Analysis of pseudoholomorphic curves on symplectization: Revisit via contact instantons

Yong-Geun Oh and Taesu Kim

Abstract In this survey article, we present the analysis of pseudoholomorphic curves $u : (\dot\Sigma, j) \to (Q \times \mathbb{R}, \widetilde{J})$ on the symplectization of contact manifold (Q, λ) as a subcase of the analysis of contact instantons $w : \dot\Sigma \to Q$, i.e., of the maps w satisfying the equation

$$\overline{\partial}^\pi w = 0, d(w^*\lambda \circ j) = 0$$

on the contact manifold (Q, λ), which has been carried out by a coordinate-free covariant tensorial calculus. The latter was initiated by Wang and the first author of the present survey in [OW18a, OW18b] for the closed string case. More recently the first author has extended the machinery to the open string case and applied it to the problems of quantitative contact topology and contact dynamics [Oh21a], [Oh22a], [OY23]. When the analysis is applied to that of pseudoholomorphic curves $u = (w, f)$ with $w = \pi \circ u$, $f = s \circ u$ on symplectization, the outcome is generally stronger and more accurate than the common results on the regularity presented in the literature in that all of our a priori estimates can be written purely in terms w not involving f. The a priori elliptic estimates for w, especially $W^{2,2}$-estimate, are largely consequences of various Weitzenböck-type formulae *with respect to the contact triad connection* introduced by Wang and the first author in [OW14], and the estimate for f is a consequence thereof by simple integration of the equation $df = w^*\lambda \circ j$.
We also derive a simple precise tensorial formulae for the linearized operator and for the asymptotic operator that admit a perturbation theory of the operators with

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respect to (adapted) almost complex structures: The latter has been missing in the analysis of pseudoholomorphic curves on symplectization in the existing literature.

Key words: Contact manifolds, Legendrian submanifolds, contact triad metric, contact triad connection, contact instantons, symplectization, locally conformal symplectic manifold, almost Hermitian manifold, canonical connection, pseudo-holomorphic curves on symplectization, Weitzenböck formula, asymptotic operator

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Part I
Introduction and overview

Let (Q,ξ) be a contact manifold. Assume ξ is coorientable. Then we can choose a contact form λ with $\ker\lambda = \xi$. With λ given, we have the Reeb vector field R_λ uniquely determined by the equation $R_\lambda \rfloor d\lambda = 0$, $R_\lambda \rfloor \lambda = 1$. Then we have decomposition $TQ = \xi \oplus \mathbb{R}\{R_\lambda\}$. We denote by $\pi : TQ \to \xi$ the associated projection and $\Pi = \Pi_\lambda : TQ \to TQ$ the associated idempotent whose image is ξ.

A *contact triad* is a triple (Q,λ,J) where λ is a contact form of ξ, i.e., $\ker\lambda = \xi$ and J is an endomorphism $J : TQ \to TQ$ that satisfies the following.

Definition 0.1 (λ-adapted CR almost complex structure)**.** A *CR almost complex structure* J is an endomorphism $J : TQ \to TQ$ satisfying $J^2 = -\Pi$, or more explicitly

$$(J|_\xi)^2 = -id|_\xi, \quad J(R_\lambda) = 0.$$

We say J is *adapted to* λ if $d\lambda(Y,JY) \geq 0$ for all $Y \in \xi$ with equality only when $Y = 0$. In this case, we all the pair (λ,J) an *adapted pair* of the contact manifold (Q,ξ).

The associated contact triad metric is given by

$$g_\lambda := d\lambda(\cdot, J\cdot) + \lambda \otimes \lambda. \tag{1}$$

The symplectization of (Q,λ) is the symplectic manifold $(Q\times\mathbb{R}, d(e^s\lambda))$ with $\mathbb{R}$-coordinate s also called the radial coordinate. We equip the symplectization with the s-translation invariant almost complex structure

$$\widetilde{J} = J \oplus J_0$$

where J_0 is the almost complex structure on the plane $\mathbb{R}\{\frac{\partial}{\partial s}, R_\lambda\}$ satisfying $J_0(\frac{\partial}{\partial s}) = R_\lambda$.

The main purpose of the present survey is to advertise the *covariant* tensorial approach to the analysis of pseudoholomorphic curves on symplectization via the study of contact instantons which was initiated by Wang and the first author of the present survey in [OW18a, OW18b], further developed by the first author in [Oha]–[Oh22a] and also by the present authors in [KO23].

Remark 0.2. When we say 'covariant tensorial', it means that we follow the way how Riemannian geometers and physicists do their tensor calculations. More specifically, we fix a 'best' connection ∇ on Q once and for all that optimizes tensor calculations. In our case, the contact triad connection is such a connection as illustrated by [OW18a, OW18b], [Oha] and [OY22]. Then for a given smooth map $w : \dot\Sigma \to Q$ or $u : \dot\Sigma \to Q \times \mathbb{R}$, we take covariant derivatives of any induced tensorial quantities only in terms of the induced connection of ∇ and the Levi-Civita connection of the domain $\dot\Sigma$.

Also the relevant Fredholm theory and compactification results are developed by the first author in [Oha] for the closed string case. More recently the theory of contact instantons has been extended in two different directions. On the one hand, in the joint work by Savelyev and the first author [OSar], they lifted the theory of contact instantons to the theory of pseudoholomorphic curves on the $\mathfrak{lcs}$-fication $(Q \times S^1_\rho, \omega_\lambda)$, Banyaga's locally conformal symplectification (which they call the $\mathfrak{lcs}$-fication) of contact manifold (Q, λ) [Ban02] on

$$(Q \times S^1_\rho, d\lambda + d\theta \wedge \lambda)$$

with the canonical angular form $d\theta$ satisfying $\int_{S^1_\rho} d\theta = 1$. According to the terminology adopted in [OSar], the authors call them the $\mathfrak{lcs}$-fication 'of nonzero temperature' on which the theory of pseudoholomorphic curves is developed. Here 'lcs' stands for the standard abbreviation of 'locally conformal symplectic'. The authors of ibid. call the relevant pseudoholomorphic curves *lcs instantons*. This family can be augmented by including the case of the product $Q \times \mathbb{R}$ as the 'zero temperature limit' with $1/\rho \to 0$,

$$(Q \times \mathbb{R}, \omega_\lambda), \quad \omega_\lambda := d\lambda + ds \otimes \lambda \tag{2}$$

(Here ρ represents the radius of the circle S^1.)

On the other hand, in [Oh21a], the first author of the present paper also extended the theory to the open-string case. It is further developed in [OY22] and applied for the construction of Legendrian contact instanton cohomology and the associated spectral invariants on the one-jet bundle in [OY23] jointly with Seungook Yu. Then the present authors have carried out precise asymptotic analyses near the punctures of finite energy contact instantons and of finite energy pseudoholomorphic curves in [KO23] in preparation by developing a generic perturbation theory of asymptotic operators over the change of adapted pairs (λ, J). The tensorial approach also clarifies the relationship between the background geometries of the contact triad (Q, λ, J), the symplectization

$$(M, d(e^s\lambda)) = (Q \times \mathbb{R}, e^s\omega_\lambda)$$

and the lcs manifold (4). Consider the decomposition

$$TM \cong \xi \oplus \mathbb{R}\left\{\frac{\partial}{\partial s}, R_\lambda\right\}. \tag{3}$$

Let $\widetilde{\nabla} := \nabla^{\text{can}}$ be the canonical connection of this almost Hermitian manifold

$$(Q \times \mathbb{R}, \widetilde{g}_\lambda, \widetilde{J}), \quad \widetilde{g}_\lambda := g_\lambda + ds \otimes ds \tag{4}$$

i.e., the unique Riemannian connection whose torsion T satisfies $T(X, \widetilde{J}X) = 0$ for all $X \in T(Q \times \mathbb{R})$ (See [Gau97, Kob03, Oh15] for its definition and basic properties.) We note that this almost Hermitian structure on $Q \times \mathbb{R}$ is translational invariant in the s-direction.

The *first upshot* of our tensorial approach utilizing the contact triad connection lies in our point of view of regarding the product

$$(Q \times \mathbb{R}, \omega_\lambda), \quad \omega_\lambda := d\lambda + ds \wedge \lambda$$

as an $\mathfrak{lcs}$-fication of contact manifold (Q, λ), and Hofer's analysis of pseudoholomorphic curves as the analysis of pseudoholomorphic curves on the $\mathfrak{lcs}$*-fication (in zero temperature)*

$$(Q \times \mathbb{R}, \omega_\lambda, \widetilde{J})$$

of contact triad (Q, λ, J). It is quite apparent that in Hofer's global analysis of pseudoholomorphic curves on symplectization *the symplectic form* $d(e^s \lambda)$ *plays little role* but $d\lambda$ and Hofer's ingenious way of considering translational invariant λ-*energy* does.

The $\mathfrak{lcs}$-fication carries a canonical translational invariant metric

$$\widetilde{g} := \omega_\lambda(\cdot, J \cdot)$$

so that the triple $(Q \times \mathbb{R}, \widetilde{g}, \widetilde{J})$ becomes an almost Hermitian (but not almost Kähler) manifold. The following geometric relationship between the contact triad connection and its lcs lifting has been implicitly exploited.

Proposition 0.3 (Canonical connection on $\mathfrak{lcs}$-fication)**.** *Let* $\widetilde{g} = \widetilde{g}_\lambda$ *be the almost Hermitan metric given above. Let* $\widetilde{\nabla}$ *be the canonical connection of the almost Hermitian manifold* (4)*, and* ∇ *be the contact triad connection for the triad* (Q, λ, J)*. Then* $\widetilde{\nabla}$ *preserves the splitting* (3) *and satisfies* $\widetilde{\nabla}|_\xi = \nabla|_\xi$*.*

The *second upshot* is our utilization of an important property of the Levi-Civita connection proved by Blair [Bla10] for the triad metric on Q. This important property has been completely unnoticed (as far as the authors are aware) in the symplectic community around pseudoholomorphic curves, including the first author until very recently at the time of preparing the paper [KO23] and this survey. (Indeed this property had been already mentioned in the paper [OW14, Proposition 4] by Wang and the first author himself!)

Proposition 0.4 (Lemma 6.1 [Bla10])**.** *Let* ∇^{LC} *be the Levi-Civita connection of the triad metric associated to the triad* (Q, λ, J)*. Then*

$$\nabla^{\mathrm{LC}}_{R_\lambda} J = 0. \tag{5}$$

This property of the Levi-Civita connection together with the usage of canonical connection on the $\mathfrak{lcs}$-fication $(Q \times \mathbb{R}, \widetilde{J}, \omega_\lambda)$ (or equivalently via contact triad connection on the triad (Q, λ, J)) plays an important role in the present authors' analysis of the asymptotic operators of finite energy contact instantons in [KO23] and hence of finite energy pseudoholomorphic curves too.

Remark 0.5. In fact if we let $\nabla^{\mathrm{LC}}_{R_\lambda}$ or ∇_{R_λ} acted upon $\Gamma(\xi) \subset \Gamma(TQ)$, then two covariant derivatives coincide, i.e.,

$$\pi \nabla^{\mathrm{LC}}_{R_\lambda}|_\xi = \pi \nabla_{R_\lambda}|_\xi,$$

even though the full connections are not the same, i.e., $\pi\nabla^{\rm LC} \neq \pi\nabla$. See [OW14, Section 6] or more clearly see its arXiv version 1212.4817(v2) Theorem 1.4 therein where ∇ is expressed as $\nabla = \nabla^{\rm LC} + B$ for some $(2,1)$ tensor B. From the expression of B there, we have $B(R_\lambda, Y) = 0$ for any $Y \in \xi$. However, while we have $\nabla^{\rm LC}_{R_\lambda} J = 0$, $\nabla_{R_\lambda} J \neq 0$ on the full tangent bundle TQ.

Roughly speaking Proposition 0.4 enables us to compute the asymptotic operator

$$A^\pi_{(\lambda,J,\nabla)} : \Gamma(\gamma^*\xi) \to \Gamma(\gamma^*\xi)$$

in the covariant tensorial way *uniformly in terms of the pull-back connection of the Levi-Civita connection of the triad metric* of the given compatible pair (λ, J).

Remark 0.6. 1. In the literature, the notion of 'asymptotic operator' of a pseudoholomorphic curve on symplectization has been used. For example, Hofer-Wysocki-Zehnder use special coordinates followed by some adjustment of the given almost complex structures along the Reeb orbit in the analysis of asymptotic operators of finite energy planes, while Siefring [Sie08] used a symmetric connection, Wendl [Wen] and Pardon [Par19] a connection obtained by declaring the Lie derivative $\mathscr{L}_{R_\lambda}$ to be the covariant differential along the Reeb orbit γ. (Compare these practices with those given in [KO23] a summary of which is given in Section 21 of the present survey.)

2. Such an important property (5) of the Levi-Civita connection has not attracted any attention from the symplectic community around pseudoholomorphic curves, because contact Hamiltonian dynamics has not attracted much attention from the researchers around pseudoholomorphic curves, and there has been no serious investigation thereof up to the level of symplectic Hamiltonian dynamics. As a consequence, there might not have been enough motivation for them to redo the analysis of pseudoholomorphic curves on symplectization *from scratch starting from its starting place, the contact triad* (Q, λ, J), especially when there is already the well-established Gromov's theory of pseudoholomorphic curves around.
3. However in relation to thermodynamics, contact completely integrable systems and new constructions of Sasaki-Einstein manifolds which is motivated by AdS/CFT correspondence and black-hole dynamics, there has been a systematic development of contact Hamiltonian calculus by a group of geometers and physicists. (See [BCT17], [dLLV19] and [Ler04] and [MS05, MS06], [Boy11] and references therein). Through their study, it has been becoming increasingly clearer that contact Hamiltonian dynamics deserves much more attention than now in many respects. We believe that the analysis and geometry of (perturbed) contact instantons will provide a flexible geometro-analytical package for the study of contact Hamiltonian dynamics and quantitative contact topology as the symplectic Floer theory does. These are illustrated by the first author and his collaborators' recent applications of the package to the problems of quantitative contact topology and contact dynamics. (See [Oh21c, Oh22a] and [OY23].)
4. The common folklore practice of lifting contact dynamics to the homogeneous Hamiltonian dynamics to the symplectization, attempting to exploit the machin-

ery of Floer's theory on the symplectization and then extracting information on the original contact dynamics therefrom does not produce optimal results because the lifting process is not reversible. Because of this, the optimal results proved in [Oh21c, Oh22a] or the Floer theoretic construction of Legendrian spectral invariants on the one-jet bundle given in [OY23] have not been obtainable by the existing machinery of pseudoholomorphic curves via symplectization at least by now. It is an interesting open problem, if possible, to recover those results belonging to purely contact realm by the symplectic machinery of pseudoholomorphic curves.

1 Pseudoholomorphic curves on symplectization

In the seminal work [Hof93], Hofer initiated the study of pseudoholomorphic curves on the symplectization $Q \times \mathbb{R} =: M$ for the study of contact topology and developed the analysis thereof in a series of papers [HWZ96b]-[HWZ02] with applications to contact dynamics in 3 dimensions. Bourgeois [Bou02] and Wendl [Wen] extend their analysis to higher dimensions, and Cant to the relative case [Can22].

Throughout the paper, we adopt the following notations.

Notation 1.1. We denote by (Σ, j) a closed Riemann surface, $\dot\Sigma$ the associated punctured Riemann surface and $\overline{\Sigma}$ the real blow-up of $\dot\Sigma$ along the punctures.

