Entrance laws for continuous-state nonlinear branching processes coming down from infinity

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Abstract We consider a class of non-negative valued, time-changed spectrally positive Lévy processes stopped whenever hitting 0, which can be identified as continuous-state branching processes with population dependent branching rates. Given the process comes down from infinity, we find expressions for Laplace transform of the first passage time and for the potential measure for the process started from infinity. Those expressions are in terms of the generalized scale functions for the corresponding spectrally positive Lévy processes.

1 Introduction

Continuous-state branching processes (CSBPs for short) are non-negative valued Markov processes satisfying the so called branching property, i.e. given two independent CSBPs with identical branching mechanism and initial values x and y, respectively, the sum of the two processes is also a CSBP with initial value $x+y$. They are the continuous-state counterparts of the Bienaymé-Galton-Watson processes. We refer to Kyprianou [9] and Li [14] for introductions on CSBPs.

In Li [13] a class of nonlinear CSBPs with power branching rate functions is introduced as solutions to the associated SDEs driven by spectrally positive Lévy processes (SPLPs for short). Nonlinear CSBPs with more general rate functions are introduced in Foucart et al. [7] by time-changing spectrally positive Lévy processes via Lamperti type transforms. The criteria for coming down from infinity and the speed of coming down from infinity for such nonlinear CSBPs are obtained in [7].

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Foucart et al. [6] further show that such a process can be extended to a $[0,\infty]$-values Markov process started from ∞. We refer to Bansaye et al. [1] for previous work on the speed of coming down from infinity for birth and death processes.

The Lamperti type transform allows to express the interested quantities of nonlinear CSBP in terms of the associated SPLP that often involves functional integral of the SPLP. As a result, one can study the nonlinear CSBPs using fluctuation theory for spectrally one-sided Lévy processes. In this note, we adopt the above approach to further derive some expressions for nonlinear CSBPs that come down from infinity. More precisely, for the process started at ∞ and coming down from ∞, we find expressions for the Laplace transform of the first passage time and for the potential measure where those expressions are in terms of generalized scale functions for SPLPs. These quantities also characterize the entrance law for the associated $[0,\infty)$-valued nonlinear CSBP. We also obtain expressions on the moments of the total progeny for the nonlinear CSBP.

In Section 2 we introduce the needed fluctuation results for SPLP, the definition of nonlinear CSBP using Lamperti transform on SPLP and some previous results on nonlinear CSBPs and the coming down from infinity. The main results and their proofs are presented in Section 3.

2 Spectrally positive Lévy processes and continuous-state nonlinear branching processes

2.1 *Spectrally positive Lévy processes*

Let ξ be a spectrally positive Lévy process, that is, a stochastic process with stationary and independent increments and without negative jumps defined on a filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$. The law of ξ for $\xi_0=x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x=0$. The Laplace component of ξ is well defined and takes the Lévy-Khintchine form, for every $\theta\geq 0$

$$\psi(\theta):=\log\mathbb{E}\big[e^{-\theta\xi_1}\big]=\gamma\theta+\frac{\sigma^2}{2}\theta^2+\int_0^\infty\big(e^{-\theta z}-1+\theta z\big)\pi(dz), \tag{1}$$

where $\gamma\in\mathbb{R},\sigma\geq 0$ are constants and π is a σ-finite measure on $(0,\infty)$ with $\int_0^\infty z\wedge z^2\nu(dz)<\infty$. We exclude the case ξ being a subordinator. It is well-known that ψ in (1) is continuous and strictly convex on $[0,\infty)$ with $-\psi'(0)=\mathbb{E}[\xi(1)]=-\gamma<\infty$. Its right inverse is defined by

$$\phi(s):=\sup\{\theta\geq 0:\psi(\theta)=s\}\quad\text{for}\quad s\geq 0.$$

In the fluctuation theory of spectrally negative Lévy processes, the following scale functions $W^{(q)}$ plays a key role, which is defined as the unique continuous and increasing function on $[0,\infty)$ satisfying

$$\int_0^\infty e^{-sy} W^{(q)}(y)\,dy = \frac{1}{\psi(s)-q} \quad \text{for } s > \phi(q) \text{ and } q \geq 0, \tag{2}$$

with extension $W^{(q)}(x) = 0$ for $x < 0$. We write $W(x) \equiv W^{(0)}(x)$ for $q = 0$ and refer the readers to [2] and [9] for more detailed discussions on the spectrally negative Lévy process and its scale functions. We also call $W^{(q)}$ the scale function of SPLP ξ for the obvious reason.

