aligned}$$

Iterating the above expression one can obtain a semi-explicit expansion of the form:

$$\begin{aligned}
\mathbb{E}[f(X_T)] =&\bar{P}_T f(x)+\sum_{p=1}^{\infty}\int_{\Delta_p}\bar{P}_{s_p}Q_{s_{p-1}-s_p}...Q_{T-s_1}f(x)ds_1...ds_p \qquad (27)\\
Q_t f(x):=&b(x)\partial_x\bar{P}_t f(x)+\left(-\partial_t\bar{P}_t f+\frac{1}{2}\sigma^2\partial_x^2\bar{P}_t f\right)(x)1_{(x\neq m)}\\
\Delta_p(T):=&\{(s_1,....,s_p);0<s_p<...<s_1<T\}.
\end{aligned}$$

Clearly, for the above expansion to be valid we need to prove convergence properties for the above infinite sum. Direct calculations using the definition of $\bar{P}$ and the assumptions on b and σ give (see (29) to guess why the estimate below is correct)

$$\|Q_t f\|_\infty \le \frac{C}{t^{(1-\eta/2)\wedge(1/2)}},$$

where η is the Hölder exponent of σ^2. Therefore iterative integration over the simplex[7] Δ_p gives that the infinite sum in (27) is absolutely convergent.

As this formula is valid for all bounded and measurable functions $f:\mathbb{R}\to\mathbb{R}$, this result also gives an expression for the density of X_T which can then be differentiated term by term in order to obtain the claimed properties. In fact, the density $p_t(x,y)$ of X_t can be written as in the following expansion

$$p_t(x,y)=\bar{p}_t(x,y)+\sum_{p=1}^{\infty}\int_{\Delta_p}\bar{P}_{s_p}Q_{s_{p-1}-s_p}...Q_{T-s_1}\delta_y(x)ds_1...ds_p. \qquad (28)$$

We remark here that

$$Q_s\delta_y(x)=b(x)\partial_x\bar{p}_t(x,y)+\frac{1}{2}(\sigma^2(y)-\sigma^2(x))\partial_x^2\bar{p}_t(x,y)1_{(x\neq m)}. \qquad (29)$$

From (28), one can easily deduce that $p_t(x,m\pm)$ exist. Moreover, by direct differentiation one also obtains for $x\neq m$

[6] Note that $\int_0^T 1_{(X_s=m)}ds=0$ due to the time-space formula for semimartingales. For more details, see formula (7.3) in Section 3.7 of [13].

[7] This requires the use of the Beta function.

$$\partial_x p_t(x,y) = \partial_x \bar{p}_t(x,y) + \sum_{p=1}^{\infty} \int_{\Delta_p} \int \partial_x \bar{p}_{s_p}(x,z) Q_{s_{p-1}-s_p} \dots Q_{T-s_1} \delta_y(z) dz ds_1 \dots ds_p.$$

This gives the existence of the derivative $\partial_x p_t(x,\pm m)$ and Gaussian type upper estimates by using the inequality $x^p e^{-cx^2} \leq C(p,c)$ for $x \geq 0$. For a probabilistic interpretation of a variation of this methodology, we refer the reader to [16].

6 Appendix 2: Explicit calculations for the expansion of $\bar{p}^{\varepsilon}$ and $\partial_x \bar{p}^{\varepsilon}$

In this section, we compute the explicit expressions for the expansion of p^{ε} up to order $\varepsilon^2 g_{\varepsilon^2 t}$.

The argument has already been explained in Section 4. The application of that argument gives (24).