Note that in the presence of contact form λ, any smooth map $u : \dot\Sigma \to Q \times \mathbb{R}$ has the form $u = (w, f)$ with

$$f = s \circ u, \, w = \pi \circ u. \tag{1}$$

We have the splitting

$$TM \cong \xi \oplus \mathbb{R}\{R_\lambda\} \oplus \mathbb{R}\left\{\frac{\partial}{\partial s}\right\}.$$

We have a canonical almost complex structure

$$J_0 : \mathbb{R}\left\{\frac{\partial}{\partial s}, R_\lambda\right\} \to \mathbb{R}\left\{\frac{\partial}{\partial s}, R_\lambda\right\}$$

defined by $J_0 \frac{\partial}{\partial s} = R_\lambda$. We equip (Q, ξ) with a triad (Q, λ, J) and the cylindrical almost complex structure $\widetilde{J} = J \oplus J_0$ which is s-translation invariant.

Remark 1.2. As mentioned above, Hofer's analysis of pseudoholomorphic curves on symplectization should be regarded as a special case for the analysis of pseudoholomorphic curves on lcs manifolds $(Q \times \mathbb{R}, d\lambda + ds \wedge \lambda)$ equipped with the above mentioned almost Hermitian structure arising from the contact triad (Q, λ, J) as illustrated by Savelyev and the first author in [OSar]. See [Oha], [OSar] for further explanation on this point of view. We would like to reiterate that the symplectic form $d(e^s \lambda)$ itself plays very little role in the compactification of punctured pseudoholomorphic curves but Hofer's translational invariant energy $E(w) = E^\pi(w) + E^\perp(w)$ does.

For $M = Q \times \mathbb{R}$, we have the decomposition of the derivative

$$du = dw \oplus \left(df \otimes \frac{\partial}{\partial s} \right)$$

viewed as a TM-valued one-form which can be further decomposed to

$$du(z) = d^\pi w \oplus (w^*\lambda \otimes R_\lambda) \oplus \left(df \otimes \frac{\partial}{\partial s} \right) \tag{2}$$

with respect to the splitting

$$\mathrm{Hom}(T_z\dot\Sigma, T_{u(z)}M) = \mathrm{Hom}(T_z\dot\Sigma, HT_{u(z)}M) \oplus \mathrm{Hom}(T_z\dot\Sigma, VT_{u(z)}M).$$

(For the notational simplicity, we often omit '$\otimes$' except the situation that could cause confusion to the readers without it.)

By definition, we have $d\pi du = dw$. It was observed by Hofer [Hof93] that u is $\widetilde{J}$-holomorphic if and only if (w, f) satisfies

$$\begin{cases} \overline{\partial}^\pi w = 0 \\ w^*\lambda \circ j = df. \end{cases} \tag{3}$$

2 Definitions of contact instantons and bordered contact instantons

Let $\dot\Sigma$ a boundary punctured Riemann surface associated a bordered compact Riemann surface (Σ, j). Then for a given map $w : \dot\Sigma \to Q$, we can decompose its derivative du, regarded as a w^*TQ-valued one-form on $\dot\Sigma$, into

$$dw = d^\pi w + w^*\lambda \otimes R_\lambda \tag{1}$$

where $d^\pi w := \pi dw$. Furthermore $d^\pi w$ is decomposed into

$$d^\pi w = \overline{\partial}^\pi w + \partial^\pi w \tag{2}$$

where $\overline{\partial}^\pi w := (dw^\pi)^{(0,1)}_J$ (resp. $\partial^\pi w := (dw^\pi)^{(1,0)}_J$) is the anti-complex linear part (resp. the complex linear part) of $d^\pi w : (T\dot\Sigma, j) \to (\xi, J|_\xi)$. (For the simplicity of notation, we will abuse our notation by often denoting $J|_\xi$ by J. We also simply write $((\cdot)^\pi)^{(0,1)}_J = (\cdot)^{\pi(0,1)}$ and $((\cdot)^\pi)^{(1,0)}_J = (\cdot)^{\pi(1,0)}$ in general.)

A contact instanton is a map $w : \dot\Sigma \to Q$ that satisfies the system of nonlinear partial differential equation

$$\overline{\partial}^\pi w = 0, \quad d(w^*\lambda \circ j) = 0 \tag{3}$$

on a contact triad (Q, λ, J). The equation itself had been first mentioned by Hofer [Hof00, p.698], and some attempt to utilize the equation to attack the Weinstein conjecture for dimension 3 was made by Abbas [Abb11], Abbas-Cieliebak-Hofer [ACH05] as well as by Bergmann [Bera, Berb].

In a series of papers, [OW18a, OW18b] jointed with Wang and in [Oha], the first named author systematically developed analysis of contact instantons (for the closed string case) *without taking symplectization* by the global covariant tensorial calculations using the notion of *contact triad connection* which was introduced in [OW14]

More recently he also studied its open string counterpart of the boundary value problem of (3) under the Legendrian boundary condition whose explanation is now in order. For the simplicity and for the main purpose of the present paper, we focus on the genus zero case so that $\dot\Sigma$ is conformally the unit disc with boundary punctures $z_0, \ldots, z_k \in \partial D^2$ ordered counterclockwise, i.e.,

$$\dot\Sigma \cong D^2 \setminus \{z_0, \ldots, z_k\}$$

Then, for a $(k+1)$-tuple $\vec R = (R_0, R_1, \cdots, R_k)$ of Legendrian submanifolds, which we call an (ordered) Legendrian link, we consider the boundary value problem

$$\begin{cases} \overline{\partial}^\pi w = 0, \quad d(w^*\lambda \circ j) = 0, \\ w(\overline{z_i z_{i+1}}) \subset R_i \end{cases} \tag{4}$$

as an elliptic boundary value problem for a map $w : \dot\Sigma \to Q$ by deriving the a priori coercive elliptic estimates. Here $\overline{z_i z_{i+1}} \subset \partial D^2$ is the open arc between z_i and z_{i+1}.

3 $W^{2,2}$-estimates, Weitzenböck formulae and contact triad connection

Let us start with stating the general Weitzenböck formula in differential geometry. A good exposition of general Weitzenböck formula is given in [FU84, Appendix C].

Let (P,h) be a Riemannian manifold and $E \to P$ is a Euclidean vector bundle with inner product $\langle \cdot, \cdot \rangle$ and assume ∇ is a connection compatible with $\langle \cdot, \cdot \rangle$. We denote by d^∇ the covariant differential and δ^∇ its Hodge dual. We also denote by

$$\Delta^\nabla := \delta^\nabla d^\nabla + d^\nabla \delta^\nabla, \quad Tr\nabla^2 := \sum_i \nabla^2_{e_i, e_i} = \nabla^*\nabla$$

the covariant Hodge Laplacian and the trace Laplacian respectively, both of which are *second-order differential operators* acted upon the section space $\Gamma(E)$. A general Weitzenböck formula provides an explicit formula for the difference between the two Laplacians in terms of the action of the Ricci curvature operator of the underlying Riemannian manifold (P,h): *An upshot of the formula is that the difference is a zero-order differential operator.*

It is well established in the analysis of geometric PDE of the types, harmonic maps, minimal surface equation and Yang-Mills equations and so on that all a priori elliptic regularity results are based on suitable applications of the following general Weitzenböck Formula and the integration by parts one way or the other. (See [SY76], [SU81], [Uhl82], [SU83], [Sch84], [PW93], [RT95], to name a few.)

Theorem 3.1 (Weitzenböck Formula)**.** *Let $E \to P$ be a vector bundle equipped with inner product $\langle \cdot, \cdot \rangle$ and an Euclidean connection ∇. Assume $\{e_i\}$ is an orthonormal frame of P, and $\{\alpha^i\}$ is the dual frame. We denote by R the curvature tensor of the bundle E with respect to the connection ∇. Then, when applied to E-valued differential forms, we have*

$$\begin{aligned}\Delta^{\nabla} &= -Tr\nabla^2 + \sum_{i,j} \alpha^j \wedge (e_i \rfloor R(e_i, e_j)(\cdot)) \\ &= -\nabla^*\nabla + \sum_{i,j} \alpha^j \wedge (e_i \rfloor R(e_i, e_j)(\cdot)).\end{aligned}$$

(For readers' convenience, we will provide a summary of the exterior calculus of E-valued differential forms and a derivation of the above Weitzenböck formula in Appendix 22.)

For each given equation, to get the optimal regularity estimates, it is important to use the 'best' connection compatible with the given geometry such as the Chern connection or the canonical connection in the harmonic theory of holomorphic vector bundles in complex geometry [Che67]. (See [Wel73] for a nice exposition on the harmonic theory of holomorphic vector bundles on complex manifolds. One may view that our calculations are largely the almost complex counterpart thereof on the Riemann surface $\dot\Sigma$.)

We specialize this general Weitzenböck formula to our purpose of tensorial study of contact instantons. For each given contact triad (Q, λ, J), we consider the vector bundles E such as

$$w^*TQ, \quad w^*\xi, \quad \Lambda^1(w^*TQ), \quad \Lambda^{(0,1)}(w^*TQ)$$

on punctured Riemann surface $(\dot\Sigma, j, h)$ equipped with Kähler metric h that is cylindrical near each puncture. As the aforementioned 'best' connection in this setting of contact triads, Wang and the first author introduced the notion of *contact triad connection* which is unique for each given contact triad (Q, λ, J). (See [OW14] for its construction and full properties. See also Section 9 for a summary.)

In the geometric PDEs of minimal surface-type map $w : S \to Q$, such as pseudo-holomorphic curves or contact instantons, the regularity estimates usually starts with computing the formula for the Laplacian of the harmonic energy density function

$$\Delta |dw|^2.$$

A priori, this quantity involves degree 3 derivatives for general smooth maps (*in the off-shell*), but which is hoped to be expressed as a sum of the terms of degree less than 3 for maps satisfying the equation (*on shell*). It will then enable one to develop a priori boot-strap arguments. We always regard dw as a w^*TQ-valued one-form on the domain $\dot\Sigma$.

Following this general practice and using the decomposition (1) and the properties $\xi \perp R_\lambda$ and $|R_\lambda| = 1$ for the contact triad metric, we can decompose the (full) harmonic energy density into the sum

$$|dw|^2 = |d^\pi w|^2 + |w^*\lambda|^2 \tag{1}$$

where $|d^\pi w|^2$ is the contribution from the ξ-direction and $|w^*\lambda|^2$ is the energy density in the Reeb direction: Recall the decomposition $dw = d^\pi w + w^*\lambda \otimes R_\lambda$ is orthogonal with respect to the induced metric from the triad metric of the target and the Kähler metric h of the domain Riemann surface $(\dot\Sigma, j)$.

The following differential identity for contact Cauchy-Riemann map plays a fundamental role for all the estimates needed for the contact instantons.

Theorem 3.2 (Fundamental Equation; Theorem 4.2 [OW18a])**.** *Let w be any contact Cauchy–Riemann map, i.e., a solution of $\overline{\partial}^\pi w = 0$. Then*

$$d^{\nabla^\pi}(d^\pi w) = -w^*\lambda \circ j \wedge \left(\frac{1}{2}(\mathscr{L}_{R_\lambda} J)\, d^\pi w\right). \tag{2}$$

Here d^∇ is the skew-symmetrization of the covariant derivative ∇ on $\dot\Sigma$.

Remark 3.3. When we apply similar calculation to a J-holomorphic map u with respect to the canonical connection on the almost Hermitian manifold (M, ω, J), then the corresponding equation is the simple harmonic map equation $d^\nabla(du) = 0$ for u. (See [Oh15, Corollary 7.3.3].) Together with the conformality of any J-holomorphic map, this provides a computational proof with the well-known fact that the image of a J-holomorphic curve is a minimal surface with respect to the compatible metric. This equation was the basis for the $W^{2,2}$-estimate for J-holomorphic map equation $\overline{\partial}_J u = 0$ on symplectic manifolds. (See [Oh15, Proposition 7.4.5].)

Then we can derive the following differential identity for the ξ-component $d^\pi w$ of the derivative dw, utilizing the Weitzenböck formula associated to the contact triad connection.

Proposition 3.4 (Equation (4.11) [OW18a])**.** *Let w be a contact Cauchy-Riemann map. Then*

$$\begin{aligned} -\frac{1}{2}\Delta e^\pi(w) = {} & |\nabla^\pi(d^\pi w)|^2 + K|d^\pi w|^2 + \langle \mathrm{Ric}^{\nabla^\pi}(d^\pi w), d^\pi w\rangle \\ & + \langle \delta^{\nabla^\pi}[(w^*\lambda \circ j) \wedge (\mathscr{L}_{R_\lambda} J) d^\pi w], d^\pi w\rangle. \end{aligned} \tag{3}$$

If w is a contact instanton, i.e., if it satisfies $d(w^*\lambda \circ j) = 0$ in addition, we also have the following identity for the Laplacian of the energy density along the Reeb direction.

Proposition 3.5 (Equation (5.4) [OW18a]; Proposition 11.8)**.** *Let w be a contact instanton. Then*

$$-\frac{1}{2}\Delta|w^*\lambda|^2 = |\nabla w^*\lambda|^2 + K|w^*\lambda|^2 + \langle *\langle \nabla^\pi d^\pi w, d^\pi w\rangle, w^*\lambda\rangle. \tag{4}$$

By adding the two identities, we obtain a formula for the Laplacian $\Delta|dw|^2$ of the full harmonic energy density function $|dw|^2$ on shell. Once these are established, the following local $W^{2,2}$-estimate is obtained by a standard trick of multiplying cut-off function and integrating by parts.

Theorem 3.6 (Theorem 1.6 [OW18a])**.** *Let $w : \dot\Sigma \to Q$ satisfy* (4)*. Then for any relatively compact domains D_1 and D_2 in $\dot\Sigma$ such that $\overline{D_1} \subset D_2$, we have*

$$\|dw\|^2_{W^{1,2}(D_1)} \leq C_4 \|dw\|^4_{L^4(D_2)}$$

where C_4 is a constant depending only on D_1, D_2 and (Q, λ, J) and R_i's.

The same estimates is also proved in the similar spirit by incorporating the Legendrian boundary condition which is a (nonlinear) elliptic boundary value problem. (See [Oh21a, Theorem 1.4] for the statement and [OY22] for the same statement with corrected proof.) This boundary estimate is rather nontrivial unlike the closed string case.

As the first step towards the analytic study of the above boundary value problem (4), we first show that the Legendrian boundary condition for the contact instanton is a free boundary value problem, i.e., it satisfies

$$\frac{\partial w}{\partial \nu} \perp TR$$

for any Legendrian submanifold. (See [Jos86] for the importance of the free boundary value condition for a general study of elliptic estimates of the minimal surface type equations.) Then we prove the elliptic $W^{2,2}$-estimate as an application of Stokes' formula combined with the Legendrian boundary condition. The global tensorial calculation deriving the a priori estimate in [OY22] illustrates how well the Legendrian boundary condition interacts with triad connection and the contact instanton equation.

3.0.1 Higher $C^{k,\delta}$ Hölder estimates

Once this $W^{2,2}$ estimates is established, we proceed with the higher boundary regularity estimates. Obviously the same estimates hold for the closed string case (3) the corresponding statement of which had been established in [OW18a]. Since this case is easier, we focus on the statements for the boundary estimates below.

Starting from Theorem 3.6 and using the embedding $W^{2,2} \hookrightarrow C^{0,\delta}$ with $0 < \delta < 1/2$, we also establish the following higher local $C^{k,\delta}$-estimates on punctured surfaces $\dot\Sigma$ in terms of the $W^{2,2}$-norms.

Theorem 3.7 (Theorem 1.4 [OY22]). *Let w satisfy* (4)*. Then for any pair of disk $D_1 \subset D_2 \subset \dot\Sigma$ of semi-disk domains $(D_1, \partial D_1) \subset (D_2, \partial D_2) \subset (\Sigma, \partial\Sigma)$ such that $\overline{D_1} \subset D_2$,*

$$\|dw\|_{C^{k,\delta}(D_1)} \leq C_\delta \left(\|dw\|_{W^{1,2}(D_2)} \right)$$

where $C_\delta = C_\delta(r) > 0$ is a function continuous at $r = 0$ and depends only on J, λ and D_1, D_2 but independent of w.

