Fractional Growth Portfolio Investment

A.E. Brockwell

Abstract We review some fundamental concepts of investment from a mathematical perspective, concentrating specifically on fractional-Kelly portfolios, which allocate a fraction of wealth to a growth-optimal portfolio, with the remaining fraction collecting (or paying) interest at a risk-free rate. Using stochastic differential equations, we lay out a coherent continuous-parameter time-series framework for analysis of these portfolios, explaining relationships between Sharpe ratios, growth rates, and leverage. We see how Kelly's criterion prescribes the same leverage as Markowitz mean-variance optimization. Furthermore, for fractional Kelly portfolios, we state a simple distributional relationship between portfolio Sharpe ratio, the fractional coefficient, and portfolio log-returns. These results provide critical insight into realistic expectations of growth for different classes of investors, from individuals to quantitative trading operations. We then illustrate application of the results by analyzing performance of various bond and equity mixes for an investor. We also demonstrate how the relationships can be exploited by a simple method-of-moments calculation to estimate portfolio Sharpe ratios and levels of risk deployment, given a fund's reported returns.

1 Introduction

This paper describes quantitative relationships which are important in investment, including the concept of leverage. In doing so, it provides a range of insights that could be useful to home investors, quantitative traders, or anyone who is managing liquid investments.

We are interested in managing a block of capital which can be invested into one or more risky *instruments*, or held in cash/debt at a "risk-free" interest rate. An instrument is simply a particular type of asset that can be bought and sold, and

---

e-mail: anthony.brockwell@gmail.com

exhibits some growth (or decay) in value over time. It is typical to aim for good long-term growth rate in total capital by investing in a mix of different instruments. Below we study the question of determining the "optimal" mix. By nature of the construction of the portfolio, we ensure theoretically, that if gains/losses on capital are re-invested continuously in time, then the pool of capital will retain non-negative value. In other words, we cannot lose more than our initial block of capital. [1]

Leverage is a critical tool in optimizing capital growth. Roughly speaking, leverage is a mechanism by which we can amplify the returns of an investment. While it might seem natural that we should amplify the returns of a "good" investment as much as possible, we will see that this is not a good idea. To give a concrete example, imagine that on two successive days we observe a 10% loss followed by a 10% gain. This leaves us with a cumulative 1% loss ($0.9 \times 1.1 = 0.99$). If we were to amplify the two returns by a factor of two, however, a 20% loss followed by a 20% gain leaves us with a 4% loss ($0.8 \times 1.2 = 0.96$). Thus, somewhat counter-intuitively, doubling the size of our returns causes our cumulative loss to be be *worse* than double the original loss. This asymmetry suggests to us a potential connection between volatility, expected growth, and optimal leverage.

In the remainder of this paper, we consider this problem from a continuous-parameter time series perspective, studying the impact of leverage analytically using stochastic calculus. It is beyond the scope of this document to provide an introduction to stochastic calculus, but for a useful starting point, see, e.g. [9], or [4] for a more comprehensive treatment. To the author's knowledge, the results stated in theorem form in this paper do not appear in existing financial literature, although components of the framework itself, along with associated topics of discussion, can be found in the literature. In particular, [1, 3, 10] give much detailed discussion of Kelly portfolios and their advantages and disadvantages. Useful related information can also be found in [7]. As in [2], we will adopt a continuous-time framework based on the use of stochastic differential equations, to analyze a portfolio of instruments whose prices follow a multivariate geometric Brownian motion. [6] (and references therein) point out the importance of avoiding "bad" outcomes in such portfolios, pointing out that fractional Kelly schemes (among other solutions) are useful in this context. In this paper, we also address this problem, obtaining a specific characterization of the trade-off between growth and safety.

2 Problem Formulation

Suppose that, at time $t = 0$, an investor has a block of capital A_0 to invest, and assume that it can be allocated to cash, which earns interest continuously at the *risk-free rate* r per unit time, or to one or more different risky instruments which generate returns over time. Over time, we will denote the total capital owned by the investor as

[1] However, it is worth noting that this is a theoretical result, since reinvestment of profits/losses continuously in time is a practical impossibility in most cases.

$$\{A_t,\ t \geq 0\}. \tag{1}$$

This quantity should be thought of as the total value of all assets, including investments in particular instruments, and cash/debt. We will assume the existence of a set of m different investment instruments (not including cash), which have price time series

$$\{P_{t,j},\ t \geq 0\},\ j = 1, 2, \ldots, m. \tag{2}$$

In what follows, we will often refer to the vector of prices by

$$\mathbf{P}_t = (P_{t,1}, \ldots, P_{t,m})^T. \tag{3}$$

The collection of investments in the m instruments as well as cash will be referred to as the *portfolio*. As mentioned above, the value of the portfolio at time t is denoted by A_t. Any cash component of the portfolio receives continuously-compounded interest at the risk-free rate r. It will be useful to define a vector version of the risk-free rate

$$\mathbf{r} = (r, r, \ldots, r)^T \in \mathbb{R}^m. \tag{4}$$

We define the *leverage* vector to be

$$\mathbf{k} = (k_1, \ldots, k_m)^T. \tag{5}$$

and we define *total-leverage*

$$\kappa = \sum_{j=1}^{m} k_j. \tag{6}$$

We will say that our portfolio is *non-leveraged* if $\kappa \leq 1$.