Define the first passage times of ξ by

$$\tau_b^+ := \inf\{t \geq 0 : \xi_t > b\} \quad \text{and} \quad \tau_a^- := \inf\{t \geq 0 : \xi_t < a\}$$

with the convention $\inf \emptyset = \infty$. Then the potential measure of ξ killed upon leaving $[a,b]$ is given by

$$\begin{aligned}\Theta^{(q)}(x,dy) &:= \int_0^\infty e^{-qt} \mathbb{P}_x\big(\xi_t \in dy; t < \tau_a^- \wedge \tau_b^+\big)\,dt \\ &= \Big(\frac{W^{(q)}(b-x)}{W^{(q)}(b-a)} W^{(q)}(y-a) - W^{(q)}(y-x)\Big)\,dy\end{aligned} \tag{3}$$

for $x, y \in (a,b)$.

Let ω be a nonnegative locally bounded Borel function and define the functional

$$\eta(t) := \int_0^t \omega(\xi_s)\,ds$$

which is called ω-weighted occupation time in Li and Palmowski [10]. We will need the following result on potential measure for process ξ.

Lemma 1 ([10] Theorem.2.2). *For $b > x, y > a$, we have*

$$\begin{aligned}\Theta^{(\omega)}(x,dy) &= \int_0^\infty \mathbb{E}_x\Big[e^{-\eta(t)}; t \leq \tau_a^- \wedge \tau_b^+, \xi_t \in dy\Big]\,dt \\ &= \Big(\frac{W^{(\omega)}(b,x)}{W^{(\omega)}(b,a)} W^{(\omega)}(y,a) - W^{(\omega)}(y,x)\Big)\,dy\end{aligned}$$

where $W^{(\omega)}(x,y)$ is defined as the unique locally bounded function satisfying

$$W^{(\omega)}(x,y) = W(x-y) + \int_y^x W^{(\omega)}(x,z)\omega(z) W(z-x)\,dz. \tag{4}$$

Remark 1. The ω-weighted scale function $W^{(\omega)}$ defined above extends the classical scale function in (3) with $W^{(\omega)}(x,x) = W(0)$ and $W^{(\omega)}(x,y) = 0$ if $x < y$. One can show that if $\omega \equiv q$, then $W^{(\omega)}(x,y) = W^{(q)}(x-y)$ for all $x, y \in \mathbb{R}$ and $\Theta^{(\omega)} = \Theta^{(q)}$ as expected.

2.2 Time-changed SPLP and nonlinear CSBP

In this paper, we are interested in the following time-changed SPLP taking values on $[0,\infty)$
$$X(t) := \xi\left(\eta^{-1}(t)\wedge\tau_0^-\right), \tag{5}$$
stopped at time $\zeta = \eta(\tau_0^-) \in [0,\infty]$ with absorbing states 0 and ∞, where
$$\eta(t) := \int_0^t \omega(\xi_s)\,ds = \int_0^t \frac{ds}{R(\xi_s)} \quad \text{for } t<\tau_0^- \text{ and } \eta^{-1}(s) := \inf\{t\geq 0 : \eta(t) > s\} \tag{6}$$
and $R(z) := \frac{1}{\omega(z)}$ is a positive and continuous function on $(0,\infty)$. The Laplace exponent ψ for ξ is called the branching mechanism for X and R is called the branching rate function. Since X is obtained by time-changing SPLP ξ, it is defined on the same probability space as ξ with $X(0) = \xi(0)$, and $\mathbb{P}_x$ can also be understood as the law of X with $X(0) = x$.

The time-changed SPLP defined above is also referred to as the Lamperti-type transformed process. It is well known that, when the function R is the identity function, X reduces to the CSBP with Ψ being the branching mechanism, and if R is an exponential function, X is related to a positive self-similar Markov process. The two transforms are called the classical Lamperti transformations in the literatures; see e.g. [9, 4].