Now, we need to expand $\partial_x \bar{p}^{\varepsilon}_t(x,y)$. This is done through the expansion of the two terms appearing in the right-hand side of (26). Let us start with the simplest of the two which is the second term. To expand this term one uses the same technique as in the above expansion. That is,

$$\begin{aligned}
\mathbb{E}\left[\delta_y(\bar{X}^{\varepsilon}_t) M^{\varepsilon}_t Y^{\varepsilon}_t\right] \approx & \varepsilon \frac{g_{\varepsilon^2 t}(F(x),F(y))}{\sigma(y)} \Bigg(\mathbb{E}\left[\int_0^t \bar{b}'_{\varepsilon}(B^{x,y}_s) dB^{x,y}_s - \varepsilon \int_0^t \bar{b}_{\varepsilon} \bar{b}'_{\varepsilon}(B^{x,y}_s) ds\right] \\
& + \varepsilon \mathbb{E}\left[\int_0^t \bar{b}'_{\varepsilon}(B^{x,y}_s) dB^{x,y}_s \int_0^t \bar{b}_{\varepsilon}(B^{x,y}_s) dB^{x,y}_s\right] \Bigg) \\
\approx & \, \varepsilon \frac{g_{\varepsilon^2 t}(F(x),F(y))}{\sigma(y)} \Bigg(\int_0^t \bar{b}'_{\varepsilon}(F_{s,t}(x,y)) \partial_s F_{s,t}(x,y) ds - \varepsilon \int_0^t \bar{b}_{\varepsilon} \bar{b}'_{\varepsilon}(F_{s,t}(x,y)) ds \\
& + \varepsilon \int_0^t \bar{b}'_{\varepsilon}(F_{s,t}(x,y)) \partial_s F_{s,t}(x,y) ds \int_0^t \bar{b}_{\varepsilon}(F_{s,t}(x,y)) \partial_s F_{s,t}(x,y) ds \Bigg).
\end{aligned}$$

Finally, for the first term in (26), we have for $G_{\varepsilon}(s,z) := \bar{b}_{\varepsilon}(z)\left(1 - s\varepsilon \bar{b}'_{\varepsilon}(z)\right)$

$$\begin{aligned}
A_1(t) := \mathbb{E}\Bigg[& \exp\left(\varepsilon \int_0^t \bar{b}_{\varepsilon}(B^{x,y}_s) dB^{x,y}_s - \frac{\varepsilon^2}{2} \int_0^t \bar{b}^2_{\varepsilon}(B^{x,y}_s) ds\right) \\
& \times \left(\frac{F(y)-F(x)}{\varepsilon} - \varepsilon \int_0^t G_{\varepsilon}(s, B^{x,y}_s) ds + s \bar{b}'_{\varepsilon}(B^{x,y}_s) dB^{x,y}_s\right)\Bigg].
\end{aligned}$$

Using the fact that $\exp(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)$ gives

$$
\begin{aligned}
&\exp\left(\varepsilon\int_0^t \bar{b}_\varepsilon(B_s^{x,y})dB_s^{x,y}-\frac{\varepsilon^2}{2}\int_0^t \bar{b}_\varepsilon^2(B_s^{x,y})ds\right)\\
&\overset{\mathbb{E}}{=} 1+\varepsilon\int_0^t \bar{b}_\varepsilon(B_s^{x,y})dB_s^{x,y}-\frac{\varepsilon^2}{2}\int_0^t \bar{b}_\varepsilon^2(B_s^{x,y})ds\\
&+\frac{1}{2}\varepsilon^2(\int_0^t \bar{b}_\varepsilon(B_s^{x,y})dB_s^{x,y}-\frac{\varepsilon}{2}\int_0^t \bar{b}_\varepsilon^2(B_s^{x,y})ds)^2\\
&+\frac{1}{6}\varepsilon^3(\int_0^t \bar{b}_\varepsilon(B_s^{x,y})dB_s^{x,y}-\frac{\varepsilon}{2}\int_0^t \bar{b}_\varepsilon^2(B_s^{x,y})ds)^3.
\end{aligned}
$$