At any given time t, the investor invests fractions k_j of his/her capital A_t into the instruments with price $P_{t,j}$, holding the remainder in cash if that remainder is positive, or maintaining the required debt otherwise. Cash is assumed to earn the interest-free rate r, while debt pays interest at the same rate. By construction, the amount of cash held at time t is clearly

$$A_t(1 - \kappa). \tag{7}$$

The leverage vector determines the manner in which returns on the m investment instruments are related to returns in the total capital. To state this property formally, let us define the infinitesimal return of an investment instrument by

$$dP_{t,j}/P_{t,j}. \tag{8}$$

Then define the infinitesimal return of our total capital by

$$dA_t/A_t. \tag{9}$$

Given initial capital A_0, the price processes $\{P_{t,j}\},\ j = 1, \ldots, m$, and a leverage vector $\mathbf{k}$, we can solve for $\{A_t,\ t \geq 0\}$ using the relationship

$$dA_t/A_t = (1-\kappa)rdt + \sum_{j=1}^{m} k_j dP_{t,j}/P_{t,j} \tag{10}$$

Equation (10) determines the distribution of the process $\{A_t\}$, but it can also be viewed as a mathematical definition of leverage. The first term on the right represents the return on capital from risk-free rate interest accrual/payment on the cash/debt portion of the portfolio, while the summation represents return contributed by the leveraged investments when the fractions of A_t invested in $P_{t,j}$ are k_j, for $j = 1,\ldots,m$.

The components of our leverage vector do not necessarily have to be less than one, or add up to one, or even be non-negative.

For example, suppose that we have $m = 2$ possible investments: an S&P500 mutual fund and a long-term bond fund. (We study a case like this in more detail later in the paper.) A leverage vector of $(0.2, 0.2)$ would indicate that at any given point in time, we keep one fifth of our total capital in the S&P500 fund, one fifth in the bond fund, and the remaining three fifths held in cash, accruing interest payment at the risk-free rate. Alternately, a leverage vector of $(0.2, 1.8)$ would indicate that we would invest one fifth of initial capital in the S&P500 fund, borrow an amount equal to the initial capital, and invest the four fifths remaining initial capital, along with the borrowed sum, in the bond fund.

2.1 Objective

There are many possible investment objectives. For example, one could attempt to minimize the probability of ultimately losing all their capital, attempt to maximize the expected return relative to some benchmark, maximize some utility function of wealth, etc. In this paper we aim for good long-term growth profiles for our capital $\{A_t\}$. To state this more precisely, we will be particularly interested in the *expected log-return per unit time*

$$L = \mathbf{E}\left[\log(A_{t+\delta}/A_t)\right]/\delta \tag{11}$$

and *log-return variance per unit time*

$$V = \mathbf{Var}(\log(A_{t+\delta}/A_t))/\delta. \tag{12}$$

As we will see below, this approach leads us to a convenient stochastic calculus-based derivation of Kelly's formula, and yields additional useful insights.

Direct maximization of L (with no regard for V) yields the so-called *growth-optimal portfolio*, also referred to as the *Kelly portfolio*. In the long-run, the growth-optimal portfolio almost-surely leads to more wealth than any competing portfolio. [2]

[2] More formally, if $\{A_t\}$ represents the capital associated with the growth-optimal portfolio, and $\{A'_t\}$ represents the capital from another portfolio in the same instruments, chosen in a different manner, then under a modest set of regularity conditions, we can show that $(A'_t/A_t) \stackrel{a.s.}{\to} 0$ as $t \to \infty$.

The price to pay for this growth dominance however, is relatively large up and down-swings in $\{A_t\}$. *Fractional Kelly schemes*, which simply apply a fractional multiplier to the growth-optimal portfolio leverage vector, provide one way to address this problem.

2.2 Mathematical Analysis

We are now in a position to analyze the impact of leverage choice. For the sake of exposition, we address the univariate case before developing the multivariate case.

2.2.1 Univariate Case

In the univariate ($m = 1$) case, we can write down a formal stochastic differential equation to describe the price $\{P_t\}$ of an instrument over time $t \in \mathbb{R}, t \geq 0$, as

$$dP_t = \mu P_t dt + \sigma P_t dW_t, \tag{13}$$
$$P_0 = 1, \tag{14}$$

where $\{W_t\}$ is a standard Brownian motion.

The solution to (13,14) is a geometric Brownian motion, for which

$$d\log(P_t) = (\mu - \sigma^2/2)dt + \sigma dW_t. \tag{15}$$

(This follows directly from an application of Itô's formula to (13,14).) The process has several important properties, which we will simply state here without proof.

1. It provides a realistic description of many real-life instruments that can be bought as investments. The parameters μ and σ vary, however, over different investments and arguably also over time for a particular investment.
2. $P_t > 0$.
3. The log-return of the price satisfies

$$\log(P_{t+\delta}/P_t) \sim \mathrm{N}(\mu\delta - \sigma^2\delta/2, \sigma^2\delta). \tag{16}$$

Now let us consider the behavior of total capital $\{A_t,\ t \geq 0\}$ over time when we invest a fraction k (leverage) in the instrument whose price P_t is given by (13,14). It follows directly from (10) that

$$dA_t = (1-k)rA_t dt + kA_t dP_t/P_t \tag{17}$$
$$= [(1-k)r + k\mu]A_t dt + kA_t \sigma dW_t. \tag{18}$$