The above positive Markov process X in (5) can also be characterized as the solution to SDE
$$X(t) = x - \gamma\int_0^t R(X_s)\,ds + \sigma\int_0^t \sqrt{R(X_s)}dB_s + \int_0^t\int_0^\infty\int_0^{R(X_{s-})} z\tilde{N}(ds,dz,du), \tag{7}$$
where B is a standard Brownian motion and $\{\tilde{N}(ds,dz,du)\}_{s,z,u>0}$ is an independent compensated Poisson random measure on $(0,\infty)^3$ with intensity $ds\nu(dz)du$, where ν is a nonnegative σ-finite measure on $(0,\infty)$ such that $\int_0^\infty (z\wedge z^2)\nu(dz) < \infty$, comparing with ψ in (1). For the above stochastic differential equation to have a unique solution, we need additional technical assumptions such as $R(\cdot)$ is locally Lipschitz on $(0,\infty)$.

Processes of this type are first introduced in [13], for power functions R, as nonlinear CSBPs that describe the evolution of populations whose reproduction rates are power functions of the population sizes instead of the identity function for the classical CSBP; also see [7] for nonlinear CSBPs with more general rate functions.

For the main result in the note we need the following Laplace transforms on downward first passage times, also see [7] for related results, where the first passage time of X is defined by
$$T_a^- := \inf\{t\geq 0, X_t < a\}$$
with the convention $\inf\emptyset = \infty$.

Lemma 2 ([12] Lemma.4.4). *If* $\int^\infty \frac{W_{\phi(0)}(x)}{R(x)}dx < \infty$, *then for* $x > a > 0$

$$\mathbb{E}_x\left[e^{-T_a^-}\right] = \frac{H^{(\omega)}(x)}{H^{(\omega)}(a)} \tag{8}$$

where $H^{(\omega)}(x)$ is defined as the unique decreasing function satisfying

$$H^{(\omega)}(x) := \lim_{b\to\infty} \frac{W^{(\omega)}(b,x)}{W(b)} = e^{-\phi(0)x} + \int_x^\infty H^{(\omega)}(z)\omega(z)W(z-x)\,dz. \tag{9}$$

Remark 2. By the definition of X in (5), the locally boundedness assumption on ω in Lemma 2 ensures the finiteness of T_a^- on the set $\{\tau_a^- < \infty\}$. By the absence of negative jumps, there always exists some function $H^{(\omega)}$ so that the ratio identity (8) holds for all $x > a > 0$.

The condition $\int^\infty W_{\phi(0)}(x)\omega(x)\,dx < \infty$ enables us to derive a proper integral equation for $H^{(\omega)}$ in (9), and whose solution can be expressed as the following infinite sum of integrals

$$\begin{aligned} H^{(\omega)}(x) = e^{-\phi(0)x}\Big(1 + \int_{z>x} \omega(z)W_{\phi(0)}(z-x)dz \\ + \iint_{z_2>z_1>x} \omega(z_2)\omega(z_1)W_{\phi(0)}(z_2-z_1)W_{\phi(0)}(z_1-x)dz_1dz_2 + \cdots\Big). \end{aligned}$$

On the other hand, if $\int^\infty W_{\phi(0)}(z)\omega(z)dz = \infty$, then $\lim_{b\to\infty} \frac{W^{(\omega)}(b,x)}{W(b)} = \infty$. Therefore, the introduction of appropriate $H^{(\omega)}$ is more involved and the associated integral equation can be different. We thus consider the following two examples to illustrate this.

If $\omega(z) = \frac{1}{z+a}$ for some $a \geq 0$, then $H^{(\omega)}$ in (9) solves a singular integral equation

$$H^{(\omega)}(x) := \lim_{b\to\infty} \frac{W^{(\omega)}(b,x)}{W(b)} = \int_x^\infty H^{(\omega)}(z)\frac{W(z-x)}{z+a}dz,$$

whose solution can be expressed, similar to [5], as

$$H^{(\omega)}(x) := \int_{\phi(0)}^\infty e^{-(x+a)y} \exp\left(\int_{\phi(0)+1}^y \frac{dz}{\psi(z)}\right)\frac{dy}{\psi(y)}, \quad \forall x > 0.$$