With some adjustment of the function C_δ, combining the two theorems, we obtain

Corollary 3.8. *Assume $k \geq 1$ and $0 < \delta < 1/2$. Let w satisfy* (4)*. Then for any pair of domains $D_1 \subset D_2 \subset \dot\Sigma$ such that $\overline{D_1} \subset D_2$,*

$$\|dw\|_{C^{k,\delta}(D_1)} \leq C_\delta \left(\|dw\|_{L^4(D_2)} \right)$$

where $C_\delta = C_\delta(r) > 0$ is a function continuous at $r = 0$ and depends only on J, λ and D_1, D_2 but independent of w.

In particular, we prove that any weak solution of (4) in $W^{1,4}_{\mathrm{loc}}$ automatically becomes a classical solution. (Compare [Oh15, Theorem 8.5.5] for a similar theorem for the Lagrangian boundary condition in symplectic geometry.)

The proof of Theorem 3.7 is carried out by an *alternating boot strap argument* by decomposing

$$dw = d^\pi w + w^*\lambda \otimes R_\lambda$$

as follows. Let $z = x + iy$ be any isothermal coordinates on $(D_2, \partial D_2) \subset (\dot\Sigma, \partial\dot\Sigma)$ adapted to the boundary, i.e., satisfying that $\frac{\partial}{\partial x}$ is tangent to $\partial\dot\Sigma$. We set

$$\begin{aligned} \zeta &:= d^\pi w(\partial_x), \\ \alpha &:= \lambda\left(\frac{\partial w}{\partial y}\right) + \sqrt{-1}\lambda\left(\frac{\partial w}{\partial x}\right) \end{aligned}$$

Then we show that the fundamental equation (2) is transformed into the following system of equations for the pair (ζ, α)

$$\begin{cases} \nabla^\pi_x \zeta + J\nabla^\pi_y \zeta + \frac{1}{2}\lambda(\frac{\partial w}{\partial y})(\mathscr{L}_{R_\lambda} J)\zeta - \frac{1}{2}\lambda(\frac{\partial w}{\partial x})(\mathscr{L}_{R_\lambda} J)J\zeta = 0 \\ \zeta(z) \in TR_i \quad \text{for } z \in \partial D_2 \end{cases} \tag{5}$$

and

$$\begin{cases} \overline{\partial}\alpha = \frac{1}{2}|\zeta|^2 \\ \alpha(z) \in \mathbb{R} \quad \text{for } z \in \partial D_2 \end{cases} \tag{6}$$

for some $i = 0, \dots, k$. With this coupled system of equations for (ζ, α) at our disposal, the proof of higher regularity results is carried out by the alternating boot strap argument between ζ and α in [Oh21a, OY22].

4 Asymptotic convergence and vanishing of asymptotic charge

Next we study the asymptotic convergence result of contact instantons of finite energy $E(w) = E^\pi(w) + E^\perp(w) < \infty$ for the closed string case (resp. with Legendrian boundary condition of pair (R_0, R_1) for the open string case) near the punctures of a Riemann surface $\dot\Sigma$. (We refer to [Oha, Oh21c, OY23] for the precise definition of total energy.)

Let $\dot\Sigma$ be a punctured Riemann surface with punctures

$$\{p_i^+\}_{i=1,\cdots,l^+} \cup \{p_j^-\}_{j=1,\cdots,l^-}$$

equipped with a metric h with cylinder-like ends (resp. *strip-like ends* for the open string case) outside a compact subset K_Σ. Let $w : \dot\Sigma \to Q$ be any such smooth map.

As in [OW18a], we define the total π-harmonic energy $E^\pi(w)$ is easy to define as

$$E^\pi(w) = E^\pi_{(\lambda,J;\dot\Sigma,h)}(w) = \frac{1}{2}\int_{\dot\Sigma} |d^\pi w|^2\, dA \tag{1}$$

where dA is the associated area form and the norm is taken in terms of the given metric h on $\dot\Sigma$ and the triad metric on Q.

4.0.1 The case of closed strings

Under the hypotheses of nondegeneracy λ (resp. of the pair $(\lambda, (R_0, R_1))$ for the open string case) and of asymptotic convergence at the punctures, we can associate two natural asymptotic invariants at each puncture defined as

$$T_w := \lim_{r\to\infty} \int_{\{r\}\times S^1} (w|_{\{r\}\times S^1})^*\lambda \tag{2}$$

$$Q_w := \lim_{r\to\infty} \int_{\{r\}\times S^1} ((w|_{\{r\}\times S^1})^*\lambda \circ j) \tag{3}$$

at each puncture. (Here we only look at positive punctures. The case of negative punctures is similar.) As in [OW18a], we call $T = T_w$ the *asymptotic contact action* and $Q = Q_w$ the *asymptotic contact charge* of the contact instanton w at the given puncture.

Remark 4.1. It is unfortunate that we ended up using the same letter Q for both contact manifold and the asymptotic charge but this practice should not confuse readers.

The proof of the following subsequence convergence result is given in [OW18a, Theorem 6.4]. A similar asymptotic convergence result in 3 dimension in the setting of pseudoholomorphic curves on symplectization, i.e., the case of $Q = 0$ is proved in [HWZ02]. (See also [HWZ96b, HWZ96a] and compare their proofs with the proof given in [OW18a].)

Theorem 4.2 (Subsequence Convergence, Theorem 6.4 [OW18a])**.** *Let $w : [0,\infty) \times S^1 \to Q$ satisfy the contact instanton equations* (4) *and Hypothesis* (2)*. Then for any sequence $s_k \to \infty$, there exists a subsequence, still denoted by s_k, and a massless instanton $w_\infty(\tau,t)$ (i.e., $E^\pi(w_\infty) = 0$) on the cylinder $\mathbb{R} \times S^1$ that satisfies the following:*

1. $\overline{\partial}^\pi w_\infty = 0$ *and*
$$\lim_{k\to\infty} w(s_k + \tau, t) = w_\infty(\tau, t)$$
in the $C^l(K \times S^1, Q)$ sense for any l, where $K \subset [0,\infty)$ is an arbitrary compact set.

2. $w_\infty^* \lambda = -Q\, d\tau + T\, dt$

In general $Q = 0$ does not necessarily hold for the closed string case. When $Q \neq 0$ combined with $T = 0$ happens, we say w has the bad limit of *appearance of spiraling instantons along the Reeb core*. It is also proven in [Oha] that If $Q = 0 = T$, then the puncture is removable.

When $Q = 0$, which is always the case when contact instanton is exact such as those arising from the symplectization case, we have the following asymptotic convergence result.

Corollary 4.3. *Assume that λ is nondegenerate. Let w be as above and assume $Q = 0, T \neq 0$ and that $w_\tau : S^1 \to Q$ converges as $|\tau| \to \infty$. Then w_τ converges to a Reeb orbit of period $|T|$ exponentially fast.*

We would like to emphasize that the asymptotic limit could be a constant path, i.e., the Reeb action $T := \int \gamma^* \lambda$ is zero, which is normally not regarded as a Reeb orbit. To unify the cases for $T \neq 0$ and $T = 0$, it is useful to utilize the following Moore-type notion of paths introduced and utilized in [Oh21c, Oh22a] for the study of Sandon-Shelukhin type conjectures.

Definition 4.4 (Isospeed Reeb trajectories)**.** A pair (γ, T) of a parameterized curve $\gamma : [0,1] \to Q$ with $T = \int \gamma^* \lambda$ is called an *isospeed Reeb trajectory* if it satisfies the equation
$$\dot\gamma(t) = T\, R_\lambda(\gamma(t)). \tag{4}$$

Note that when $T \neq 0$, each such pair gives rise to a Reeb trajectory on the domain $[0, |T|]$, while when $T = 0$ the resulting path is a constant path. From now on, we will say $w(\tau,\cdot)$ *converges to an isospeed Reeb trajectory* (γ, T) if $w(\tau,\cdot) \to \gamma$ as $|\tau| \to \infty$ with $\int \gamma^* \lambda = T$.

4.0.2 The case of open strings

Now we make the corresponding statement for the open string case proved in [Oh21a].

Theorem 4.5 (Subsequence Convergence; the case of open strings)**.** *Let $w : [0,\infty) \times [0,1] \to Q$ satisfy the contact instanton equations* (4)*. Then for any sequence $s_k \to \infty$, there exists a subsequence, still denoted by s_k, and a massless instanton $w_\infty(\tau,t)$ (i.e., $E^\pi(w_\infty) = 0$) on the cylinder $\mathbb{R} \times [0,1]$ such that*

$$\lim_{k\to\infty} w(s_k + \tau, t) = w_\infty(\tau,t)$$

in the $C^l(K \times [0,1], Q)$ sense for any l, where $K \subset [0,\infty)$ is an arbitrary compact set. Furthermore, w_∞ has $Q = 0$ and the formula $w_\infty(\tau,t) = \gamma(t)$ with asymptotic action T, where (γ,t) is an isospeed Reeb chord with its action $T = \int \gamma^*\lambda$ joining R_0 and R_1 of period $|T|$.

Corollary 4.6 (Vanishing Charge)**.** *Assume the pair $(\lambda, \vec{R})$ is nondegenerate. Let w be as above with finite energy. Suppose that $w(\tau,\cdot)$ converges as $\tau \to \infty$ in the strip-like coordinate at a puncture $p \in \partial\Sigma$ with associated Legendrian pair (R,R'). Then its asymptotic charge Q vanishes at p.*

5 Asymptotic operators and their analysis

We first mention a few differences between the way how we study the asymptotic operators and those of [HWZ96b] and of other literature such as [RS01, Appendix C], [Sie08, Sie11], [Wen], [Can22].

Remark 5.1. In [RS01, Appendix E], [Sie08, Sie11], [Wen, Section 3.3], there have been attempts to give a coordinate-free definition of the asymptotic operator along the associated asymptotic Reeb orbit for a pseudoholomorphic curve $u = (w, f)$ on symplectization. However both fall short of a seamless definition of the 'asymptotic operator' of the Reeb orbits because the Reeb orbit lives on Q while the pseudoholomorphic curves live on the product $Q \times \mathbb{R}$ and the asymptotic limit of pseudoholomorphic curve live at infinity $Q \times \{\pm\infty\}$ *where only the contact structure makes sense, i.e., is canonically defined, but not the contact form itself*. Cant studies the asymptotic operator for the relative context in [Can22, Section 6.3] by adapting Wendl's. What these literature (e.g.[Wen, Section 3.3]) are describing is actually the asymptotic operator of the contact instanton w but trying to describe it in terms of the pseudoholomorphic curves which prevents them from being able to give a seamless definition. (See our definition of the asymptotic operator of contact instantons given in Definition 21.2 and compare it therewith. See also [OW18b, Section 11.2 & 11.5] for the precursor of our definition.)

In their series of works [HWZ95] – [HWZ02] in 3 dimension, Hofer-Wysocki-Zehnder carried out fundamental analytic study of pseudoholomorphic curves on symplectization utilizing *special coordinates followed by some local adjustment of given almost complex structure along the Reeb orbit of interest.* This practice has been propagated to other literature, such as [Bou02], [Hut02, Section 1.3] giving rise to some unnecessary restrictions on the choice of almost complex structure beyond the natural λ-adaptedness. [1]

Largely, thanks to the property $\nabla^{\mathrm{LC}}_{R_\lambda} J = 0$ from Proposition 0.4 combined with our usage of contact triad connection in the derivation of the formula for the asymptotic operator, our asymptotic analysis provided in [KO23] does not need any of those special coordinates and so Hutchings' assumption (or similar ansatz in other literature) is really not needed.

Theorem 5.2 (Corollary 21.7)**.** *Let (λ, J) be any adapted pair and let ∇^{LC} be the Levi-Civita connection of the triad metric of (Q, λ, J). For given contact instanton w with its action $\int \gamma^*\lambda = T$ at a puncture, let $A_{(\lambda,J,\nabla)}$ be the asymptotic operator of w written in cylindrical coordinate (τ, t). Then*

1. $[\nabla^{\mathrm{LC}}_t, J] (= \nabla^{\mathrm{LC}}_t J) = 0$,
2. We have

$$\begin{aligned} A^\pi_{(\lambda,J,\nabla)} &= -J\nabla_t + \frac{T}{2}\mathscr{L}_{R_\lambda} J J \\ &= -J\nabla^{\mathrm{LC}}_t - \frac{T}{2} Id + \frac{T}{2}\mathscr{L}_{R_\lambda} J J. \end{aligned}$$

We also have the following formula of $A^\pi_{(\lambda,J,\nabla)}$ that is independent of the choice of connections.

Proposition 5.3 (Proposition 1.7 [KO23])**.** *We have*

$$A^\pi_{(\lambda,J,\nabla)} = T\left(-\frac{1}{2}\mathscr{L}_{R_\lambda} J - \mathscr{L}_{R_\lambda} + \frac{1}{2}(\mathscr{L}_{R_\lambda} J) J\right). \tag{1}$$

Because the existing literature on the pseudoholomorphic curves on symplectization lack this kind of explicit formula of the asymptotic operator (together with the commuting property $[\nabla^{\mathrm{LC}}_t, J] = 0$), it has been the case that the general abstract perturbation theory of linear operators Kato [Kat95] is just quoted in their study of asymptotic operators which prevents one from making any statement on specific dependence on the adapted almost complex structures. (See [Wen, Lemma 3.17 & Theorem 3.35], for example, which in turn follows the statements and arguments given in [HWZ95].)

Remark 5.4. 1. The formula of an asymptotic operator canonically applied to every contact instanton w of the adapted pair (λ, J) (as well as that of a pseudoholomorphic curve $u = (w, f)$ on symplectization) depends not only on the adapted

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[1] Hutchings has informed the first author that this part of his paper is now obsolete in that Siefring proved the asymptotics he needed without the extra assumptions in [Sie08] quoted above. Siefring's paper is now quoted for this in Hutching's later paper [Hut09] which is a kind of update to [Hut02].

pair (λ, J) but also on the connection that is used to compute the linearization operator of w (or (u, f)). (See Definition 21.2 and Remark 21.3 for the explanation for why.) In this regard, we denote the asymptotic operator by

$$A^{\pi}_{(\lambda,J,\nabla)}(u) := A^{\pi}_{(\lambda,J,\nabla)}(w)$$

So it is conceivable to expect that a good choice of connection will give rise to a formula of the asymptotic operator that is easier to analyze.

2. In fact, the lack of precise definition together with the non-commuting property $[\nabla_t, J] \neq 0$ in the literature combined with the practice of *using special coordinates followed by adjusting the almost complex structure along the Reeb orbits* is bound to make the analysis of asymptotic operators very complicated as seen from [HWZ02], [Sie08], and prevents one from developing any perturbation theory of asymptotic operators under the perturbation of J's such as those developed by the present authors in [KO23]. The latter is summarized in Section 21 of the present survey.

On the other hand, our explicit formula of the asymptotic operator given in Theorem 5.2, which simultaneously applies to all closed Reeb orbits, enables us to prove the following natural generic perturbation result [KO23].

Theorem 5.5 (Generic simpleness of eigenvalues; [KO23])**.** *Let (Q, ξ) be a contact manifold. Assume that λ is nondegenerate. For a generic choice of λ-adapted CR almost complex structures J, all eigenvalues μ_i of the asymptotic operator are simple for all closed Reeb orbits of λ.*

See Section 21 for our derivation of the formula of the asymptotic operator and a summary of our analysis in [KO23] of the asymptotic operators. We believe that this kind of perturbation result will play some role in the construction of Kuranishi structures on the moduli space of finite energy contact instantons so that certain natural functor can be defined in our Fukaya-type category of contact manifolds [KO], [Ohc] (See Remark 9.10 (2) for the relevant remark.)

6 Comparison of compactifications of two moduli spaces

(The materials in this subsection is borrowed from [KO], [Ohc] which are in preparation.)