Using the previous expansion technique, we expand the above expression. This gives

$$
\begin{aligned}
A_1(t)\approx&\frac{F(y)-F(x)}{\varepsilon}\left(1+\varepsilon\int_0^t \bar{b}_\varepsilon(F_{s,t}(x,y))\partial_s F_{s,t}(x,y)ds-\frac{\varepsilon^2}{2}\int_0^t \bar{b}_\varepsilon^2(F_{s,t}(x,y))ds\right.\\
&\left.-\varepsilon^3\int_0^t \bar{b}_\varepsilon'(F_{s,t}(x,y))\frac{s}{t}ds+\frac{\varepsilon^3}{2}\int_0^t \bar{b}_\varepsilon''(F_{s,t}(x,y))\frac{s(t-s)}{t}\partial_s F_{s,t}(x,y)ds\right)\\
&-\varepsilon\left(1+\varepsilon\int_0^t \bar{b}_\varepsilon(F_{s,t}(x,y))\partial_s F_{s,t}(x,y)ds\right)\\
&\times\int_0^t \left(G_\varepsilon(s,F_{s,t}(x,y))-s\bar{b}_\varepsilon'(F_{s,t}(x,y))\partial_s F_{s,t}(x,y)\right)ds.
\end{aligned}
$$

Now we need to put all these estimates together into the first order expression in $I_1^\varepsilon(t,x,y)$ defined in Theorem 2.1. That is,

$$
\begin{aligned}
&\alpha\sigma(0)^2\varepsilon^{2+\ell}\int_0^t \bar{p}_s^\varepsilon(x,0)\partial_x\bar{p}_{t-s}^\varepsilon(0,y)ds\\
\approx&\alpha\varepsilon^\ell\left(\sigma(y)^{-1}g_{\varepsilon^2 t}(|F(x)|,|F(y)|)+\varepsilon B_1+\varepsilon^2 B_2\right).
\end{aligned}
$$

Here, we have used the identity (proven in Lemma 10 in [7])

$$
\int_0^t g_{\varepsilon^2 s}(F(x),0)g_{\varepsilon^2(t-s)}(0,F(y))\frac{F(y)}{\varepsilon^2(t-s)}ds=g_{\varepsilon^2 t}(|F(x)|,|F(y)|).
$$

The following terms in the expansions can be computed and give:

$$
\begin{aligned}
B_1 &:= \int_0^t g_{\varepsilon^2 s}(F(x),0) g_{\varepsilon^2(t-s)}(0,F(y)) \frac{F(y)}{\sigma(y)} \frac{B_1^1(s)}{t-s} ds \\
B_1^1(s) &:= \int_0^s \bar{b}_\varepsilon(F_{u,s}(x,0)) \partial_u F_{u,s}(x,0) du + \int_s^t \bar{b}_\varepsilon(F_{u,t}(0,y)) \partial_u F_{u,t}(0,y) du. \\
B_2 &:= \int_0^t g_{\varepsilon^2 s}(F(x),0) g_{\varepsilon^2(t-s)}(0,F(y)) \frac{F(y)B_2^1(s)+B_2^2(s)}{\sigma(y)(t-s)} ds \\
B_2^1(s) &:= -\frac{1}{2}\int_0^s \bar{b}_\varepsilon^2(F_{u,s}(x,0)) du - \frac{1}{2}\int_s^t \bar{b}_\varepsilon^2(F_{u,t}(0,y)) du \\
&\quad + \int_0^s \bar{b}_\varepsilon(F_{u,s}(x,0)) \partial_u F_{u,s}(x,0) du \int_s^t \bar{b}_\varepsilon(F_{u,t}(0,y)) \partial_u F_{u,t}(0,y) du \\
B_2^2(s) &:= \int_s^t \bar{b}_\varepsilon(F_{u,t}(0,y) - (u-s)\bar{b}'_\varepsilon(F_{u,t}(0,y)) \partial_u F_{u,t}(0,y) du.
\end{aligned}
$$

In fact, with the above formulas one can also compute B_3 but as the reader can guess the expressions are longer. Still, one can see some algebraic structure in these constants. This is inherent to the parametrix approach and it will be interesting to understand this structure which is clearly linked to the so called skeleton of Wiener functionals.

As we have previously explained, our goal here is to explain the technique rather than providing full technical details. For example, one may entertain the possibility that condition **(R')** is not satisfied and expand the density with the same technique if one has an expansion of $\bar{b}_\varepsilon$ in powers of ε.

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