However, if $\int^\infty W_{\phi(q)}(x)\big|\omega(x)-q\big|dx < \infty$ for some ω and $q > 0$, then one would expect the following equation from [10, equation (2.3)] and [11, Lem.4.3]

$$W^{(\omega)}(x,y) = W^{(q)}(x-y) + \int_y^x W^{(\omega)}(x,z)\big(\omega(z)-q\big)W^{(q)}(z-y)dz,$$

and $H^{(\omega)}$ can be defined as a limit that solves equation

$$H^{(\omega)}(x) := \lim_{b\to\infty} \frac{W^{(\omega)}(b,x)}{W^{(q)}(b)} = e^{-\phi(q)x} + \int_x^\infty H^{(\omega)}(z)(\omega(z)-q)W^{(q)}(z-x)dz,$$

which can be further expressed as an infinite sum of integrals.

Remark 3. If we assume $\omega(z) = \int_0^\infty e^{-zx}\mu(dx)$ for some positive measure μ on $(0,\infty)$ such that

$$\int^\infty \frac{W_{\phi(0)}(x)}{R(x)}dx \asymp \int_a^\infty \frac{W_{\phi(0)}(x)}{R(x+a)}dx = \int_{0+}^\infty e^{-at}\frac{\mu(dt)}{\psi_{\phi(0)}(t)} \asymp \int_{0+} \frac{\mu(dt)}{\psi_{\phi(0)}(t)} < \infty,$$

then $H^{(\omega)}(x) = \int_0^\infty e^{-xy}\nu(dy)$ for some positive measure ν on $[\phi(0),\infty)$ so that

$$\nu(ds) = \delta_{\{\phi(0)\}}(ds) + \frac{\nu * \mu(ds)}{\psi(s)} \quad \text{for } s \geq \phi(0),$$

where δ denotes the delta measure and $\nu * \mu$ denotes the convolution of ν and μ.

2.3 Nonlinear CSBPs coming down from infinity

By saying a stochastic process *comes down from infinity*, we mean intuitively that the process drops dramatically within arbitrary short time when initiated from a very high level. The following definitions can be found in [8] and in recent papers [13, 6].

Definition 1. Let $(Y_t, t\geq 0)$ be a positive Markov process. The boundary ∞ is an instantaneous entrance boundary for the process Y if the process does not explode and

$$\forall t > 0, \quad \lim_{a\to\infty}\liminf_{x\to\infty}\mathbb{P}_x\big(S_a^- \leq t\big) = 1,$$

where $S_x^- := \inf\{t > 0, Y(t) < x\}$ denotes the first downcrossing time of Y.

If a strong Markov process comes down from infinity in the above sense, it is natural to extend the state space and define the process starting at ∞. Kallenberg [8] considers such an extension for regular diffusions. For positive-valued strong Markov processes with no negative jumps, besides some equivalent conditions [6, Lem.1.2.] for the entrance boundary, the following results for Markov process coming down from infinity are also shown in the paper, see also [13] and references there in. Recalling that for a positive Markov process Y, its Markov semigroup $\big(P_t, t\geq 0\big)$ is a family of linear operators on $L^\infty(\mathbb{R}^+)$ indexed by $t\geq 0$ and given by, c.f. [2, I.2]

$$P_t f(x) = \mathbb{E}_x\big[f(Y_t)\big] = \int_0^\infty f(y)\mathbb{P}_x(Y_t\in dy).$$

Definition 2 (C_b-Feller property). For $E = [0,\infty)$ or $[0,\infty]$, a semigroup $\big(P_t, t\geq 0\big)$ satisfies the *Feller property* if

(i) for any $f \in C_b(E)$, $x \to P_t f(x) \in C_b(E)$ for every $t \geq 0$,
(ii) for any $f \in C_b(E)$ and $x \in E$, $P_t f(x) \to f(x)$, as $t \to 0$,

where $C_b(E)$ denotes the space of bounded continuous functions on E. A Markov process Y on E is called a Feller process if its transition group satisfies the Feller property.