Now let us consider contact instantons w arising from a pseudoholomorphic curves on symplectization (w, f). In particular all such w has its the *charge class* vanishes $[w^*\lambda \circ j] = 0$ in $H^1(\dot\Sigma, \mathbb{Z})$. (See [OSar] or Subsection 19.2 for its definition.)

Remark 6.1. For the boundary punctured case, the charge class can be lifted to $H^1(\overline{\Sigma}, \partial_\infty \dot\Sigma)$ where $\overline{\Sigma}$ is the real blow-up of $\dot\Sigma$ along the punctures. See Subsection 19.2 for its definition.

Let Σ be a closed Riemann surface of genus g and $\dot\Sigma$ be the associated punctured Riemann surface $\dot\Sigma = \Sigma \setminus \{z_1, \cdots, z_\ell\}$. We denote the moduli space of such contact instantons $w : \dot\Sigma \to Q$ of finite energy by

$$\widetilde{\mathscr{M}}_{g,\ell}^{\text{exact}}(Q,J;\vec\gamma^-,\vec\gamma^+)$$

and

$$\mathscr{M}_{g,k,\ell}^{\text{exact}}(Q,J;\vec\gamma^-,\vec\gamma^+) := \widetilde{\mathscr{M}}_{g,\ell}^{\text{exact}}(Q,J;\vec\gamma^-,\vec\gamma^+)/\mathrm{Aut}(\dot\Sigma)$$

the set of isomorphism classes thereof. We have the natural forgetful map $(w,f) \mapsto w$ which descends to

$$\mathfrak{forget} : \mathscr{M}_{g,\ell}(M,\widetilde J;\vec\gamma^-,\vec\gamma^+) \to \mathscr{M}_{g,\ell}^{\text{exact}}(Q,J;\vec\gamma^-,\vec\gamma^+). \tag{1}$$

By definition of the equivalence relation on $\widetilde{\mathscr{M}}_{g,\ell}(M,\widetilde J;\vec\gamma^-,\vec\gamma^+)$ defined in [EGH00], [BEHZ03], it follows that this forgetful map is a bijective correspondence, *provided* $\dot\Sigma$ *is connected*.

However when one considers the SFT compactification, one needs to also consider the case of pseudoholomorphic curves with disconnected domains. So let us consider such cases. Suppose that the Riemann surface $\dot\Sigma$ is the union

$$\dot\Sigma = \bigsqcup_{i=1}^k \dot\Sigma_i$$

of connected components with $k \geq 2$. We denote by

$$\overline{\mathscr{M}}_{g,\ell}(M,\widetilde J;\vec\gamma^-,\vec\gamma^+)$$

and

$$\overline{\mathscr{M}}_{g,\ell}^{\text{exact}}(Q,J;\vec\gamma^-,\vec\gamma^+)$$

the relevant stable map compactifications, respectively. The following proposition shows the precise relationship between the two. We know that each story carries at least one non-cylindrical component.

Proposition 6.2 (Proposition 19.7)**.** *Let* $1 \leq \ell \leq k$ *be the number of connected components which are not cylinderical. The forgetful map* $\mathfrak{forget}$ (1) *extends to the compactified moduli spaces, and is a principle* $\mathbb{R}^{\ell-1}$ *fibration.*

So the above mentioned bijective correspondence still holds when there is exactly one non-trivial component. (See [BEHZ03, p. 835] for a relevant remark.)

This proposition clearly shows that the compactification proposed in [EGH00], [BEHZ03] have spurious strata that contract in the compactification of (exact) contact instantons under the forgetful map. In this regard, our compactification of exact contact instanton moduli spaces is closely related to Pardon's compactification of moduli spaces of pseudoholomorphic curves on symplectization [Par19], which is slightly different from that of [EGH00], [BEHZ03]. We will make precise their relationship in [KO] and [Ohc]. Pardon [Par19] used his compactification for his construction of Kuranishi structure on the moduli space of pseudoholomorphic curves on symplectization to define contact homology.

7 Fredholm theory and the index formula

Next, we study another crucial component, the relevant Fredholm theory and the index formula for the equation (4) by adapting the one from [Oha], [OSar] to the current case of contact instantons *with boundary*. The relevant Fredholm theory in general has been developed by the first named author in [Oha] for the closed string case and [Ohb] for the case with boundary.

In the present paper, we state the index formula for general disk instanton w with finite number of boundary punctures.

We recall the following Fredholm property of the linearized operator that is proved in [Ohb].

Proposition 7.1 (Proposition 3.18 & 3.20 [Ohb])**.** *Suppose that w is a solution to* (4)*. Consider the completion of $D\Upsilon(w)$, which we still denote by $D\Upsilon(w)$, as a bounded linear map from $\Omega^0_{k,p}(w^*TQ,(\partial w)^*T\vec{R})$ to $\Omega^{(0,1)}(w^*\xi)\oplus\Omega^2(\Sigma)$ for $k\geq 2$ and $p\geq 2$. Then*

1. *The off-diagonal terms of $D\Upsilon(w)$ are relatively compact operators against the diagonal operator.*
2. *The operator $D\Upsilon(w)$ is homotopic to the operator*
$$\begin{pmatrix} \overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} & 0 \\ 0 & -\Delta(\lambda(\cdot))\,dA \end{pmatrix} \tag{1}$$
via the homotopy
$$s\in[0,1]\mapsto \begin{pmatrix} \overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)} & \frac{s}{2}\lambda(\cdot)(\mathscr{L}_{R_\lambda}J)J(\pi dw)^{(1,0)} \\ s\,d\left((\cdot)\rfloor d\lambda)\circ j\right) & -\Delta(\lambda(\cdot))\,dA \end{pmatrix} =: L_s \tag{2}$$
which is a continuous family of Fredholm operators.
3. *And the principal symbol*
$$\sigma(z,\eta): w^*TQ|_z \to w^*\xi|_z\oplus\Lambda^2(T_z\Sigma), \quad 0\neq\eta\in T^*_z\Sigma$$
of (1) *is given by the matrix*
$$\begin{pmatrix} \frac{\eta+i\eta\circ j}{2}Id & 0 \\ 0 & |\eta|^2 \end{pmatrix}.$$

Then we have the Fredholm index of $D\Upsilon(w)$ is given by

$$\mathrm{Index}\,D\Upsilon(w) = \mathrm{Index}(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}) + \mathrm{Index}(-\Delta_0).$$

For this purpose of computing the Fredholm index of the linearized operator in terms of a topological index, especially in terms of those who will equip the relevant Floer-type complex with the absolute grading, we need the notion of anchored Legendrian submanifolds [OY22] which is an adaptation of that of Lagrangian submanifolds studied in [FOOO10] in symplectic geometry.

Definition 7.2. Fix a base point y of ambient contact manifold (Q,ξ). Let R be a Legendrian submanifold of (Q,ξ). We define an *anchor* of R to y is a path $\ell : [0,1] \to Q$ such that $\ell(0) = y$, $\ell(1) \in R$. We call a pair (R,ℓ) an *anchored* Legendrian submanifold. A chain $\mathscr{E} = ((R_0,\ell_0),\dots,(R_k,\ell_k))$ is called an *anchored Legendrian chain*.

We refer readers to [OY22] for the details of derivation of the relevant index formula which expresses the analytical index of the linearized problem of (4) in terms of a topological index of the Maslov-type. This index is made more explicit in [OY23] for the case of Hamiltonian isotopes of the zero section of the one-jet bundle.

Remark 7.3. While we are preparing this survey, Dylan Cant informed the first author of his thesis work [Can22] in which he studied the Fredholm theory and index formula *for the open string case* of pseudoholomorphic curves for the symplectization of *general pair* $(Q\times\mathbb{R}, R\times\mathbb{R})$ with Legendrian submanifold R in general dimension. As mentioned in [OY22, Section 1.5], this case is included as a part of the study of pseudoholomorphic curves on the $\mathfrak{lcs}$-fication of contact manifold as the exact case or as the 'zero-temperature limit' thereof.

8 Finer asymptotics

Finally we describe our work on the precise fine asymptotics of contact instantons (and so of pseudoholomorphic curves on symplectization) through the coordinate-free analysis of asymptotic operators and the study of their eigenvalues under the perturbation of adapted pairs (λ, J) in our work [KO23]. We refer readers to Section 21.3 for more detailed summary of the materials below.

Remark 8.1. A study of precise asymptotic formula of pseudholomorphic curves on symplectization in the more general context of stable Hamiltonian structures is given by Siefring in [Sie08, Sie11] which is applied to develop local intersection theory and embedding controls (in 4 dimension) of two embedded pseudoholomorphic curves with the same asymptotic limits.

Our study provides a precise generic description of the spectral behavior of asymptotic operators under the change $d\lambda$-compatible CR almost complex structures J when λ is fixed. For example, we prove the following in [KO23].

An upshot of our study of asymptotic behavior of finite contact instantons (and hence that of pseudoholomorphic curves on symplectization) is our usage of the following variable

$$\zeta(\tau,t) := \left(\frac{\partial w}{\partial \tau}\right)^{\pi}$$

on the cylindrical coordinates of $\dot\Sigma$ near a puncture in our various study of contact instantons which enables us to avoid any usage of special target coordinates that has been essential in Hofer-Wyosocki-Zehnder's approach, when we take the covariant tensorial approach using the contact triad connection.

We will derive in (4) that ζ satisfies

$$\nabla_\tau^\pi \zeta + J\nabla_t^\pi + S\zeta = 0 \tag{1}$$

where S is the zero operator

$$S\zeta = \frac{1}{2} w^*\lambda(\partial_\tau)(\mathscr{L}_{R_\lambda} J)\zeta + \frac{1}{2} w^*\lambda(\partial_t)(\mathscr{L}_{R_\lambda} J)J\zeta.$$

We write the loop $w_\tau := w(\tau,\cdot)$ and

$$A^\tau = (-J\nabla_t^\pi - S)|_{w_\tau^*\xi}$$

which we know converges to the asymptotic operator $A^\pi_{(\lambda,J,\nabla)}$. We define

$$\nu_\tau := \frac{\zeta_\tau}{\|\zeta_\tau\|_{L^2}}$$

so that it has unit L^2-norm on S^1 for all $\tau \geq \tau_0$. With this definition, one can following essentially verbatim the proof of [HWZ01, Theorem 2.8] using the tensorial way considering parallel transport of sections of $w^*\xi$ with respect to the canonical connection.

Then the following theorem describes the asymptotic behavior of w as $|\tau| \to \infty$, which is the analog to [HWZ96b, Theorem 1.4]. Our proof is the covariant tensorial version of the Hofer-Wysocki-Zehnder's proof of [HWZ01, Theorem 2.8] given in Section 3 therein via coordinate calculations in 3 dimension.

Theorem 8.2 (Asymptotic behavior)**.** *Assume that (γ,T) is nondegenerate. Consider the unit vectors.*

$$\nu(\tau,t) := \frac{\zeta(\tau,t)}{\|\zeta(\tau)\|_{L^2(w_\tau^*\xi)}}, \quad \alpha(\tau) := \frac{d}{d\tau}\log\|\zeta(\tau)\|_{L^2(w_\tau^*\xi)}.$$

Then we have the following:

1. *Either $\zeta(\tau,t) = 0$ for all $(\tau,t) \in [\tau_0,\infty)$ or*
2. *the following hold:*

   a. *There exists an eigenvector e of $A^\pi_{(\lambda,J,\nabla)}(\gamma_T)$ of eigenvalue μ such that $\nu_\tau \to e$ as $\tau \to \infty$ and*

$$\zeta(\tau,t) = e^{\int_{\tau_0}^\tau \alpha(s)\,ds}(e(t) + \widetilde{r}(\tau,t))$$

where $\widetilde{r}(\tau,t) \to 0$ *in* C^∞ *topology.*

b. *For any* $0 < r < \mu$*, there exist constants* $\delta > 0$*,* $\tau_0 > 0$ *and* C_β *such that for all multi-indices* $\beta = (\beta_1, \beta_2)) \in \mathbb{N}^2$ *such that*

$$\sup_{(\tau,t)} |(\nabla^\beta \zeta)(\tau,t)| \leq C_\beta e^{-r\tau}$$

for all $\tau \geq \tau_0$ *and* $t \in S^1$*. The same exponential decay also holds for the function* $\widetilde{r}$ *appearing in Statement (a) above.*

The exponential convergence here appearing can be derived by a boot-strap argument using the exponential convergence result on ζ as $\tau \to \infty$ given in Section 17. (See [OW18b], [OY22] for the details of this exponential convergence and the boot-strap argument.)

Note that the above representation formula of ζ implies the following convergence of the tangent plane.

Corollary 8.3 (Convergence of tangent plane)**.** *Assume the second alternative in Theorem 8.2 and denote*

$$P(\tau,t) := \mathrm{Image}\, dw(\tau,t) \in \mathrm{Gr}_2(\xi_{w(\tau,t)})$$

where $\mathrm{Gr}_2(\xi_x)$ *is the set of 2 dimensional subspaces of the contact hyperplane* $\xi_x \subset T_x Q$*. Then* $P(\tau,t) \to \mathrm{span}_{\mathbb{R}}\{e(t), Je(t) + TR_\lambda(\gamma(t))\}$ *exponentially fast in* C^∞ *topology uniformly in* $t \in S^1$*.*

This convergence statement is a trivial vacuous statement in dimension 3 since $\dim \xi = 2$ but is a nontrivial statement for higher dimensions. We refer readers to [Sie08, Sie11] for a precise local intersection theory of two pseudoholomorphic curves at the punctures and topological controls on the intersection number of two curves with the same 'fine' asymptotic limit.

Finally we would like to just mention that the same asymptotic study can be made in a straightforward way by incorporating the boundary condition by now as done in [OY22], [Ohb], [Oh22b].

This article is largely a survey of the first author and his collaborators' series of works on the analysis of contact instantons, focusing mainly on the *unperturbed ones* [OW14], [OW18a, OW18b], [Oha] and [Oh21a].

This survey does not touch upon the case of *Hamiltonian-perturbed contact instantons* studied in [Oh21b] for the elliptic regularity theory generalizing that of [Oh21a, OY22] in which does lie the real power of our approach through the interplay between geometric analysis of perturbed contact instantons and the calculus of Hamiltonian geometry and dynamics. We refer readers to [Oh21c], [Oh22a] in which the interplay has been exhibited by the proof of Sandon-Shelukhin type conjectures.

The case of unperturbed contact instantons corresponds to the case of Gromov's original pseudoholomorphic curves [Gro85] while the perturbed ones correspond to solutions of Floer's Hamiltonian-perturbed trajectory equations [Flo89, SZ92] in symplectic geometry. We refer interested readers to [Oh21b] for the case of perturbed equation. We also refer readers to [OW18b] for the tensorial proof of exponential convergence result in the Morse-Bott nondegenerate case of contact instantons again utilizing Weitezenböck-type formulae with respect to contact triad connections.

Throughout the paper, we freely use the (covariant) exterior calculus of E-valued differential forms and Weitzenböck formula with respect to the contact triad connection ∇. For the convenience of the prospective readers of this survey article, we duplicate [OW18a, Appendix A & B] here which summarizes the exterior calculus of vector-valued forms and the derivation of Weitzenböck formula in Appendix of the present paper.

There is one exception of the usage of contact triad connection: This is for the study of asymptotic operators along the Reeb orbits for the closed case (or along the Reeb chords in the open string case) where the derivation of asymptotic operators is done using the triad connection but converted the formula in the final conclusion to one involving the Levi-Civita connection of the triad metric *along the Reeb orbits (or along the Reeb chords)* because of our desire to more widely advertize the wonderful property of the Levi-Civita connection given in Proposition 0.4.

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Conventions: All the conventions regarding the definition of Hamiltonian vector fields, canonical symplectic forms on the cotangent bundle and definition of contact Hamiltonians and others are the same as those adopted and listed in [Oh21a, Conventions]. These also coincide with the conventions used in [dLLV19].