In other words, like the underlying instrument price, our amount of capital A_t also follows a geometric Brownian motion process, but with different parameters. Consequently,

$$d\log(A_t) = [(1-k)r + k\mu - k^2\sigma^2/2]dt + k\sigma dW_t. \quad (19)$$

It is very important to note[3] that

$$d\log(A_t) \neq k \cdot d\log(P_t). \quad (20)$$

It follows directly from (19) that

1. $A_t > 0$. This is our guarantee that we do not lose more than our initial capital. However, note that it relies on the unrealistic assumption that we are able to carry out continuous reinvestment of profits/losses.
2. The log of total amount of capital (including reinvested profits/losses) satisfies

$$\log(A_{t+\delta}/A_t) \sim \mathrm{N}(r\delta + k(\mu - r)\delta - k^2\sigma^2\delta/2, k^2\sigma^2\delta). \quad (21)$$

At any time t, equation (21) provides a predictive distribution for our amount of capital at a time point $(t+\delta)$ in the future. Specifically, it is log-normal.

Following on from (21), we see that the *expected log-return per unit time* of our capital is

$$L(k) = \mathbf{E}[\log(A_{t+\delta}/A_t)]/\delta = r + k(\mu - r) - k^2\sigma^2/2. \quad (22)$$

Differentiating this with respect to k and setting to zero gives us a formula for the value of leverage k that maximizes expected log-return per unit time. This is simply

$$k^* = (\mu - r)/\sigma^2, \quad (23)$$

and the corresponding expected log-return per unit time is

$$L(k^*) = r + \frac{1}{2}(\mu - r)^2/\sigma^2. \quad (24)$$

The expression (23) is well-known, and is typically referred to as *Kelly's formula*, for Kelly's analysis of a closely-related problem (see [5]).

2.2.2 Multivariate Case

It is straightforward to extend both (22) and (23) to the multivariate case when we have multiple investments ($m > 1$), and we allocate proportions $\mathbf{k} = (k_1, k_2, \ldots, k_m)^T$ of capital to the respective investments. In this case, we define $\mu = (\mu_1, \ldots, \mu_m)^T$, $\sigma = (\sigma_1, \ldots, \sigma_m)^T$, as well as a correlation matrix $R \in \mathbb{R}^{m \times m}$. The multivariate analog of (13) becomes

$$d\mathbf{P}_t = \mathrm{diag}(\mu)\mathbf{P}_t dt + \mathrm{diag}(\sigma)\mathrm{diag}(\mathbf{P}_t) d\mathbf{U}_t, \quad (25)$$

[3] It is a common mistake to presume, for example, that a triple-leverage investment yields triple the log-returns over time. This is not the case.

where $\{\mathbf{U}_t\}$ is a multivariate Brownian motion with correlation matrix R, so that

$$\mathbf{E}[\mathbf{U}_t] = 0, \quad \mathbf{Var}(\mathbf{U_t}) = Rt, \tag{26}$$

and diag$(\cdot)$ represents a square matrix with diagonal elements given by the vector argument, and zeros in all off-diagonal positions. It will be convenient to define

$$\Sigma = [\mathrm{diag}(\sigma)]R[\mathrm{diag}(\sigma)]. \tag{27}$$

Also, recall the definition $\kappa = \sum k_j = \mathbf{k} \cdot \mathbf{e}_m$, where $\mathbf{e}_m = (1, 1, \ldots, 1)^T \in \mathbb{R}^m$.

Applying (10), it is straightforward to show that there is another standard Brownian motion $\{W_t\}$ such that the total capital $\{A_t\}$ satisfies

$$dA_t = ((1 - \mathbf{k} \cdot \mathbf{e}_m)r + \mathbf{k} \cdot \mu)A_t dt + (\mathbf{k}^T \Sigma \mathbf{k})^{1/2} A_t dW_t \tag{28}$$
$$= (r - \mathbf{k} \cdot \mathbf{r} + \mathbf{k} \cdot \mu)A_t dt + (\mathbf{k}^T \Sigma \mathbf{k})^{1/2} A_t dW_t, \tag{29}$$

and it follows directly from application of Itô's formula that

$$d\log(A_t) = (r + \mathbf{k} \cdot (\mu - \mathbf{r}) - \mathbf{k}^T \Sigma \mathbf{k}/2)dt + (\mathbf{k}^T \Sigma \mathbf{k})^{1/2} dW_t, \tag{30}$$

so the expected log-return per unit time is given by

$$L(\mathbf{k}) = \mathbf{E}[\log(A_{t+\delta}/A_t)]/\delta = r + \mathbf{k} \cdot (\mu - \mathbf{r}) - \mathbf{k}^T \Sigma \mathbf{k}/2, \tag{31}$$

Consequently Kelly's formula becomes

$$\mathbf{k}^* = \Sigma^{-1}(\mu - \mathbf{r}), \tag{32}$$

with maximum expected log-return per unit time

$$L(\mathbf{k}^*) = r + \frac{1}{2}(\mu - \mathbf{r})^T \Sigma^{-1} (\mu - \mathbf{r}). \tag{33}$$

We will also be interested in the variance of the log-return per unit time. From (30), this is easily seen to be

$$V(\mathbf{k}^*) = \mathbf{Var}(\log(A_{t+\delta}/A_t))/\delta = \mathbf{k}^T \Sigma \mathbf{k}. \tag{34}$$

Readers may recognize (32) as the solution of a standard Markowitz mean-variance portfolio weight selection problem [8].