Lemma 3 ([6] Theorem 2.2). *Let Y be a non-explosive Markov process taking values in $[0,\infty)$ with no negative jumps, satisfying the Feller property and with an entrance boundary at ∞, in the sense of Definitions 1 and 2, respectively. Then Y can be extended to a Feller process valued in $[0,\infty]$ such that under $\mathbb{P}_\infty$, it starts from ∞, leaves from it instantaneously and stays finite almost-surely:*

$$\mathbb{P}_\infty\big(Y_0=\infty\big)=1 \quad \textit{and} \quad \mathbb{P}_\infty\big(\forall t>0, Y_t<\infty\big)=1.$$

Lemma 4 ([6] Theorem 2.5). *Assume that Y is a Feller process on $[0,\infty]$. Let $Y^{(x)}$ be the Markov process started from $x \in [0,\infty]$ with càdlàg sample paths. Then the family of processes $\big(Y^{(x)}\big)_{x\in[0,\infty)}$ converges weakly, in the Skorohod topology, as $x \to \infty$ towards $Y^{(\infty)}$.*

Lemma 3 shows the convergence of semigroups, that is,

$$P_t f(x) \to P_t f(\infty) = \mathbb{E}_\infty\big[f(X_t)\big]$$

as $x \to \infty$ for every $f \in C_b[0,\infty)$ to some probability measures $\big(\mathbb{P}_\infty(t,dy)\big)_{t>0}$ on $[0,\infty)$. Lemma 4 deals with the convergence of processes as elements in cádlág space. As far as the convergence of stopping times is concerned, the following useful regularity property of the first passage times under the probability measures $\big(\mathbb{P}_x\big)_{x\geq 0}$ is also proved in the above mentioned papers.

Lemma 5 ([6] Proposition 2.4). *Suppose that the conditions in Lemma 3 hold. Let $h \in C_b[0,\infty)$ be bounded or nonnegative increasing. Then*

(a) for any $\theta>0$, there exists $b_\theta>0$ such that for all $b \geq b_\theta, \mathbb{E}_\infty\big[e^{\theta S_b^-}\big]<\infty$;
(b) for any $b>0$, if $S_b^-<\infty$, $\mathbb{P}_\infty$-a.s. and $\mathbb{E}_\infty\big[h(S_b^-)\big]<\infty$, then

$$\mathbb{E}_x\big[h(S_b^-)\big] \to \mathbb{E}_\infty\big[h(S_b^-)\big], \quad \textit{as } x \to \infty.$$

For the time-changed SPLP X in (5), the Feller property as well as the necessary and sufficient condition for the process to come down from infinity has been shown in [7].

Lemma 6 ([7] Proposition 2.1). *Assume that function R is strictly positive and continuous function on $(0,\infty)$, and ξ is a spectrally positive Lévy process. Then for any $x>0$ and $t \in [0,\tau_0^-)$, we have $\eta(t)<\infty$ $\mathbb{P}_x$-a.s. The process X is well-defined, strong Markov and with cádlág paths, whose semigroup $\big(P_t, t\geq 0\big)$ satisfies the Feller property.*

Lemma 7 ([7] Theorem 3.1). *Assume that $\mathbb{E}[-\xi_1] \geq 0$. The boundary ∞ is an instantaneous entrance boundary for the process X in the sense of Definition 1 if and only if*

$$\int^{\infty} \frac{1}{x\psi(1/x)R(x)}\, dx < \infty.$$

Moreover, $\mathbb{E}_\infty\big[T_b\big] = \int_b^\infty \frac{W(x-b)}{R(x)}\, dx$ for the measure $\mathbb{P}_\infty$ in Lemma 3.

Since $W(x) \asymp \frac{1}{x\psi(1/x)}$ by Proposition VII.10 in [2], the above integral condition is equivalent to

$$\int^{\infty} \frac{W(x)}{R(x)}\, dx < \infty.$$

3 Main results

In this note we take use of the results from [6, 7] to compute quantities related to X that starts from infinity and comes down from infinity. Therefore, for the rest of the note we always assume that the conditions in Lemma 7 hold, that is, the following assumption holds.