Part II
Contact triads and their $\mathfrak{lcs}$-fications

9 Contact triad connection and canonical connection

Let (Q,ξ) be a given contact manifold. When a contact form λ is given, we have the projection $\pi = \pi_\lambda$ from TQ to ξ associated to the decomposition

$$TQ = \xi \oplus \mathbb{R}\langle R_\lambda\rangle.$$

We denote by $\Pi = \Pi_\lambda : TQ \to TQ$ the corresponding idempotent, i.e., the endomorphism of TQ satisfying $\Pi^2 = \Pi$, $\operatorname{Im}\Pi = \xi$, $\ker\Pi = \mathbb{R}\{R_\lambda\}$.

9.1 Contact triads and triad connections

Let (Q,λ,J) be a contact triad of dimension $2n+1$ for the contact manifold (Q,ξ), and equip with it the contact triad metric $g = g_\xi + \lambda\otimes\lambda$. In [OW14], Wang and the first author introduced the *contact triad connection* associated to every contact triad (Q,λ,J) with the contact triad metric and proved its existence and uniqueness and naturality.

Theorem 9.1 (Contact Triad Connection [OW14])**.** *For every contact triad (Q,λ,J), there exists a unique affine connection ∇, called the contact triad connection, satisfying the following properties:*

1. *The connection ∇ is metric with respect to the contact triad metric, i.e., $\nabla g = 0$;*
2. *The torsion tensor T of ∇ satisfies $T(R_\lambda,\cdot) = 0$;*
3. *The covariant derivatives satisfy $\nabla_{R_\lambda} R_\lambda = 0$, and $\nabla_Y R_\lambda \in \xi$ for any $Y \in \xi$;*
4. *The projection $\nabla^\pi := \pi\nabla|_\xi$ defines a Hermitian connection of the vector bundle $\xi \to Q$ with Hermitian structure $(d\lambda|_\xi, J)$;*
5. *The ξ-projection of the torsion T, denoted by $T^\pi := \pi T$ satisfies the following property:*
$$T^\pi(Y,JY) = 0 \tag{1}$$
for all X tangent to ξ;
6. *For $X \in \xi$, we have the following*
$$\partial^\nabla_Y R_\lambda := \frac{1}{2}(\nabla_Y R_\lambda - J\nabla_{JY} R_\lambda) = 0.$$

From this theorem, we see that the contact triad connection ∇ canonically induces a Hermitian connection ∇^π for the Hermitian vector bundle (ξ, J, g_ξ), and we call it the *contact Hermitian connection*. This connection will be used to study estimates for the π-energy in later sections.

Moreover, the following fundamental properties of the contact triad connection was proved in [OW14]

Corollary 9.2 (Naturality)**.** *1. Let ∇ be the contact triad connection of the triad (Q,λ,J). Then for any diffeomorphism $\phi : Q \to Q$, the pull-back connection $\phi^*\nabla$*

is the triad connection associated to the triad $(Q, \phi^*\lambda, \phi^*J)$ associated to the pull-back contact structure $\phi^*\xi$.

2. *In particular if ϕ is contact, i.e., $d\phi(\xi) \subset \xi$, then $(Q, \phi^*\lambda, \phi^*J)$ is a contact triad of ξ and $\phi^*\nabla$ the contact triad connection (Q, ξ).*

The following identities are also very useful to perform tensorial calculations in the study of a priori elliptic estimates and in the derivation of the linearization formula.

Corollary 9.3. *Let ∇ be the contact triad connection. Then*

1. *For any vector field X on Q,*

$$\nabla_X R_\lambda = \frac{1}{2}(\mathscr{L}_{R_\lambda} J)JX; \tag{2}$$

2. *For the Reeb component of the torsion T, we have $\lambda(T) = d\lambda$.*

We refer readers to [OW14] for more discussion on the contact triad connection and its relation with other related canonical type connections.

9.2 Canonical connection on almost Hermitian manifold

In this subsection, we give the definition of canonical connection on general almost Hermitian manifolds and apply it to the case of $\mathfrak{lcs}$-fication of contact triad (Q, λ, J). See [Oh15, Chapter 7] for the exposition of canonical connection for almost Hermitian manifolds and its relationship with the Levi-Civita connection, and in relation to the study of pseudoholomorphic curves on general symplectic manifolds emphasizing the Weitzenböck formulae in the study of elliptic regularity in the same spirit of the present survey.

Let (M, J) be any almost complex manifold.

Definition 9.4. A metric g on (M, J) is called Hermitian, if g satisfies

$$g(Ju, Jv) = g(u, v), \quad u, v \in T_xM, x \in M.$$

We call the triple (M, J, g) an almost Hermitian manifold.

For any given almost Hermitian manifold (M, J, g), the bilinear form

$$\Phi := g(J\cdot, \cdot)$$

is called the fundamental two-form in [KN96], which is nondegenerate.

Definition 9.5. An almost Hermitian manifold (M, J, g) is an *almost Kähler manifold* if the two-form Φ above is closed.

Definition 9.6. A (almost) Hermitian connection ∇ is an affine connection satisfying

$$\nabla g = 0 = \nabla J.$$

Existence of such a connection is easy to check. In general the torsion $T = T_\nabla$ of the almost Hermitian connection ∇ is not zero, even when J is integrable. The following is the almost complex version of the Chern connection in complex geometry.

Theorem 9.7 ([Gau97], [Kob03])**.** *On any almost Hermitian manifold* (M,J,g)*, there exists a unique Hermitian connection* ∇ *on* TM *satisfying*

$$T(X,JX) = 0 \tag{3}$$

for all $X \in TM$.

In complex geometry [Che67] where J is integrable, a Hermitian connection satisfying (3) is called the Chern connection.

Definition 9.8. A *canonical connection* of an almost Hermitian connection is defined to be one that has the torsion property (3).

The triple (4)

$$(Q\times\mathbb{R},\widetilde{J},\widetilde{g}_\lambda)$$

is a natural example of an almost Hermitian manifold associated to the contact triad (Q,λ,J).

Let $\widetilde{\nabla}$ be the canonical connection thereof. Then we have the following which also provides a natural relationship between the contact triad connection and the canonical connection.

Proposition 9.9 (Canonical connection versus contact triad connection)**.** *Let* $\widetilde{g} = \widetilde{g}_\lambda$ *be the almost Hermitan metric given above. Let* $\widetilde{\nabla}$ *be the canonical connection of the almost Hermitian manifold* (4)*, and* ∇ *be the contact triad connection for the triad* (Q,λ,J)*. Then* $\widetilde{\nabla}$ *preserves the splitting* (3) *and satisfies* $\widetilde{\nabla}|_\xi = \nabla|_\xi$.

Remark 9.10. 1. In fact it was shown in [OW14] that for each real constant c, there is the unique connection that satisfies all properties (1)–(5) and (6) replaced by (6;c)

$$\nabla_{JY}R_\lambda + J\nabla_Y R_\lambda = cY, \text{or equivalently } \partial^\nabla_Y R_\lambda = \frac{c}{2}Y \tag{4}$$

for all $Y \in \xi$. Our canonical connection corresponds to $c = 0$. In particular all of these connections, temporarily denoted by ∇^c and called c-triad connection, induce the same Hermitian connection on ξ, i.e.,

$$\nabla^{c;\pi} = \nabla^\pi$$

for all c. With this choice of connection ∇^c, (2) is replaced by

$$\nabla^c_Y R_\lambda = -\frac{c}{2}JY + \frac{1}{2}(\mathscr{L}_{R_\lambda}J)JY.$$

It seems to us that this constant c is related to the way how one lifts triad connection to a canonical connection on the symplectization. Recall if one starts from a Liouville manifold M with cylindrical end, only the contact structure ξ is naturally induced from the Liouville structure, but not the contact form, on its ideal boundary $\partial_\infty M$.

2. We suspect that Proposition 9.9 will be important to make the Kuranishi structure constructed in [KO] compatible with Kuranishi structure on the moduli spaces of pseudoholomorphic curves on noncompact symplectic manifolds M of contact-type boundary such as on Liouville manifolds. In this way, we conjecture existence of an A_∞-type functor

$$D^\pi \mathscr{W}(M)/D^\pi \mathscr{F}(M) \to D^\pi \mathfrak{Leg}(\partial_\infty M):$$

Here the codomain $D^\pi \mathfrak{Leg}(Q,\xi)$ is the derived category of the Fukaya-type category $\mathfrak{Leg}(Q) = \mathfrak{Leg}(Q,\xi)$ generated by Legendrian submanifolds that will be constructed in [KO] and [Ohc]. For the domain, $D^\pi \mathscr{W}(M)$ and $D^\pi \mathscr{F}(M)$ are the derived wrapped Fukaya category and the derived compact Fukaya category of the Liouville manifold M respectively. See [BJK] for the description of the quotient category associated to a pluming space in terms of a cluster category associated. According thereto, Ganatra-Gao-Venkatesh relate this quotient to a derived Rabinowitz Fukaya category.

10 Contact instantons and pseudoholomorphic curves on symplectization

Denote by $(\dot\Sigma, j)$ a punctured Riemann surface (including the case of closed Riemann surfaces without punctures).

10.1 Analysis of pseudoholomorphic curves on symplectization in the literature

We would like to make it clear that the analytical results themselves we describe on the symplectization in the present article are mostly known and established by Hofer-Wysocki-Zehnder's in a series of papers in [HWZ96a] - [HWZ99] in 3 dimension and some in higher dimensions by Bourgeois [Bou02] and Siefring [Sie08, Sie11]. Wendl's book manuscript [Wen] also describes the analysis in general dimension and Cant's thesis [Can22] explains the relative case in general dimension.

To highlight the main differences between the above and Wang and the first author's analysis of contact instantons [OW18a, OW18b], [Oh21a], we start here with quoting a few sample statements of the main results from [HWZ96a]-[HWZ99], which are propagated to other later literature.

The following is the prototype of the statements made in [HWZ96a] - [HWZ99] on the description of finer asymptotics of pseudoholomorphic curves near punctures, and repeated in other later literature on the pseudoholomorphic curves on symplectization.

10.1.1 Choice of special coordinates

Let γ be a closed Reeb orbit with action $T \neq 0$.

Hofer-Wysocki-Zehnder first take special coordinates (a, ϑ, z) in an S^1*-invariant neighborhood* $W \subset Q$ of the given Reeb orbit $x(T\cdot) \in C^\infty(S^1, Q)$ in such a way that

1. Image $v \subset W$ for all s large enough,
2. They choose the coordinates $(\theta, x, y) \in S^1 \times \mathbb{R}^2$ by considering contact diffeomorphism onto its image contained in $(S^1 \times \mathbb{R}^2, f \cdot \lambda_0)$ to (Q, λ) where
   - the periodic solution corresponds to $S^1 \times \{0\}$,
   - f a positive function,
   - $f \cdot \lambda_0$ is a contact form with $\lambda_0 = d\theta + x\,dy$ the standard contact form on $S^1 \times \mathbb{R}^2$.
   - They lift the map v to $\widetilde{v}$ defined on the universal covering space $[s_0, \infty) \times \mathbb{R}$ of $[s_0, \infty) \times S^1$.

With these preparations, the authors therefrom start with a finite energy cylinder $\widetilde{v} : \mathbb{R} \times S^1 \to Q \times \mathbb{R}$ with $\widetilde{v} = (v, a)$ for a given contact manifold (Q, λ), and derive the following estimates.

Theorem 10.1 (Theorem 2.8 [HWZ96b], Asymptotic behavior of nondegenerate finite energy planes)**.** *Assume the functions* $(a, u); [s_0, \infty) \times \mathbb{R} \to \mathbb{R}^4$ *meet the above conditions. Then either*

1. There exists $c \in \mathbb{R}$ *such that*

$$(a(s,t), \vartheta(s,t), z(s,t)) = (Ts + c, kt, 0)$$

or

2. There are constants $c \in \mathbb{R}$, $d > 0$ *and* $M_\alpha > 0$ *for all* $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N} \times \mathbb{N}$ *such that*

$$|\partial^\alpha [a(s,t) - Ts - c]| \leq M_\alpha e^{-d\cdot s}$$
$$|\partial^\alpha [\vartheta(s,t) - kt]| \leq M_\alpha e^{-d\cdot s}$$

for all $s \geq s_0$, $t \in \mathbb{R}$*. Moreover*

$$z(s,t) = e^{\int_{s_0}^{s} \gamma(\tau)\, d\tau} [e(t) + r(s,t)].$$

Here $e \neq 0$ is an eigenvector of the self-adjoint operator A_∞ corresponding to a negative eigenvalue $\lambda < 0$ and $\gamma : [s_0, \infty) \to \mathbb{R}$ is a smooth function satisfying $\gamma(s) \to \lambda$ as $\lambda \to \infty$. In particular $e(t) \neq 0$ pointwise and the remainder $r(s,t)$ satisfies $\partial^\alpha r(s,t) \to 0$ for all derivatives $\alpha = (\alpha_1, \alpha_2)$, uniformly in $t \in \mathbb{R}$.

10.1.2 Local adjustment of J along the Reeb orbits

Next, they put some constraints on the almost complex structure in $W \times \mathbb{R} \subset Q \times \mathbb{R}$ which has troubled the first author much: Along the way, they *change the given almost complex structure* on W *in the way that depends on the given Reeb orbit* as follows. Consider the symplectic inner product $d\lambda$ on the $S^1 \times \mathbb{R}^2$ family of 2-dimensional contact plane $\xi_m = \ker \lambda_m$ given by

$$\xi_m \subset \mathbb{R}^3 \cong T_m(S^1 \times \mathbb{R}^2)$$

where $\xi_m = \operatorname{span}\langle e_1, e_2 \rangle$ with

$$e_1 = (0,1,0), \quad e_2 = (-x_1, 0, 1).$$

Then the 2×2 matrix $\Omega = \Omega(\theta, x, y)$ associated to $d\lambda$ on ξ_m is given by

$$\Omega = f J_0, \quad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where J_0 is the standard complex structure on $\mathbb{R}^2$. Then the almost complex structure $j_m : \xi_m \to \xi_m$ is the pull-back of a standard λ-adapted CR almost complex structure on Q restricted to its contact distributions. Then they put a 'very special' almost complex structure [HWZ96b, In the middle of p.351] denoted by $\widetilde{J}$.

10.1.3 Weak points of the aforementioned coordinate approach

This usage of special coordinates and adjustment on J along the Reeb orbits *depending on the Reeb orbits* prevents one from enabling to study the J-dependence on the generic properties of asymptotic behaviors of pseudoholomorphic curves, e.g., about the generic properties of eigenvalues of the asymptotic operators in terms of the choice of J. In this regard, Siefring [Sie08, Sie11] studied an operator that he calls the asymptotic operator in the coordinate-free way but did not develop its spectral perturbation theory under the change of CR almost complex structures J.

The first author's attempt to find a more canonical way of doing the aforementioned asymptotic study of Reeb orbits, especially that of doing those practiced in the higher dimensional Morse-Bott case given in [Bou02], had motivated Wang and him to pursue the current tensorial approach at the time of writing [OW14, OW18a, OW18b] in the beginning of 2010's. Furthermore our tensorial approach also helps the understanding of the background geometry of contact triads

(Q,λ,J) and the symplectization irrespective of the analytic study of pseudoholomorphic curves as described in Section 9.

10.2 Contact Cauchy-Riemann maps

The following definition is introduced in [OW18a].