2.3 Sharpe Ratios and Fractional Kelly Investment

In the finance industry, in the univariate case, it is common practice to refer to the quantity $(\mu - r)/\sigma$, as the *Sharpe ratio* of the investment. [4] We will work with the closely-related *maximal Sharpe ratio* of a single investment or a set of investments, which we define by

$$S_M = [(\mu - \mathbf{r})^T \Sigma^{-1} (\mu - \mathbf{r})]^{1/2}, \quad (35)$$

where μ and Σ are the parameters defining the growth and volatility of the component investments, and $\mathbf{r}$ is the vector whose elements are all equal to the risk-free rate. This is "maximal" in the sense that it represents the maximum Sharpe ratio over all possible linear combinations of portfolio components. It is easily verified that in the univariate case, it reduces to $S_M = |(\mu - r)/\sigma|$, which is indeed the maximum Sharpe ratio one can obtain by scaling a single investment with Sharpe ratio $(\mu - r)/\sigma$.

The maximal Sharpe S_M is crucially important, because the maximum growth rates in (24, 33) increase quadratically with respect to S_M.

We have seen that Kelly's formula prescribes the leverage that maximizes expected log-return per unit time L of an investment. However, to limit up/downswings in capital, it is desirable to operate with a lower total level of risk. To achieve this, we can apply a multiplier to (32),

$$\mathbf{k}_\alpha = \alpha \Sigma^{-1} (\mu - \mathbf{r}), \quad \alpha \in [0, 1]. \quad (36)$$

This quantity is often referred to as *fractional Kelly* leverage. In this light, fractional Kelly investment can be viewed as a particular means of leverage selection, with α specifying the level of risk relative to that required for the growth-optimal portfolio.

The following result makes a formal connection between leverage, log-returns and Sharpe ratios.

Theorem 1 (Sharpe-Leverage Performance Profile). *Suppose that we apply fractional Kelly leverage (36) with multiplier $\alpha \in [0,1]$ to the portfolio with prices governed by (25), earning/paying the risk-free interest rate r on the cash/debt component. Then the resulting capital process $\{A_t\}$ is a geometric Brownian motion satisfying*

$$d \log(A_t) = [r + (\alpha - \alpha^2/2) S_M^2] dt + \alpha S_M dW_t, \quad (37)$$

where $S_M = [(\mu - \mathbf{r})^T \Sigma^{-1} (\mu - \mathbf{r})]^{1/2}$ denotes the maximal Sharpe ratio of the portfolio. Consequently, $\{A_t\}$ has expected log-return per unit time

$$L(\mathbf{k}_\alpha) = \mathbf{E}[\log(A_{t+\delta}/A_t)]/\delta = r + (\alpha - \alpha^2/2) S_M^2 \quad (38)$$

and log-return variance per unit time

$$V(\mathbf{k}_\alpha) = \mathbf{Var}(\log(A_{t+\delta}/A_t))/\delta = \alpha^2 S_M^2. \quad (39)$$

[4] It is important to differentiate between this quantity, and estimators thereof. Unfortunately, people often use the same term to refer to both the quantity and its estimator(s).

Proof. The derivation of (30) above establishes that $\{A_t\}$ is a geometric Brownian motion. Substituting (36) into (30) and making use of the definition (35) yields (37). The remaining results then follow directly.

When the conditions of Theorem 1 are met and $\alpha = 1$, we obtain the growth-optimal portfolio. More generally, equations (38, 39) show the trade-off we obtain with different values of α. Values closer to one yield higher expected log-return per unit time on capital, but values closer to zero give better (reduced) variance of the log-return around the mean.

3 Examples and Applications

We now consider some practical implications of the theory outlined above, beginning with analysis that would be of interest to a typical individual with some savings to invest.

3.1 Equity and Bonds Mix

Let us consider a portfolio consisting of only two instruments: $\{P_{t,1}\}$ will represent the price of Vanguard 500 Mutual Fund Index shares (VFIAX), and $\{P_{t,2}\}$ will represent the price of Vanguard Long-Term Treasury Fund shares (VUSUX).

Figure 1 shows differences of logarithms of adjusted daily closing prices for these two instruments from 2002 to 2023. This time period covers the dot-com bubble of 2002, the sub-prime crisis of 2008, the more recent market turbulence at the onset of the coronavirus pandemic in early 2020, and the interest-rate reset period beginning at the end of 2021.

In what follows, we will assume that the risk-free rate r is equal to zero. This is a fairly reasonable approximation for the post-sub-prime period from 2008 to 2022, although results here can be easily adapted if r is assumed to be non-zero.