Assumption $\mathbb{H}$: $-\mathbb{E}[\xi_1] \geq 0$, R is a strictly positive, continuous function on $(0,\infty)$ and

$$\int^{\infty} \frac{1}{x\psi(1/x)R(x)}\, dx \asymp \int^{\infty} \frac{W(x)}{R(x)}\, dx = \int^{\infty} \omega(z)W(z)\, dz < \infty.$$

Under the above assumption, $\phi(0) = 0$ and $H^{(\omega)}$ in Lemma 2 is the unique decreasing solution to equation

$$H^{(\omega)}(x) = 1 + \int_x^{\infty} H^{(\omega)}(z)\omega(z)W(z-x)\, dz,$$

which converges to 1 as $x \to \infty$.

Theorem 1. *Under Assumption $\mathbb{H}$ we have for any $q, a > 0$, $T_a^- < \infty$ $\mathbb{P}_\infty$-a.s. and*

$$\mathbb{E}_\infty\Big[e^{-qT_a^-}\Big] = \Big(H^{(q\omega)}(a)\Big)^{-1}.$$

In addition, the resolvent measure of X is

$$U_a^{(q)}(\infty, dy) := \int_0^{\infty} e^{-qt}\mathbb{P}_\infty\big(X_t \in dy, t \leq T_a^-\big)\, dt = \omega(y)\frac{W^{(q\omega)}(y,a)}{H^{(q\omega)}(a)}\, dy.$$

Proof. It follows from Assumption $\mathbb{H}$ that $T_a^- < \infty$ $\mathbb{P}_\infty$-a.s. Applying Lemma 5 (b) to Lemma 2 directly with weight function $q\omega$ and noticing that $\phi(0) = 0$, we obtain the Laplace transform for T_a^-.

For the resolvent measure of X, notice that for every $f \in C_b[0,\infty)$ we have from Lemma 1

$$\begin{aligned}
&\int_0^\infty e^{-qt}\mathbb{E}_x\big[f(X(t)); t \le T_a^- \wedge T_b^+\big]\,dt \\
=&\int_0^\infty \mathbb{E}_x\Big[e^{-q\eta(s)}\omega(\xi_s)f(\xi(s)); s \le \tau_a^- \wedge \tau_b^+\Big]\,dt \\
=&\int_a^b f(y)\omega(y)\Big(\frac{W^{(q\omega)}(b,x)}{W^{(q\omega)}(b,a)}W^{(q\omega)}(y,a) - W^{(q\omega)}(y,x)\Big)\,dy.
\end{aligned}$$

Letting $b \to \infty$ first and then $x \to \infty$, we have

$$\begin{aligned}
U_a^{(q)}f(x) :=& \int_0^\infty e^{-qt}\mathbb{E}_x\big[f(X_t); t \le T_a^-\big]\,dt \\
=& \int_a^\infty f(y)\omega(y)\Big(\frac{H^{(q\omega)}(x)}{H^{(q\omega)}(a)}W^{(q\omega)}(y,a) - W^{(q\omega)}(y,x)\Big)\,dy \\
\to& \int_a^\infty f(y)\omega(y)\frac{W^{(q\omega)}(y,a)}{H^{(q\omega)}(a)}\,dy.
\end{aligned}$$

Thus, as a function of x, $U_a^{(q)}f(x)$ has a limit as $x \to \infty$.

On the other hand, to give a probabilistic interpretation of the limit of $U_a^{(q)}f(x)$. Let e_q be an exponential random variable with parameter $q > 0$ independent of X. For every fixed $t > 0$, applying the Markov property of X and the memoryless property of e_q at t under $\mathbb{P}_x$, we have for $x \in (0,\infty)$

$$\begin{aligned}
qU_a^{(q)}f(x) =& \mathbb{E}_x\Big[f\big(X(e_q)\big); e_q \le T_a^-\Big] \\
=& e^{-qt}\mathbb{E}_x\Big[U_a^{(q)}f(X_t); t \le T_a^-\Big] + \mathbb{E}_x\Big[f\big(X(e_q)\big); e_q < T_a^- \wedge t\Big].
\end{aligned} \tag{10}$$

Applying Lemma 3 to the first term above, for the Feller semigroup of $\big(\mathbb{P}_x(X_s \in dy, s < T_a^-), s \ge 0\big)$ and function $U_a^{(q)}f \in C_b[0,\infty)$, we have as $x \to \infty$

$$\begin{aligned}
e^{-qt}\mathbb{E}_x\Big[U_a^{(q)}f(X_t); t \le T_a^-\Big] &\to e^{-qt}\mathbb{E}_\infty\Big[U_a^{(q)}f(X_t); t \le T_a^-\Big] \\
&= \mathbb{E}_\infty\Big[f\big(X(e_q)\big); e_q \le T_a^-\Big] - \mathbb{E}_\infty\Big[f\big(X(e_q)\big); e_q < T_a^- \wedge t\Big].
\end{aligned}$$