Definition 10.2 (Contact Cauchy-Riemann map)**.** A smooth map $w : \dot\Sigma \to Q$ is called a *contact Cauchy-Riemann map* (with respect to the contact triad (Q,λ,J), if w satisfies the following Cauchy–Riemann equation

$$\overline{\partial}^\pi w := \overline{\partial}^\pi_{j,J} w := \frac{1}{2}(\pi dw + J\pi dw \circ j) = 0.$$

Recall that for a fixed smooth map $w : \dot\Sigma \to Q$, the triple

$$(w^*\xi, w^*J, w^*g_\xi)$$

becomes a Hermitian vector bundle over the punctured Riemann surface $\dot\Sigma$. This introduces a Hermitian bundle structure on $Hom(T\dot\Sigma, w^*\xi) \cong T^*\dot\Sigma \otimes w^*\xi$ over $\dot\Sigma$, with inner product given by

$$\langle \alpha \otimes \zeta, \beta \otimes \eta \rangle = h(\alpha,\beta) g_\xi(\zeta,\eta),$$

where $\alpha,\beta \in \Omega^1(\dot\Sigma)$, $\zeta,\eta \in \Gamma(w^*\xi)$, and h is the Kähler metric on the punctured Riemann surface $(\dot\Sigma, j)$.

Let ∇^π be the contact Hermitian connection. Combining the pulling-back of this connection and the Levi–Civita connection of the Riemann surface, we get a Hermitian connection for the bundle $T^*\dot\Sigma \otimes w^*\xi \to \dot\Sigma$, which we still denote by ∇^π by a slight abuse of notation. *This is the setting where we apply the harmonic theory and Weitzenböck formulae to study the global a priori $W^{1,2}$-estimate of $d^\pi w$ mentioned in the introduction of the present survey:* The smooth map w has an associated π-harmonic energy density, the function $e^\pi(w) : \dot\Sigma \to \mathbb{R}$, defined by

$$e^\pi(w)(z) := |d^\pi w|^2(z).$$

(Here we use $|\cdot|$ to denote the norm from $\langle\cdot,\cdot\rangle$ which should be clear from the context.)

Similar to standard Cauchy–Riemann maps for almost Hermitian manifolds (i.e., pseudo-holomorphic curves), we have the following whose proofs are straightforward and so omitted. (See [OW18a, Lemma 3.2] for the proofs.)

Lemma 10.3. *Fix a Kähler metric h on $(\dot\Sigma, j)$, and consider a smooth map $w : \dot\Sigma \to Q$. Then we have the following equations*

1. $e^\pi(w) := |d^\pi w|^2 = |\partial^\pi w|^2 + |\overline{\partial}^\pi w|^2$*;*

2. $2w^*d\lambda = (-|\overline{\partial}^\pi w|^2 + |\partial^\pi w|^2)\,dA$ *where* dA *is the area form of the metric* h *on* $\dot\Sigma$*;*
3. $w^*\lambda \wedge w^*\lambda \circ j = -|w^*\lambda|^2\,dA$.

As a consequence, if w *is a contact Cauchy-Riemann map, i.e.,* $\overline{\partial}^\pi w = 0$*, then*

$$|d^\pi w|^2 = |\partial^\pi w|^2 \quad \textit{and} \quad w^*d\lambda = \frac{1}{2}|d^\pi w|^2\,dA. \tag{1}$$

However the contact Cauchy-Riemann equation itself $\overline{\partial}^\pi w = 0$ does not form an elliptic system because it is degenerate along the Reeb direction: Note that the rank of w^*TQ has $2n+1$ while that of $w^*\xi \otimes \Lambda^{0,1}(\Sigma)$ is $2n$. Therefore to develop suitable deformation theory and a priori estimates, one needs to lift the equation to an elliptic system by incorporating the data of the Reeb direction. In hindsight, the pseudoholomorphic curve system of the pair (a,w) is one of many possible such liftings by introducing an auxiliary variable a, *when the one-form* $w^*\lambda \circ j$ *is exact.* Hofer [Hof93] did this by lifting the equation to the symplectization $Q \times \mathbb{R}$ and considering the pull-back function $f := s \circ w$ of the $\mathbb{R}$-coordinate function s of $Q \times \mathbb{R}$. By doing so, he *added one more variable* to the equation $\overline{\partial}^\pi w = 0$ while *adding 2 more equations* $w^*\lambda \circ j = df$ which becomes Gromov's pseudoholomorphic curve system on the product $Q \times \mathbb{R}$.

We now introduce two other possible elliptic liftings of Cauchy-Riemann maps.

10.3 Gauged sigma model lifting of contact Cauchy-Riemann map

We first recall that any contact manifold (Q,ξ), whether it is coorientable or not, carries a natural real line bundle $TQ/\xi \to Q$. This bundle is trivial when ξ is coorientable. By complexifying the line bundle, we look for a lifting of the Cauchy-Riemann map equation by coupling a section of complex line bundle over Q

$$\mathscr{L}_\lambda \to Q \tag{2}$$

whose fiber at $q \in Q$ is given by

$$\mathscr{L}_{\lambda,q} = \mathbb{R}_{\lambda,q} \otimes \mathbb{C}$$

where $\mathbb{R}_\lambda \to Q$ is the trivial real line bundle whose fiber at q is given by

$$\mathbb{R}_{\lambda,q} = \mathbb{R}\{R_\lambda(q)\}.$$

Now let $w : \Sigma \to Q$ be a smooth map where $\dot\Sigma$ is either closed or a punctured Riemann surface, and χ be a section of the pull-back bundle $w^*\mathscr{L}_\lambda$.

Definition 10.4. We call a triple (w, j, χ) consisting of a complex structure j on Σ, $w : \Sigma \to Q$ and a $\mathbb{C}$-valued one-form χ a *gauged contact instanton* if they satisfy

$$\begin{cases} \overline{\partial}^\pi w = 0 \\ \overline{\partial}\chi = 0, \quad \operatorname{Im}\chi = w^*\lambda. \end{cases} \tag{3}$$

This system is a coupled system of the contact Cauchy-Riemann map equation and the well-known Riemann-Hilbert problem of the type which solves the real part in terms of the imaginary part of holomorphic functions in complex variable theory.

10.4 Contact instanton lifting of contact Cauchy-Riemann map

By augmenting the closedness condition $d(w^*\lambda \circ j) = 0$ to Contact Cauchy-Riemann map equation $\overline{\partial}^\pi w = 0$, we arrive at an elliptic system (4) which is of our main interest in the present survey. The current contact instanton map system

$$\overline{\partial}^\pi w = 0, \quad d(w^*\lambda \circ j) = 0 \tag{4}$$

is another such an elliptic lifting which is more natural in some respect in that it does not introduce any additional variable and keeps the original 'bulk', the contact manifold Q.

To illustrate the effect of the closedness condition on the behavior of contact instantons, we look at them on *closed* Riemann surface and prove the following classification result. The following proposition is stated by Abbas as a part of [Abb11, Proposition 1.4]. We refer readers [OW18a, Proposition 3.4] for another proof which is somewhat different from Abbas' in [Abb11].

Proposition 10.5 (Proposition 1.4, [Abb11])**.** *Assume $w : \Sigma \to Q$ is a smooth contact instanton from a closed Riemann surface. Then*

1. *If $g(\Sigma) = 0$, w is a constant map;*
2. *If $g(\Sigma) \geq 1$, w is either a constant or the locus of its image is a* closed *Reeb orbit.*

Part III
A priori estimates

11 Weitzenböck formulae

In this section, we use the contact triad connection, first to derive Weitzenböck-type formula associated to the π-harmonic energy *for contact Cauchy–Riemann maps*, and then to derive another formula associated to the full harmonic energy density *for the contact instantons*. The contact triad connection fits well for this purpose which will be seen clearly in this section.

11.1 Weitzenböck formulae for contact Cauchy-Riemann maps

We start with by looking at the (Hodge) Laplacian of the π-harmonic energy density of an arbitrary smooth map $w : \dot\Sigma \to M$, i.e., in the *off-shell level* in physics terminology. As the first step, we apply the standard Weitzenböck formula to the connection ∇^π on $T^*\dot\Sigma \otimes w^*\xi$ that is induced by the pull-back connection on bundle $w^*\xi$ and the Levi–Civita connection on $T\dot\Sigma$, and obtain the following formula

$$-\frac{1}{2}\Delta e^\pi(w) = |\nabla^\pi(d^\pi w)|^2 - \langle \Delta^{\nabla^\pi} d^\pi w, d^\pi w\rangle + K\cdot |d^\pi w|^2 + \langle \mathrm{Ric}^{\nabla^\pi}(d^\pi w), d^\pi w\rangle. \tag{1}$$

Here $e^\pi(w) = |d^\pi w|^2$ is the π-harmonic energy density, K the Gaussian curvature of the Kähler manifold $(\dot\Sigma, h)$, and $\mathrm{Ric}^{\nabla^\pi}$ is the Ricci tensor of the connection ∇^π on the vector bundle $w^*\xi$. (We refer to [OW18a, Appendix A] for the he proof of (1).)

The following fundamental identity is derived in [OW18a, Lemma 4.1] to which we refer readers for its derivation. This is an analog to a similar formula [Oh15, Lemma 7.3.2] in the symplectic context.

Lemma 11.1. *Let $w : \dot\Sigma \to M$ be any smooth map. Denote by T^π the torsion tensor of ∇^π. Then as a two form with values in $w^*\xi$, $d^{\nabla^\pi}(d^\pi w)$ has the expression*

$$d^{\nabla^\pi}(d^\pi w) = T^\pi(\Pi dw, \Pi dw) + w^*\lambda \wedge \left(\frac{1}{2}(\mathscr{L}_{R_\lambda} J) J d^\pi w\right). \tag{2}$$

We now restrict the above lemma to the case of contact Cauchy–Riemann map, i.e., maps satisfying $\overline\partial^\pi w = 0$. In this case, by the property $T(Y, JY) = 0$ of the torsion T of the contact triad connection, we derive the following formula as an immediate corollary of the previous lemma.

Theorem 11.2 (Fundamental Equation; Theorem 4.2 [OW18a]). *Let w be a contact Cauchy–Riemann map, i.e., a solution of $\overline\partial^\pi w = 0$. Then*

$$d^{\nabla^\pi}(d^\pi w) = -w^*\lambda \circ j \wedge \left(\frac{1}{2}(\mathscr{L}_{R_\lambda} J) d^\pi w\right). \tag{3}$$

The following elegant expression of Fundamental Equation in any *isothermal coordinates* (x,y), i.e., one such that $z = x+iy$ provides a complex coordinate of $(\dot\Sigma, j)$ such that $h = dx^2+dy^2$, will be useful for the study of higher a priori $C^{k,\alpha}$ Hölder estimates.

Corollary 11.3 (Fundamental Equation in Isothermal Coordinates)**.** *Let (x,y) be an isothermal coordinates. Write $\zeta := \pi\frac{\partial w}{\partial\tau}$ as a section of $w^*\xi \to Q$. Then*

$$\nabla^\pi_x\zeta + J\nabla^\pi_y\zeta - \frac{1}{2}\lambda\left(\frac{\partial w}{\partial x}\right)(\mathscr{L}_{R_\lambda}J)\zeta + \frac{1}{2}\lambda\left(\frac{\partial w}{\partial y}\right)(\mathscr{L}_{R_\lambda}J)J\zeta = 0. \tag{4}$$

Proof. We denote $\pi\frac{\partial w}{\partial x}$ by ζ and $\pi\frac{\partial w}{\partial y}$ by η. By the isothermality of the coordinate (x,y), we have $j\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$. Using the (j,J)-linearity of $d^\pi w$, we derive

$$\eta = dw^\pi\left(\frac{\partial}{\partial y}\right) = dw^\pi\left(j\frac{\partial}{\partial x}\right) = Jdw^\pi\left(\frac{\partial}{\partial x}\right) = J\zeta.$$

Now we evaluate each side of (3) against $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. For the left hand side, we get

$$\nabla^\pi_x\eta - \nabla^\pi_y\zeta = \nabla^\pi_x J\zeta - \nabla^\pi_y\zeta = J\nabla^\pi_x\zeta - \nabla^\pi_y\zeta.$$

For the right hand side, we get

$$\begin{aligned}&\frac{1}{2}\lambda\left(\frac{\partial w}{\partial x}\right)(\mathscr{L}_{R_\lambda}J)J\eta - \frac{1}{2}\lambda\left(\frac{\partial w}{\partial y}\right)(\mathscr{L}_{R_\lambda}J)J\zeta\\ &= -\frac{1}{2}\lambda\left(\frac{\partial w}{\partial x}\right)(\mathscr{L}_{R_\lambda}J)\zeta - \frac{1}{2}\lambda\left(\frac{\partial w}{\partial y}\right)(\mathscr{L}_{R_\lambda}J)J\zeta\end{aligned}$$

where we use the equation $\eta = J\zeta$ for the equality. By setting them equal and applying J to the resulting equation using the fact that $\mathscr{L}_{R_\lambda}J$ anti-commutes with J, we obtain the equation. □

Remark 11.4. The fundamental equation in cylindrical (or strip-like) coordinates is nothing but the linearization equation of the contact Cauchy-Riemann equation in the direction $\frac{\partial}{\partial\tau}$. This plays an important role in the derivations of the exponential decay of the derivatives at cylindrical ends and of finer asymptotic study of the asymptotic operators. (See [OW18b, Part II].)

Now we can convert the general Weitzenböck formula (1) into the following in our case.

Proposition 11.5 (Equation (4.11) [OW18a])**.** *Let w be a contact Cauchy-Riemann map. Then*

$$\begin{aligned}-\frac{1}{2}\Delta e^\pi(w) &= |\nabla^\pi(\partial^\pi w)|^2 + K|\partial^\pi w|^2 + \langle \mathrm{Ric}^{\nabla^\pi}(\partial^\pi w), \partial^\pi w\rangle\\ &\quad + \langle \delta^{\nabla^\pi}[(w^*\lambda\circ j)\wedge(\mathscr{L}_{R_\lambda}J)\partial^\pi w], \partial^\pi w\rangle. \end{aligned}\tag{5}$$

The upshot of the equation is that the *a priori the third derivatives involving the LHS for general smooth maps can be written in terms of those involving at most the second derivatives for the contact Cauchy-Riemann maps*. This enables us to perform the bootstrap arguments to obtain the higher regularity results.