3.1.1 Parameter Estimation

From the raw data shown in the plots in Figure 1, we can easily compute estimators of $\mu = (\mu_1, \mu_2)$,

Let the unit of time be one year, assume there are 260 trading days each year, and define the daily log-returns (shown in the left part of Figure 1) as

$$D_{t,j} = \log(P_{(t+1)/260,j}) - \log(P_{t/260,j}), \quad t = 1, 2, \ldots, n-1, \tag{40}$$

where n is the total number of trading days of data. We know from (16) that

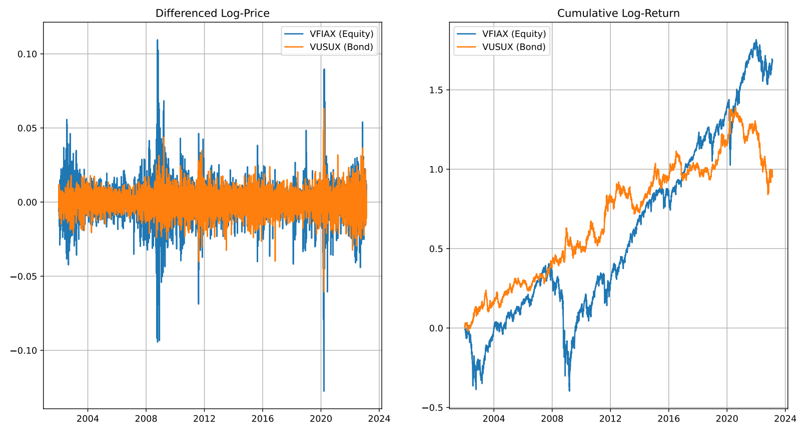

Fig. 1 Equity (VFIAX) and Bond (VUSUX) performance from Jan. 2nd, 2002 to Feb. 13th, 2023. Data source: Yahoo Finance.

$$D_{t,j} \sim \mathrm{N}((\mu_j - \sigma_j^2/2)/260, \sigma^2/260). \tag{41}$$

Furthermore,

$$\mathrm{Corr}(D_{t,1}, D_{t,2}) = R_{12}. \tag{42}$$

We can therefore take sample mean and variance from our two daily price series, compute sample correlation, and use them to construct method-of-moments estimators

$$\hat{\sigma}_j^2 = \frac{260}{n-2} \sum_{t=1}^{n-1} (D_{t,j} - \bar{D}_j)^2, \quad \bar{D}_j = \frac{1}{n-1} \sum_{t=1}^{n-1} D_{t,j}, \tag{43}$$

$$\hat{\sigma}_j = \sqrt{\hat{\sigma}_j^2}, \tag{44}$$

$$\hat{\mu}_j = 260\bar{D}_j + \hat{\sigma}_j^2/2, \tag{45}$$

$$\hat{R}_{jk} = \frac{260}{\hat{\sigma}_j \hat{\sigma}_k (n-2)} \sum_{t=1}^{n-1} (D_{t,j} - \bar{D}_j)(D_{t,k} - \bar{D}_k) \tag{46}$$

With the raw data described above, we obtain

$$\hat{\mu} = (0.102, 0.055)^T, \quad \hat{\sigma} = (0.200, 0.127)^T, \quad \hat{R}_{12} = -0.343. \tag{47}$$

From a practical perspective, we also want to consider the different impact of taxation on these two instruments. In the United States, tax on a mutual fund that tracks the S&P500 is typically close to the long-term capital gain rate of 20%, while tax on bond funds, although more complex to compute, is typically closer to standard income tax, which is often around 40%. To approximate the impact of taxation, we will therefore adjust the estimates (47) above by multiplying components of μ by one minus the corresponding tax rate, giving us final estimates

$$\hat{\mu} = (0.082, 0.033), \quad \hat{\Sigma} = \begin{bmatrix} 0.0399 & -0.0087 \\ -0.0087 & 0.0160 \end{bmatrix}. \tag{48}$$

From these estimates, in turn, we can determine the Kelly leverage for the growth-optimal portfolio by applying (32), obtaining

$$\mathbf{k}^* = \hat{\Sigma}^{-1}\hat{\mu} \simeq (2.83, 3.58)^T. \tag{49}$$

If μ and Σ were indeed the same as these estimates, and if we could borrow money to provide leverage, this says we could maximize growth over time of capital by borrowing 5.41 times our initial capital (assuming a no-interest loan) to obtain total leverage of $2.83 + 3.58 = 6.41$. We would then invest the original capital along with the borrowed capital into the two components of the portfolio, rebalancing to hold the fraction of capital in the two components constant. The resulting expected annual log-return on initial (non-borrowed) capital would be

$$\mathbf{k}^* \cdot \hat{\mu} - \mathbf{k}^{*T}\hat{\Sigma}\mathbf{k}^*/2 \simeq 0.175 \tag{50}$$

In this argument we have pretended that μ and Σ are known precisely. In fact, estimation error on these quantities can have a significant impact, but that analysis is beyond the scope of this paper.

3.1.2 Leverage Value Configurations

We now consider a range of different investment possibilities. Some of these will be applied to only one of our two instruments, and for these cases we will use $m = 1$ along with the corresponding (marginal) components of $\hat{\mu}$, $\hat{\Sigma}$ and the corresponding estimate $\hat{S}$ of Sharpe ratio. In the more interesting cases we will examine portfolios consisting of both instruments. We consider the following cases.

Cases are constructed as follows.

- *Non-leveraged/Double/Triple Equities/Bonds:* The non-leveraged approach invests all capital directly into the equities/bonds instrument. Double and triple versions invest all capital in the same instrument but with leverage 2 or 3, respectively.
- *Fractional Kelly (0.31):* This applies leverage $\alpha = 0.31$ times the full-Kelly leverage. In this example, it results in total leverage κ approximately equal to 2.0.
- *Constrained Kelly (2.0):* Here we choose $\mathbf{k}$ so as to maximize expected growth L subject to the constraint that $\kappa = 2$. This can be done using Lagrange multipliers, as described later in Section 5.1.