For the second term in (10), by the boundedness of f and Lemma 5 we have

$$\begin{aligned}
&\limsup_{x\to\infty}\mathbb{E}_x\Big[f\big(X(e_q)\big); e_q < T_a^- \wedge t\Big] \le ||f||_\infty \cdot \limsup_{x\to\infty}\mathbb{P}_x\big(e_q < T_a^- \wedge t\big) \\
&= ||f||_\infty \cdot \lim_{x\to\infty}\mathbb{E}_x\big[1 - e^{-q(T_a^- \wedge t)}\big] = ||f||_\infty \cdot \mathbb{E}_\infty\big[1 - e^{-q(T_a^- \wedge t)}\big].
\end{aligned}$$

Finally, letting $t \to 0$ gives

$$\lim_{x\to\infty}\mathbb{E}_x\Big[f\big(X(e_q)\big);e_q\le T_a^-\Big]=\mathbb{E}_\infty\Big[f\big(X(e_q)\big);e_q\le T_a^-\Big]$$

and this finishes the proof. □

Recall the *total progeny* until time $t\ge 0$

$$J(t):=\int_0^t X(s)\,ds$$

defined in [9, Sec.12.2.3.]. Note that $T_a^-=\eta(\tau_a^-)<\infty$. By a change of variable, we have

$$J(T_a^-)=\int_0^{\eta(\tau_a^-)}\xi\big(\eta^{-1}(t)\big)\,dt=\int_0^{\tau_a^-}\xi(s)\eta(ds)=\int_0^{\tau_a^-}\frac{\xi(s)}{R(\xi(s))}\,ds.$$

Applying Theorem 1, we have the following Laplace transform.

Corollary 1. *If Assumption* $\mathbb{H}$ *holds with* $R(x)$ *replaced by* $R(x)/x$*, then we have* $J(T_a^-)<\infty$ $\mathbb{P}_\infty$*-a.s. and for* $\omega^*(x):=x\omega(x)$*,*

$$\mathbb{E}_\infty\Big[e^{-qJ(T_a^-)}\Big]=\Big(H^{(q\omega^*)}(a)\Big)^{-1}.$$

We can also derive the moments for $J(T_a^-)$.

Theorem 2. *Under Assumption* $\mathbb{H}$*, for every* $a>0$ *denote by*

$$m_{a,0}=1 \text{ and } m_{a,n}:=\mathbb{E}_\infty\big[(T_a^-)^n\big]$$

the n^{th} *moment of* T_a^- *under* $\mathbb{P}_\infty$*. Then* $\{m_{a,n}\}_{n\ge1}$ *satisfies equation*

$$m_{a,n}=n!\sum_{k=1}^{n}(-1)^{k-1}a_k m_{a,n-k}=n!\sum_{k=0}^{n-1}(-1)^{n-k-1}m_{a,k}a_{n-k}, \tag{11}$$

where the sequence $(a_n)_{n\ge1}$ *is defined by* $a_1:=\int_a^\infty\frac{W(x-a)}{R(x)}\,dx$ *and*

$$a_n:=\iint_{x_n>x_{n-1}>\cdots>x_1>a}\frac{W(x_n-x_{n-1})W(x_{n-1}-x_{n-2})\cdots W(x_1-a)}{R(x_n)R(x_{n-1})\cdots R(x_1)}\,dx_n\cdots dx_1$$

that are finite for every $a>0$ *and* $n\ge1$*.*

Proof. Fix $a>0$. Under Assumption $\mathbb{H}$, the following functions are well-defined for $n\in\mathbb{N}$.

$$f_0(x)=1\quad\text{and}\quad f_n(x)=\int_x^\infty f_{n-1}(z)\frac{W(z-x)}{R(z)}\,dz. \tag{12}$$