Outline of the proof. The following formula expresses $\Delta^{\nabla^\pi} d^\pi w$, which involves the third derivatives of w, in terms of the terms involving derivatives of order at most two. Here Δ^{∇^π} is the covariant harmonic Laplacian

$$\Delta^{\nabla^\pi} = \delta^{\nabla^\pi} d^{\nabla^\pi} + d^{\nabla^\pi} \delta^{\nabla^\pi}.$$

Lemma 11.6. *For any contact Cauchy–Riemann map w,*

$$\begin{aligned} -\Delta^{\nabla^\pi} d^\pi w &= \delta^{\nabla^\pi}[(w^*\lambda \circ j) \wedge (\mathscr{L}_{R_\lambda} J)\partial^\pi w] \\ &= - * \langle (\nabla^\pi(\mathscr{L}_{R_\lambda} J))\partial^\pi w, w^*\lambda\rangle \\ &\quad - * \langle (\mathscr{L}_{R_\lambda} J)\nabla^\pi \partial^\pi w, w^*\lambda\rangle - *\langle (\mathscr{L}_{R_\lambda} J)\partial^\pi w, \nabla w^*\lambda\rangle. \end{aligned}$$

Proof. The first equality immediately follows from the fundamental equation, Theorem 11.2, for contact Cauchy–Riemann maps. For the second equality, we calculate by writing

$$\delta^{\nabla^\pi}[(w^*\lambda \circ j) \wedge (\mathscr{L}_{R_\lambda} J)\partial^\pi w] = - * d^{\nabla^\pi} * [(\mathscr{L}_{R_\lambda} J)\partial^\pi w \wedge (*w^*\lambda)],$$

and then by applying the definition of the Hodge $*$ to the expression $*[(\mathscr{L}_{R_\lambda} J)\partial^\pi w \wedge (*w^*\lambda)]$, we further get

$$\begin{aligned} &\delta^{\nabla^\pi}[(w^*\lambda \circ j) \wedge (\mathscr{L}_{R_\lambda} J)\partial^\pi w] \\ &= - * d^{\nabla^\pi} \langle (\mathscr{L}_{R_\lambda} J)\partial^\pi w, w^*\lambda\rangle \\ &= - * \langle (\nabla^\pi(\mathscr{L}_{R_\lambda} J))\partial^\pi w, w^*\lambda\rangle - *\langle (\mathscr{L}_{R_\lambda} J)\nabla^\pi \partial^\pi w, w^*\lambda\rangle - *\langle (\mathscr{L}_{R_\lambda} J)\partial^\pi w, \nabla w^*\lambda\rangle. \end{aligned}$$

□

This leads us to the following formula

Corollary 11.7. *For any contact Cauchy-Riemann map w, we have*

$$\begin{aligned} -\langle \Delta^{\nabla^\pi} d^\pi w, d^\pi w\rangle &= \langle \delta^{\nabla^\pi}[(w^*\lambda \circ j) \wedge (\mathscr{L}_{R_\lambda} J) d^\pi w], d^\pi w\rangle \\ &= -\langle *\langle (\nabla^\pi(\mathscr{L}_{R_\lambda} J))\partial^\pi w, w^*\lambda\rangle, d^\pi w\rangle \\ &\quad -\langle *\langle (\mathscr{L}_{R_\lambda} J)\nabla^\pi \partial^\pi w, w^*\lambda\rangle, d^\pi w\rangle \\ &\quad -\langle *\langle (\mathscr{L}_{R_\lambda} J)\partial^\pi w, \nabla w^*\lambda\rangle, d^\pi w\rangle. \end{aligned}$$

Here in the above lemma $\langle \cdot, \cdot \rangle$ denotes the inner product induced from h, i.e., $\langle \alpha_1 \otimes \zeta, \alpha_2\rangle := h(\alpha_1, \alpha_2)\zeta$, for any $\alpha_1, \alpha_2 \in \Omega^k(P)$ and ζ a section of E. This inner product should not be confused with the inner product of the vector bundles.

By applying δ^{∇^π} to (2) and the resulting expression of $\Delta^{\nabla^\pi}(d^\pi w)$ thereinto, we can convert the Weitzenböck formula (1) to (5). □

11.2 The case of contact instantons

Now we consider contact instantons which are Cauchy–Riemann maps satisfying $d(w^*\lambda \circ j) = 0$ in addition.

Proposition 11.8. *Let w be a contact instanton. Then*

$$-\frac{1}{2}\Delta|w^*\lambda|^2 = |\nabla w^*\lambda|^2 + K|w^*\lambda|^2 + \langle *\langle \nabla^\pi \partial^\pi w, \partial^\pi w\rangle, w^*\lambda\rangle. \tag{6}$$

Proof. In this case again by using the Bochner–Weitzenböck formula (for forms on Riemann surface), we get the following inequality

$$-\frac{1}{2}\Delta|w^*\lambda|^2 = |\nabla w^*\lambda|^2 + K|w^*\lambda|^2 - \langle \Delta(w^*\lambda), w^*\lambda\rangle. \tag{7}$$

Write

$$\Delta(w^*\lambda) = d\delta(w^*\lambda) + \delta d(w^*\lambda),$$

in which the first term vanishes since w satisfies the contact instanton equation which includes $0 = d(w^*\lambda \circ j) = -\delta(w^*\lambda)$ in addition. Then a straightforward calculation gives rise to

$$\begin{aligned}
\langle \Delta(w^*\lambda), w^*\lambda\rangle &= \langle \delta d(w^*\lambda), w^*\lambda\rangle \\
&= -\frac{1}{2}\langle *d|\partial^\pi w|^2, w^*\lambda\rangle \\
&= -\langle *\langle \nabla^\pi \partial^\pi w, \partial^\pi w\rangle, w^*\lambda\rangle.
\end{aligned}$$

Substituting this into (7) we have finished the proof. □

12 A priori $W^{2,2}$-estimates for contact instantons

In this section, we derive basic estimates for the harmonic energy density $|dw|^2$ of contact instantons w. These estimates are important for the derivation of local regularity and the ε-regularity needed for the compactification of moduli spaces in general.

We first remark that the total harmonic energy density is decomposed into

$$e(w) := |dw|^2 = e^\pi(w) + |w^*\lambda|^2 :$$

This follows from the decomposition $dw = d^\pi w + w^*\lambda \otimes R_\lambda$ and the orthogonality of the two summands with respect to the triad metric and $|R_\lambda| \equiv 1$.

Therefore we will derive the Laplacian of each summand, $|d^\pi w|^2$ and $|w^*\lambda|^2$.

12.1 Computation of $\Delta|dw|^2$ and Weitzenböck formulae

Recall the formula (3) of the Laplacian $\Delta e^\pi(w)$ from the last section. The following estimate is proved in [OW18a, Equation (5.3)].

Lemma 12.1. *For any constant $c > 0$, we have*

$$\begin{aligned}
&|\langle \delta^{\nabla^\pi}[(w^*\lambda\circ j)\wedge(\mathscr{L}_{R_\lambda}J)\partial^\pi w], \partial^\pi w\rangle| \\
&\leq \frac{1}{2c}\left(|\nabla^\pi(\partial^\pi w)|^2 + |\nabla w^*\lambda|^2\right) + \left(c\|\mathscr{L}_{R_\lambda}J\|^2_{C^0(Q)} + \|\nabla^\pi(\mathscr{L}_{R_\lambda}J)\|_{C^0(Q)}\right)|dw|^4
\end{aligned} \tag{1}$$

This gives rise to the following differential inequality

$$\begin{aligned}
-\frac{1}{2}\Delta e^\pi(w) \leq & \left(1+\frac{1}{2c}|\nabla^\pi(d^\pi w)|^2+\frac{1}{2c}\right)+\frac{1}{2c}\left(|\nabla w^*\lambda|^2\right) \\
&+(\|K\|_{C^0(Q)}+\|\operatorname{Ric}^{\nabla^\pi}\|_{C^0(Q)})|d^\pi w|^2 \\
&+\left(c\|\mathscr{L}_{R_\lambda}J\|^2_{C^0(Q)}+\|\nabla^\pi(\mathscr{L}_{R_\lambda}J)\|_{C^0(Q)}\right)|dw|^4
\end{aligned} \tag{2}$$

Similarly as in the previous estimates for the Laplacian term of $\partial^\pi w$, we can bound

$$\begin{aligned}
|-\langle\Delta(w^*\lambda), w^*\lambda\rangle| &= |\langle *\langle\nabla^\pi\partial^\pi w, \partial^\pi w\rangle, w^*\lambda\rangle| \\
&\leq |\nabla^\pi\partial^\pi w||dw|^2 \\
&\leq \frac{1}{2c}|\nabla^\pi\partial^\pi w|^2+\frac{c}{2}|dw|^4.
\end{aligned} \tag{3}$$

We add (2) and (7), and then apply the estimates (1) and (3) respectively. This yields the following differential inequality for the total energy density

$$\begin{aligned}
&-\frac{1}{2}\Delta e(w) \\
&\geq \left(1-\frac{1}{c}\right)|\nabla^\pi(\partial^\pi w)|^2+\left(1-\frac{1}{2c}\right)|\nabla w^*\lambda|^2 \\
&\quad-\left(c\|\mathscr{L}_{R_\lambda}J\|^2_{C^0(Q)}+\|\nabla^\pi(\mathscr{L}_{R_\lambda}J)\|_{C^0(Q)}+\frac{c}{2}+\|\operatorname{Ric}\|_{C^0(Q)}\right)e(w)^2+Ke(w)
\end{aligned} \tag{4}$$

$$\geq -\left(c\|\mathscr{L}_{R_\lambda}J\|^2_{C^0(Q)}+\|\nabla^\pi(\mathscr{L}_{R_\lambda}J)\|_{C^0(Q)}+\frac{c}{2}+\|\operatorname{Ric}\|_{C^0(Q)}\right)e(w)^2+Ke(w),$$

for any $c > 1$. We fix $c = 2$ and get the following

Theorem 12.2 (Theorem 5.1 [OW18a])**.** *For a contact instanton w, we have the following total energy density estimate*

$$\Delta e(w) \leq Ce(w)^2+\|K\|_{L^\infty(\dot\Sigma)}e(w), \tag{5}$$

where

$$C = 2\|\mathscr{L}_{R_\lambda} J\|^2_{C^0(Q)} + \|\nabla^\pi(\mathscr{L}_{R_\lambda} J)\|_{C^0(Q)} + \|\operatorname{Ric}\|_{C^0(Q)} + 1$$

which is a positive constant independent of w.

An immediate corollary of (5) is the following density estimate which is derived by the standard argument from [Sch84]. (Also see the proof of [Oh15, Theorem 8.1.3] given in the context of pseudoholomorphic curves.) It is in turn a consequence of an application of the mean value inequality of Morrey (See [GT70, Problems 4.5 in p.67] for the relevant extension of the mean-value inequality for the Poisson equation.)

Corollary 12.3 (ε-regularity and interior density estimate; Corollary 5.2 [OW18a])**.** *There exist constants* C, ε_0 *and* $r_0 > 0$*, depending only on* J *and the Hermitian metric* h *on* $\dot\Sigma$*, such that for any* C^1 *contact instanton* $w : \dot\Sigma \to Q$ *with*

$$E(r_0) := \frac{1}{2}\int_{D(r_0)} |dw|^2 \leq \varepsilon_0,$$

and discs $D(2r) \subset \operatorname{Int}\Sigma$ *with* $0 < 2r \leq r_0$*,* w *satisfies*

$$\max_{\sigma\in(0,r]}\left(\sigma^2 \sup_{D(r-\sigma)} e(w)\right) \leq CE(r) \tag{6}$$

for all $0 < r \leq r_0$*. In particular, letting* $\sigma = r/2$*, we obtain*

$$\sup_{D(r/2)} |dw|^2 \leq \frac{4CE(r)}{r^2} \tag{7}$$

for all $r \leq r_0$.

Now we rewrite (4) into

$$\begin{aligned}
&\left(1-\frac{1}{c}\right)|\nabla^\pi(\partial^\pi w)|^2 + \left(1-\frac{1}{2c}\right)|\nabla w^*\lambda|^2 \\
\leq\; & -\frac{1}{2}\Delta e(w) - Ke(w) \\
&\quad + \left(c\|\mathscr{L}_{R_\lambda} J\|^2_{C^0(Q)} + \|\nabla^\pi(\mathscr{L}_{R_\lambda} J)\|_{C^0(Q)} + \frac{c}{2} + \|\operatorname{Ric}\|_{C^0(Q)}\right) e^2
\end{aligned} \tag{8}$$

Taking $c = 2$, we obtain the following differential inequality, which is to used derive coercive elliptic estimates for contact instantons.

Proposition 12.4 (Equation (5.13) [OW18a])**.**

$$|\nabla(dw)|^2 \leq C_1|dw|^4 - 4K|dw|^2 - 2\Delta e(w) \tag{9}$$

where

$$C_1 := 9\|\mathscr{L}_{R_\lambda} J\|^2_{C^0(Q)} + 4\|\nabla^\pi(\mathscr{L}_{R_\lambda} J)\|_{C^0(Q)} + 4\|\operatorname{Ric}\|_{C^0(Q)} + 4$$

denotes a constant.

Then by multiplying a cut-off function and doing integration by parts, we obtain the following local $W^{2,2}$ estimate.

Proposition 12.5 (Proposition 5.3 [OW18a]). *For any pair of open domains D_1 and D_2 in $\dot\Sigma$ such that $\overline{D}_1 \subset \mathrm{Int}(D_2)$,*

$$\|\nabla(dw)\|^2_{L^2(D_1)} \leq C_1(D_1,D_2)\|dw\|^2_{L^2(D_2)} + C_2(D_1,D_2)\|dw\|^4_{L^4(D_2)}$$

for any contact instanton w, where $C_1(D_1,D_2)$, $C_2(D_1,D_2)$ are some constants which depend on D_1, D_2 and (Q,λ,J), but are independent of w.

12.2 Local boundary $W^{2,2}$-estimate

Now let us consider the contact instantons with Legendrian boundary condition. Let $\vec{R} = (R_0,\cdots,R_k)$ be a $(k+1)$-component Legendrian link.

Consider the equation

$$\begin{cases} \overline{\partial}^\pi w = 0, \ d(w^*\lambda\circ j) = 0 \\ w(\overline{z_i z_{i+1}}) \subset R_i \end{cases} \tag{10}$$

for a smooth map $w : (\dot\Sigma, \partial\dot\Sigma) \to (Q,\vec{R})$. We will simplify writing of the boundary condition as

$$w(\partial\dot\Sigma) \subset \vec{R}$$

in a single equation.

The following local boundary a priori estimate is established in [Oh21a], [OY22].

Theorem 12.6. *Let $w : \mathbb{R}\times[0,1] \to Q$ satisfy* (4). *Then for any relatively compact domains D_1 and D_2 of the semi-disc type in $\dot\Sigma$ such that $\overline{D_1} \subset D_2$, we have*

$$\|dw\|^2_{W^{1,2}(D_1)} \leq C_1\|dw\|^2_{L^2(D_2)} + C_2\|dw\|^4_{L^4(D_2)} \tag{11}$$

where C_1, C_2 are some constants which depend only on D_1, D_2 and (Q,λ,J) and C_3 is a constant which also depends on R_i with $w(\partial D_2) \subset R_i$ as well.

Leaving the details of the proof to [OY22], we just outline the strategy of the proof here (See [Oh15, Section 8.2 & 8.3] for the same strategy used for pseudo-holomorphic curves with Lagrangian boundary condition.):

1. As the first step, we utilize the contact triad connection ∇ for the study of boundary value problem to derive the following differential inequality

$$\|dw\|^2_{W^{1,2}(D_1)} \leq C_1\|dw\|^2_{L^2(D_2)} + C_2\|dw\|^4_{L^4(D_2)} + \int_{\partial D_2} |C(\partial D_2)| \tag{12}$$

where we have
$$C(\partial D_2) := -8\left\langle B\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial x}\right), \frac{\partial w}{\partial y} \right\rangle \tag{13}$$
for the second fundamental form $B = B^\nabla$ of ∇ in the isothermal coordinate $z = x + iy$ adapted to $\partial\dot{\Sigma} \cap D_2$.

2. Once we derive (12), noting that Legendrian boundary condition $\vec{R} = (R_0, \cdots, R_k)$ for the contact instanton is automatically a free boundary value problem, i.e., a solution w of
$$\frac{\partial w}{\partial \nu} \perp TR_i,$$
one can use the Levi-Civita connection ∇^{LC} of a metric for which each component of Legendrian link $\vec{R}$ becomes *totally geodesic* (i.e., $B^{\nabla^{\mathrm{LC}}} = 0$) which will eliminate the boundary contribution appearing above in (12). Then recalling the standard fact that $\nabla = \nabla^{\mathrm{LC}} + P$ for a $(2,1)$ tensor P, we can convert the inequality (12) into (11) with some adjustment of constants C_1, C_2. (See [Oh15, Section 8.3] for such detail.)

The following lemma is an important ingredient entering in the proof.

Lemma 12.7. *Let $e = |dw|^2$ be the total harmonic energy density function. Then we have*
$$*de = -4\left\langle B\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial x}\right), \frac{\partial w}{\partial y}\right\rangle \tag{14}$$
on ∂D.

13 $C^{k,\delta}$ coercive estimates for $k \geq 1$: alternating boot-strap

Once we have established $W^{2,2}$ estimate, we can proceed with the $W^{k+2,2}$ estimate $k \geq 1$ inductively as in [OW18a, Section 5.2]. Because of the effect of the Legendrian boundary condition on the higher derivative estimate, it is not quite straightforward to boot-strap using the Sobolev norms but is better to work with $C^{k,\delta}$ Hölder norms as in [Oh21a], [OY22].