	Leverage	Drift/Diffusion			Kelly-Fraction
Name	k	$\hat{\mu}$	$\hat{\Sigma}$	$\hat{S}$	$\hat{\alpha}$
Single-instrument ($m=1$) portfolios					
Non-Leveraged Equity	1.00	0.082	0.200^2	0.410	0.502
Double-Equity	2.00	0.082	0.200^2	0.410	1.003
Triple-Equity	3.00	0.082	0.200^2	0.410	1.505
Non-Leveraged Bonds	1.00	0.033	0.127^2	0.260	0.488
Double-Bonds	2.00	0.033	0.127^2	0.260	0.976
Two-instrument ($m=2$) portfolios					
Fractional Kelly(0.31)	(0.88,1.12)	(0.082,0.033)	$\hat{\Sigma}$ (eqn 48)	0.592	0.31
Constr. Kelly(2.0)	(1.30,0.70)	(0.082,0.033)	$\hat{\Sigma}$ (eqn 48)	0.592	NA
Full-Kelly	(2.83,3.58)	(0.082,0.033)	$\hat{\Sigma}$ (eqn 48)	0.592	1.0

Table 1 Example portfolios.

- *Full-Kelly:* This choice, given by (49) leads to maximum growth rate over time, at the cost of high variance and large draw-downs.

Table 1 also shows a term

$$\hat{\alpha} = \mathbf{k}/(\hat{\Sigma}^{-1}\hat{\mu}), \tag{51}$$

which is only defined when the leverage vector $\mathbf{k}$ is a multiple of $\hat{\Sigma}^{-1}\hat{\mu}$, and in that case represents an estimate of the ratio of portfolio leverage to growth-optimal leverage. Values less than 1.0 are desirable in this context, since it is obvious from (38) and (39) that $\alpha > 1$ implies we are taking unnecessary extra risk for a return that can also be achieved with $\alpha < 1$.

3.1.3 Simulation Results

It is straightforward to simulate hypothetical capital invested in the portfolios corresponding to the test cases. Using data ranging from Jan. 1st, 2002 to Feb. 13th, 2023, we compute the daily portfolio returns using the recorded data. In order to approximate the tax-effect on the two components, we also apply a negative drift by subtracting the tax adjustment used on μ in (48) on a daily basis from the log-returns of the data.

Figure 2 shows cumulative capital $\{A_t\}$ over time for a range of different leverage settings. The full-Kelly leverage case is omitted from the plot since it distorts the scale, but the full-Kelly case is included in Figure 3, which shows the log of cumulative capital.

Table 2 shows corresponding summary statistics, including the sample (annualized) growth rate $\hat{L}$, the corresponding sample variance $\hat{V}$, as well as the final value and maximum "draw-down" over the time period, which is defined as the maximum peak-to-trough drop such that the trough occurs after the peak. From a psychological perspective, investors pay significant attention to difference between the historical

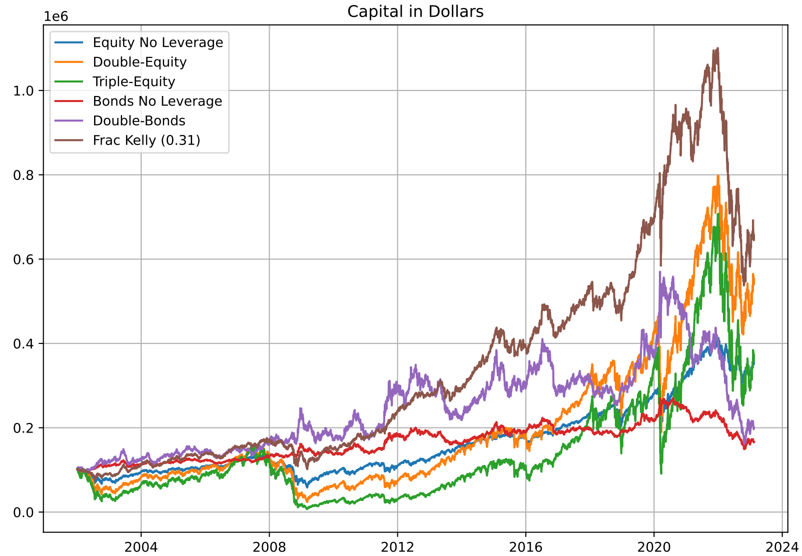

Fig. 2 Simulated post-tax capital of hypothetical portfolios built of VFIAX and VUSUX, assuming starting capital of \$100000, for a range of different leverage vectors.

maximum and the current value of an investment. Large values of draw-down typically induce great anxiety.

Several important observations can be made from the results. First, the full-Kelly case and the triple-equity case both exhibit very large maximum draw-downs over this time period - 92.4% and 95.6%, respectively. Few investors would hold onto an investment after losses that large. Both the fractional Kelly and the constrained Kelly options appear to give a fairly good compromise between greater growth and higher variance/draw-downs. Of course different investors may different preferences, but it is notable that of the choices in Table 2, apart from the non-leveraged bond case, the fractional Kelly option has the smallest value of $\hat{V}$ and still yields a respectable growth rate $\hat{L}$.