Then we have from [3, Lem.8.11.1.] (see also [12, Prop.4.7.]) that

$$
\begin{aligned}
m_{a,n}(x) := \mathbb{E}_x\Big[\big(T_a^-\big)^n; T_a^- < \infty\Big] &= n\int_a^\infty \frac{m_{a,n-1}(y)}{R(y)}\big(W(y-a)-W(y-x)\big)\,dy \\
&= n\Big(\int_a^\infty m_{a,n-1}(y)\frac{W(y-a)}{R(y)}\,dy - \int_x^\infty m_{a,n-1}(y)\frac{W(y-x)}{R(y)}\,dy\Big)
\end{aligned}
\tag{13}
$$

for the moments of T_a^+ under $\mathbb{P}_x$ and again the finiteness assumption from $\mathbb{H}$. One can check from the identity that for $n \in \mathbb{N}$ and $x > a$

$$
m_{a,n}(x) \le n!\Big(\int_a^\infty \frac{W(y-a)}{R(y)}\,dy\Big)^n.
$$

Given the identity (13) and $f_n(x)$ defined in (12), one can also prove by induction on n that the moments $m_{a,n}$ can be written as

$$
m_{a,n}(x) = n!\sum_{k=0}^n (-1)^k f_k(x)\cdot c_{n,k} \tag{14}
$$

for some constant array $\{c_{n,k}, k=0,\cdots,n\}_{n\ge 0}$. Plugging (14) into (13) and then applying (12), we have

$$
\begin{aligned}
\frac{m_{a,n+1}(x)}{(n+1)!} &= \sum_{k=0}^n (-1)^k c_{n,k}\Big(\int_a^\infty f_k(z)\frac{W(z-a)}{R(z)}\,dz - \int_x^\infty f_k(z)\frac{W(z-x)}{R(z)}\,dz\Big) \\
&= \sum_{k=0}^n (-1)^k c_{n,k}\big(f_{k+1}(a) - f_{k+1}(x)\big).
\end{aligned}
\tag{15}
$$

Comparing (14) with (15) we have $c_{0,0} = 1$ for $n \in \mathbb{N}$

$$
c_{n,0} = \sum_{k=1}^n (-1)^{k-1} f_k(a) c_{n-1,k-1} \quad \text{and} \quad c_{n,k} = c_{n-1,k-1}.
$$

Therefore, letting $x \to \infty$ in (13) and applying Lemma 5 (b), we have

$$
m_{a,n} = \lim_{x\to\infty} m_{a,n}(x) = n!\cdot c_{n,0} = n!\sum_{k=1}^n (-1)^{k-1} a_k m_{a,n-k}.
$$

This finishes the proof. □

Remark 4. If $\omega(z) = R^{-1}(z) = \hat{\mu}(z)$ for some measure μ on $(0,\infty)$ as described in Remark 3, one can have from (12) in the proof that

$$
f_1(x) = \int_x^\infty dz \int_0^\infty e^{-zy} W(z-x)\mu(dy) = \int_0^\infty \frac{e^{-xy}}{\psi(y)}\mu(dy)
$$

and then by induction on n,

$$
f_n(x) = \iint_{y_1<y_2<\cdots<y_n} e^{-xy_n}\frac{\mu(dy_1)\mu(dy_2-y_1)\cdots\mu(dy_n-y_{n-1})}{\psi(y_1)\psi(y_2)\cdots\psi(y_n)}.
$$

In particular, if $R(x) = e^{-\lambda x}$ for some $\lambda > 0$, then $\mu(dz) = \delta_\lambda(dz)$ for delta measure δ_λ. We have

$$a_n = f_n(a) = e^{-na\lambda} \prod_{k=1}^{n} \frac{1}{\psi(k\lambda)} \quad \text{for } n \geq 1.$$

If $\psi(x) = x^\alpha$ and $R(x) = x^\beta$ for some $\beta > \alpha > 1$, then $\mu(dz) = \frac{z^{\beta-1}}{\Gamma(\beta)}\, dz$ and

$$a_n = f_n(a) = a^{n(\alpha-\beta)} \prod_{k=1}^{n} \frac{\Gamma(k(\beta-\alpha))}{\Gamma\big(k(\beta-\alpha)+\alpha\big)} \quad \text{for } n \geq 1.$$

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