3.2 Application to Fund Evaluation

Hedge fund managers generally release very little information about the nature of their trading operations. However, they often release yearly or quarterly statments

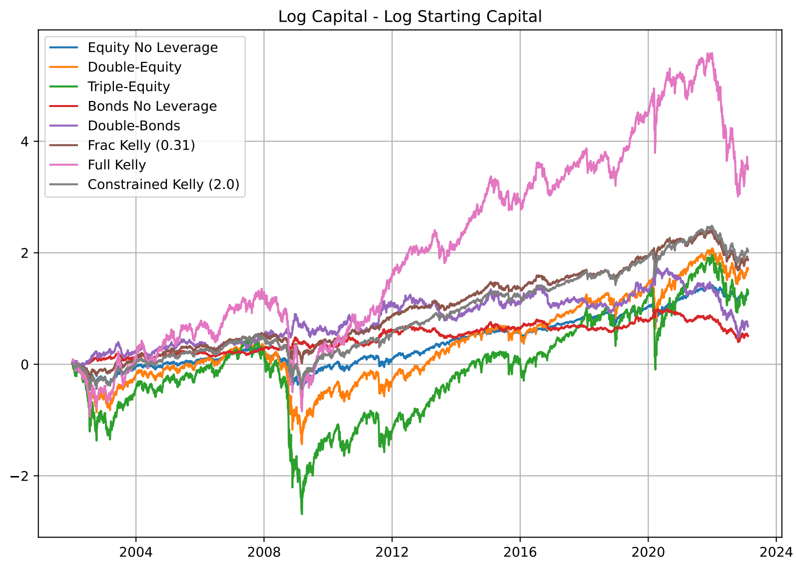

Fig. 3 Simulated post-tax log-return of hypothetical portfolios built of VFIAX and VUSUX, for a range of different leverage vectors. The full-Kelly case clearly dominates the others in terms of growth, but it also loses 92.4% of its capital value from Dec. 23rd, 2021 to Oct. 19th, 2022.

of actual returns on their hedge funds. We have already seen that Theorem 1 assists in choosing a desirable leverage vector. Here we see how it leads to a simple method-of-moments approach to estimate portfolio maximal Sharpe ratio and risk deployment of a fund, using only such publicly available return data. The approach relies on an implicit assumption that the fund is applying a fractional Kelly approach, or something similar. In light of the previous observation at the end of Subsection 2.2.2 that fractional Kelly leverage is equivalent to industry-standard Markowitz mean-variance optimization, this is not an unrealistic assumption. This "reverse-engineering" allows us to evaluate funds in a more nuanced way than simple inspection of past returns. For example, we can easily differentiate between funds that obtain high returns by taking excessive risk, and those that obtain high returns by deployment of high-Sharpe ratio portfolios.

Consider, for example, the annual returns of the well-known (and very strongly performing) Renaissance Medallion fund. These returns are publicly available in Appendix 1 of [11], and can easily be translated into log-returns and viewed in light of equations (38) and (39). Without re-printing the full table itself, we give summary statistics of their resulting (before-fee) log-returns in Table 3 below.

	Growth		Draw-down			
Name	$\hat{L}$	$\hat{V}^{1/2}$	Max.	Start	End	Final Value
Single-instrument ($m = 1$) portfolios						
Non-Leveraged Equity	0.062	0.200	56.4%	Oct 8 2007	Mar 6 2009	\$354,781
Double-Equity	0.083	0.401	84.6%	Oct 8 2007	Mar 6 2009	\$553,393
Triple-Equity	0.063	0.605	95.6%	Jul 18 2007	Mar 6 2009	\$372,336
Non-Leveraged Bonds	0.025	0.127	44.9%	Aug 3 2020	Oct 21 2022	\$166,362
Double-Bonds	0.033	0.253	71.7%	Mar 6 2020	Oct 21 2022	\$199,389
Two-instrument ($m = 2$) portfolios						
Fractional Kelly(0.31)	0.091	0.184	51.2%	Dec 23 2021	Oct 19 2022	\$656,615
Constr. Kelly(2.0)	0.099	0.244	62.5%	Oct 8 2007	Mar 6 2009	\$768,512
Full-Kelly	0.172	0.596	92.4%	Dec 23 2021	Oct 19 2022	\$3,473,850

Table 2 Summary of simulated performance of example portfolios derived from the VFIAX and VUSUX instruments, simulated from Feb. 12th, 2001 to Aug. 24th, 2021, with initial capital \$100,000. Risk-free interest rate is assumed to be zero. $\hat{L}$ denotes the sample average of the annualized log-return, $\hat{V}^{1/2}$ denotes the annualized sample standard deviation of those log-returns, and the maximum *draw-down* is defined as the maximum drop from peak to trough with trough occurring after the peak.

Date Range	1988-2018 inclusive
# Data Points	31
Average Log-Return	0.490
Std. Dev. (Log-Return)	0.187

Table 3 Summary statistics of Renaissance Medallion Fund returns as given in [11].

Re-arranging the pair of equations (38,39), we see that for a fractional Kelly investor,

$$\alpha = 2V(2L+V)^{-1} \tag{52}$$

$$S_M^2 = \alpha^{-1}(L+V/2). \tag{53}$$

By simply matching $L = 0.490$ and $V = 0.187^2$ from the table above, we see that the Medallion returns are consistent with deployment of a portfolio with maximal Sharpe ratio $S_M \simeq 2.72$ at approximately $\alpha = 0.068$ times the Kelly-optimal leverage. The (estimated) value of 0.068 places them comfortably below the obvious danger point. A value equal to 1.0 would indicate deployment of risk at the growth-optimal level, with its extreme volatility. Such a value is not palatable to a typical investor. Any value larger than 1.0 would indicate a sub-optimal deployment of risk. A value larger than 2.0 would indicate that the fund will likely eventually collapse in the sense that its value will converge in probability to zero.

4 Concluding Remarks

Given a portfolio of instruments with risk, some capital A_0, and a risk-free interest rate r, we have seen that the Kelly portfolio maximizes long-term growth of capital, but with distressingly large draw-downs along the way. This undesirable quality can be mitigated by the use of fractional Kelly portfolios, at the cost of a reduction in long-term growth rate. In Theorem 1, we have directly quantified this distributional relationship, and given explicit expressions that help us to find a desirable balance between growth and variance of returns. We have also seen how, under a set of reasonable assumptions, fractional Kelly investment is the same as Markowitz mean-variance optimization.

The stochastic differential equation methodology used in this paper has been used by others in the same context, notably [2], but appears not to be widely appreciated. Within this framework, one could potentially generalize the geometric Brownian motion price models (13, 25) to jump-diffusion models, thereby allowing for the more realistic case of skewed and heavy-tailed log-returns. Using multivariate stochastic calculus, it would also possible to carry out analysis of situations where drift and/or diffusion coefficients μ and Σ are not known and must be estimated.

On the practical side, the relationships given in Theorem 1 have broad applicability. They apply equally well to simple investment of an individual's personal funds as to the more sophisticated operations of a quantitative trading organization or other fund. For example, individual investors can use the implied return distributions to assess the quality of leveraged mutual funds under various assumptions. At the more sophisticated end of the spectrum, a quantitative trading operation with a porfolio of known Sharpe ratio could use the result to determine how much capital/leverage to deploy, while remaining within distributional constraints imposed by their investors. More generally, the stochastic differential equation framework provides a solid foundation with which to address important yet-unsolved practical investment problems.

5 Supporting Results

5.1 Constrained Maximization of Expected Log-Return per Unit Time

Suppose we wish to choose the leverage vector $k \in \mathbb{R}^m$ to maximize expected log-return per unit time (restating equation (31))

$$L(\mathbf{k}) = \mathbf{k} \cdot \mu - \frac{1}{2}\mathbf{k}^T \Sigma \mathbf{k}, \tag{54}$$

subject to the constraint that total-leverage is

$$\kappa = \sum_{j=1}^{m} k_j = \mathbf{k} \cdot \mathbf{e}_m = \kappa_0, \tag{55}$$

where $\mathbf{e}_m = (1, \ldots, 1)^T \in \mathbb{R}^m$. We can express the constraint as

$$(\mathbf{k} \cdot \mathbf{e}_m - \kappa_0) = 0. \tag{56}$$

Thus the Lagrangian is

$$\mathscr{L}(\mathbf{k}) = \mathbf{k} \cdot \mu - \frac{1}{2}\mathbf{k}^T \Sigma \mathbf{k} - \lambda(\mathbf{k} \cdot \mathbf{e}_m - \kappa_0), \tag{57}$$

Differentiating with respect to the vector $\mathbf{k}$, and setting to zero, we find that the value maximizing (54) takes the form

$$\mathbf{k} = \Sigma^{-1}(\mu - \lambda \mathbf{e}_m). \tag{58}$$

Substituting (58) into the constraint (56), we can solve for λ, obtaining

$$\lambda = (\mathbf{e}_m^T \Sigma^{-1} \mu - \kappa_0)(\mathbf{e}_m^T \Sigma^{-1} \mathbf{e}_m)^{-1}. \tag{59}$$

Equations (58) and (59) together specify the leverage vector maximizing L subject to the required constraint.

Acknowledgements The author is grateful to Peter Brockwell, Jonathan Baxter, and Peter Dodds for their valuable comments and suggestions, as well as an anonymous reviewer. This work has benefited substantially from their feedback.

References

1. Breiman, L.: Optimal gambling systems for favorable games. Fourth Berkeley Symposium on Probability and Statistics **1**, 65–78 (1961)
2. Browne, S.: Survival and growth with a liability: Optimal portfolio strategies in continuous time. Mathematics of Operations Research **22**(2), 468–498 (1997)
3. Hakansson, N.: Capital growth and the mean-variance approach to portfolio selection. Journal of Financial and Quantitative Analysis **26**, 857–884 (1971)
4. Karatzas, I., Shreve, S.: Brownian Motion and Stochastic Calculus. Springer (1991)
5. Kelly, J.L.: A new interpretation of information rate. Bell System Technical Journal **35**(4), 917–926 (1956)
6. MacLean, L., Sanegre, R., Zhao, Y., Ziemba, W.: Capital growth with security. Journal of Economic Dynamics and Control **28**, 937–954 (2004)
7. MacLean, L., Thorp, E., Ziemba, W. (eds.): The Kelly Capital Growth Investment Criterion, *World Scientific Handbook in Financial Economic Series*, vol. 3. World Scientific (2012)
8. Markowitz, H.: Portfolio selection. Journal of Finance **7**(1), 77–91 (1952)
9. Øksendal, B.: Stochastic Differential Equations, 6th edn. Springer (2003)
10. Thorpe, E.: Portfolio choice and the Kelly criterion. In: W. Ziemba, T. Vickson (eds.) Stochastic Optimization Models in Finance, pp. 599–619. Academic Press (1975)
11. Zuckerman, G.: The Man Who Solved the Market: How Jim Simons Launched the Quant Revolution. Penguin Random House LLC (2019)