Here we provide an outline of the main steps of the *alternating boot-strap arguments* from [OY22, Section 4] to establish higher $C^{k,\delta}$ regularity results.

We start with the fundamental equation in isothermal coordinates $z = x + iy$
$$\nabla_x^\pi \zeta + J\nabla_y^\pi \zeta + \frac{1}{2}\lambda\left(\frac{\partial w}{\partial y}\right)(\mathscr{L}_{R_\lambda} J)\zeta - \frac{1}{2}\lambda\left(\frac{\partial w}{\partial x}\right)(\mathscr{L}_{R_\lambda} J)J\zeta = 0.$$
(See Corollary 11.3.) By writing
$$\overline{\nabla}^\pi := \nabla^{\pi(0,1)} = \frac{\nabla^\pi + J\nabla^\pi_{j(\cdot)}}{2}$$

which is the anti-complex linear part of ∇^π, and the linear operator

$$P_{w^*\lambda}(\zeta) := \frac{1}{4}\lambda\left(\frac{\partial w}{\partial y}\right)(\mathscr{L}_{R_\lambda}J)\zeta - \frac{1}{4}\lambda\left(\frac{\partial w}{\partial x}\right)(\mathscr{L}_{R_\lambda}J)J\zeta,$$

the equation becomes

$$\overline{\nabla}^\pi\zeta + P_{w^*\lambda}(\zeta) = 0 \tag{1}$$

which is a linear first-order PDE for ζ of Cauchy-Riemann type once $w^*\lambda$ is given. By the Sobolev embedding, we have $W^{2,2} \subset C^{0,\delta}$ for $0 \le \delta < 1/2$. Therefore we start from $C^{0,\delta}$ bound with $0 < \delta < 1/2$ and will inductively bootstrap it to get $C^{k,\delta}$ bounds for $k \ge 1$.

WLOG, we assume that $D_2 \subset \dot\Sigma$ is a semi-disc with $\partial D_2 \subset \partial\dot\Sigma$ and equipped with an isothermal coordinates (x,y) such that

$$D_2 = \{(x,y) \mid |x|^2 + |y|^2 < \varepsilon,\, y \ge 0\}$$

for some $\varepsilon > 0$ and so $\partial D_2 \subset \{(x,y) \in D_2 \mid y = 0\}$. Assume $D_1 \subset D_2$ is the semi-disc with radius $\varepsilon/2$. We denote $\zeta = \pi\frac{\partial w}{\partial x}$, $\eta = \pi\frac{\partial w}{\partial y}$ as in [OW18a], and consider the complex-valued function

$$\alpha(x,y) = \lambda\left(\frac{\partial w}{\partial y}\right) + \sqrt{-1}\left(\lambda\left(\frac{\partial w}{\partial x}\right)\right) \tag{2}$$

as in [OW18b, Subsection 11.5].

Remark 13.1. In [OW18b, Subsection 11.5], the global isothermal coordinate (τ,t) of $[0,\infty)\times S^1$ with circle-valued flat coordinate t is used and the function α defined by

$$\alpha(x,y) = \lambda\left(\frac{\partial w}{\partial y}\right) - T + \sqrt{-1}\left(\lambda\left(\frac{\partial w}{\partial x}\right)\right)$$

is used for the exponential convergence result. See the displayed formula right above in Lemma 11.19 of [OW18b, Subsection 11.5].

We note that since w satisfies the Legendrian boundary condition $w(\partial\dot\Sigma) \subset \vec{R}$, we have

$$\lambda\left(\frac{\partial w}{\partial x}\right) = 0 \tag{3}$$

on ∂D_2. The following formula is crucially used in [OW18b, Subsection 11.5] for the exponential decay result, and in [Oh21a, OY22] for the alternating boot-strap argument for the higher regularity but without detailed proof. Because the proof well demonstrates how important closedness of $w^*\lambda \circ j$ and the equality (1) are in the study of a priori elliptic estimates of contact instanton w and also because how the Legendrian boundary condition interacts with the equation, we give the full details of its proof here.

Proposition 13.2. *The complex-valued function α satisfies the equations*

$$\begin{cases} \overline{\partial}\alpha = \nu, & \nu = \frac{1}{2}|\zeta|^2 \\ \alpha(z) \in \mathbb{R} & \text{for } z \in \partial D_2 \end{cases} \tag{4}$$

Proof. For the equation, we first recall

$$d(w^*\lambda \circ j) = 0, \quad dw^*\lambda = \frac{1}{2}|d^\pi w|^2\, dA$$

where the second equality is from (1). In isothermal coordinate (x,y), we have $dA = dx \wedge dy$ and hence we have

$$*(dw^*\lambda) = \frac{1}{2}|d^\pi w|^2.$$

By the isothermality of (x,y), $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is an orthonormal frame of $T\dot{\Sigma}$ and hence

$$|d^\pi w|^2 = \left|\pi \frac{\partial w}{\partial x}\right|^2 + \left|\pi \frac{\partial w}{\partial y}\right|^2 = 2\left|\pi \frac{\partial w}{\partial x}\right|^2$$

where the last equality follows since $0 = \overline{\partial}^\pi w = (\overline{\partial} w)^\pi$ for the Cauchy-Riemann operator $\overline{\partial}$ for the standard complex structure $J_0 = \sqrt{-1}$.

Therefore if we write $\zeta = \pi \frac{\partial w}{\partial x}$,

$$*d(w^*\lambda) = |\zeta|^2.$$

On the other hand, using this identity, the isothermality of the coordinate again (x,y) and the equation $d(w^*\lambda \circ j) = 0$, we derive

$$\begin{aligned} \overline{\partial}\alpha &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\lambda\left(\frac{\partial w}{\partial y}\right) + i\left(\lambda\left(\frac{\partial w}{\partial x}\right)\right)\right) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x}\left(\lambda\left(\frac{\partial w}{\partial y}\right)\right) - \frac{1}{2}\frac{\partial}{\partial y}\left(\lambda\left(\frac{\partial w}{\partial x}\right)\right)\right) \\ &\qquad + \frac{i}{2}\left(\frac{\partial}{\partial x}\left(\lambda\left(\frac{\partial w}{\partial x}\right)\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\lambda\left(\frac{\partial w}{\partial y}\right)\right)\right) \\ &= \frac{1}{2}\left(d(w^*\lambda) - id(w^*\lambda \circ j)\right)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \\ &= \frac{1}{2}d(w^*\lambda)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{1}{2} * d(w^*\lambda). \end{aligned}$$

Combining the two, we have derived $\overline{\partial}\alpha = \frac{1}{2}|\zeta|^2$. This finishes the proof of the equation.

The boundary condition $\alpha(z) \in \mathbb{R}$ for $z \in \partial D_2$ follows from the Legendrian boundary condition: We have

$$\operatorname{Im}\alpha(z) = \lambda\left(\frac{\partial w}{\partial x}\right) = 0$$

since the vector $\frac{\partial w}{\partial x}$ is tangent to the given Legendrian submanifold by the adaptedness $\frac{\partial}{\partial x} \in \partial D_2$ of the isothermal coordinate (x,y) to the boundary ∂D_2. This finishes the proof. □

Then we arrive at the following system of equations for the pair (ζ, α)

$$\begin{cases} \nabla_x^\pi \zeta + J\nabla_y^\pi \zeta + \frac{1}{2}\lambda(\frac{\partial w}{\partial y})(\mathscr{L}_{R_\lambda}J)\zeta - \frac{1}{2}\lambda(\frac{\partial w}{\partial x})(\mathscr{L}_{R_\lambda}J)J\zeta = 0 \\ \zeta(z) \in TR_i \quad \text{for } z \in \partial D_2 \end{cases} \tag{5}$$

for some $i = 0, \ldots, k$, and

$$\begin{cases} \overline{\partial}\alpha = \frac{1}{2}|\zeta|^2 \\ \alpha(z) \in \mathbb{R} \quad \text{for } z \in \partial D_2. \end{cases} \tag{6}$$

These two equations form a nonlinear elliptic system for (ζ, α) which are coupled: α is fed into (5) through its coefficients and then ζ provides the input for the right hand side of the equation (6) and then back and forth. Using this structure of coupling, we can derive the higher derivative estimates by alternating boot strap arguments between ζ and α which is now in order.

Theorem 13.3 (Theorem 1.4, [OY22]). *Assume $k \geq 1$ and $0 < \delta < 1/2$. Let w be a contact instanton satisfying* (4). *Then for any pair of domains $D_1 \subset D_2 \subset \dot{\Sigma}$ such that $\overline{D_1} \subset D_2$, we have*

$$\|dw\|_{C^{k,\delta}(D_1)} \leq C_\delta(\|dw\|_{W^{1,2}(D_2)})$$

for some positive function $C_\delta = C_\delta(r)$ that is continuous at $r = 0$ which depends on J, λ and D_1, D_2 but independent of w.

The rest of the proof of this theorem given in [OY22] consist of the following steps:

1. Start of alternating boot-strap: $W^{1,2}$-estimate for dw.
2. $C^{1,\delta}$-estimate for $w^*\lambda = f\,dx + g\,dy$.
3. $C^{1,\delta}$-estimate for $d^\pi w$.
4. $C^{2,\delta}$-estimate for $w^*\lambda$.
5. $C^{2,\delta}$-estimate for $d^\pi w$.
6. Wrap-up of the alternating boot-strap argument: We repeat the above alternating boot strap arguments between ζ and α back and forth by taking the differential with respect to ∇_x^{LC} to inductively derive the $C^{k,\delta}$-estimates both for ζ and α in terms of $\|\zeta\|_{L^4(D_2)}$ and $\|\alpha\|_{L^4(D_2)}$ which is equivalent to considering the full $\|dw\|_{L^4}$. This completes the proof of Theorem 13.3.

We refer readers to [OY22] for complete details of this for the alternating boot strap arguments which go back and forth between ζ and α.

Part IV
Asymptotic convergence and charge vanishing

In this part, we study the most basic asymptotic behavior of finite energy contact instantons near the punctures. We divide the study into two cases separately, one the case of closed strings, i.e., on the cylinderical region $[0,\infty)\times S^1$ near interiors punctures, and the other the case of open strings, i.e., on the strip-like region $[0,\infty)\times [0,1]$ with Legendrian boundary condition of the pair (R,R').

14 Generic nondegeneracy of Reeb orbits and of Reeb chords

Nondegeneracy of closed Reeb orbits or of Reeb chords is fundamental in the Fredholm property of the linearized operator of contact instanton equations as well as of pseudoholomorphic curves on symplectization. The main conclusion of the present subsection will be the statement on the generic nondegeneracy under the perturbation of contact forms or of Legendrian boundary conditions. The study of the case of closed Reeb orbits is standard (see [ABW10]), and our exposition on the results for the case of open strings is based on [Ohb, Appendix B].

14.1 The case of closed Reeb orbits

Let γ be a closed Reeb orbit of period $T>0$. In other words, $\gamma:\mathbb{R}\to Q$ is a solution of $\dot x = R_\lambda(x)$ satisfying $\gamma(T)=\gamma(0)$. By definition, we can write $\gamma(T)=\phi^T_{R_\lambda}(\gamma(0))$ for the Reeb flow $\phi^T=\phi^T_{R_\lambda}$ of the Reeb vector field R_λ. Therefore if γ is a closed orbit, then we have
$$\phi^T_{R_\lambda}(\gamma(0))=\gamma(0)$$
i.e., $p=\gamma(0)$ is a fixed point of the diffeomorphism ϕ^T. Since $\mathscr{L}_{R_\lambda}\lambda=0$, $\phi^T_{R_\lambda}$ is a (strict) contact diffeomorphism and so induces an isomorphism
$$d\phi^T(p)|_{\xi_p}:\xi_p\to\xi_p$$
which is the linearization restricted to ξ_p of the Poincaré return map.

Definition 14.1. We say a T-closed Reeb orbit (T,λ) is *nondegenerate* if $d\phi^T(p)|_{\xi_p}:\xi_p\to\xi_p$ with $p=\gamma(0)$ has not eigenvalue 1.

Denote $\mathscr{L}(Q)=C^\infty(S^1,Q)$ the space of loops $z:S^1=\mathbb{R}/\mathbb{Z}\to Q$. We consider the assignment
$$\Phi:(T,\gamma,\lambda)\mapsto\dot\gamma-T\,R_\lambda(\gamma)$$
which we would like to consider a section of some Banach vector bundle over $(0,\infty)\times\mathscr{L}^{1,2}(Q)\times\mathscr{C}(Q,\xi)$ where $\mathscr{L}^{1,2}(Q)$ is the $W^{1,2}$-completion of $\mathscr{L}(Q)$. We note the value
$$\dot\gamma-T\,R_\lambda(\gamma)\in\Gamma(\gamma^*TQ).$$

We denote by $L^2(\gamma^*TQ)$ the space of L^2-sections of the vector bundle γ^*TQ. Then we define the vector bundle

$$\mathscr{L}^2 \to (0,\infty) \times \mathscr{L}^{1,2}(Q) \times \mathrm{Cont}(Q,\xi)$$

whose fiber at (T,γ,λ) is $L^2(\gamma^*TQ)$. We denote by π_i, $i = 1, 2, 3$ the corresponding projections.

We denote $\mathfrak{Reeb}(Q,\xi) = \Phi^{-1}(0)$. Then the set *Reeb*$(\lambda)$ of λ-Reeb orbits (γ,T) is nothing but $\mathfrak{Reeb}(Q,\xi) \cap \pi_3^{-1}(\lambda)$.

Proposition 14.2. *A T-closed Reeb orbit (T,γ) is nondegenerate if and only if the linearization*

$$d_{(T,\gamma)}\Phi : \mathbb{R} \times W^{1,2}(\gamma^*TQ) \to L^2(\gamma^*TQ)$$

is surjective.

The following generic nondegeneracy result is proved by Albers-Bramham-Wendl in [ABW10]. We denote by

$$\mathscr{C}(Q,\xi)$$

the set of contact forms of (Q,ξ) equipped with C^∞-topology.

Theorem 14.3 (Albers-Bramham-Wendl)**.** *Let (Q,ξ) be a contact manifold. Then there exists a residual subset $\mathscr{C}^{\mathrm{reg}}(Q,\xi) \subset \mathscr{C}(Q,\xi)$ such that for any contact form $\lambda \in \mathscr{C}^{\mathrm{reg}}(Q,\xi)$ all Reeb orbits are nondegenerate for $T > 0$.*

(The case $T = 0$ can be included as the Morse-Bott nondegenerate case if we allow the action $T = 0$ by extending the definition of Reeb trajectory to *isospeed Reeb trajectories* of the pairs (γ,T) with $\gamma : [0,1] \to Q$ with $T = \int \gamma^*\lambda$ as done in [Oh21c, Ohb].)

14.2 The case of Reeb chords

We first recall the notion of iso-speed Reeb trajectories used in [Oh21c] and recall the definition of nondegeneracy of thereof.

Consider contact triads (Q,λ,J) and the boundary value problem for (γ,T) with $\gamma : [0,1] \to Q$

$$\begin{cases} \dot\gamma(t) = TR_\lambda(\gamma(t)), \\ \gamma(0) \in R_0, \quad \gamma(1) \in R_1. \end{cases} \tag{1}$$

Definition 14.4 (Isospeed Reeb trajectory; Definition 2.1 [Ohb])**.** We call a pair (γ,T) of a smooth curve $\gamma : [0,1] \to Q$ and $T \in \mathbb{R}$ an *iso-speed Reeb trajectory* if they satisfy

$$\dot\gamma(t) = TR_\lambda(\gamma(t)), \quad \int \gamma^*\lambda = T \tag{2}$$

for all $t \in [0,1]$. We call (γ, T) an iso-speed closed Reeb orbit if $\gamma(0) = \gamma(1)$, and an iso-speed Reeb chord of (R_0, R_1) it $\gamma(0) \in R_0$ and $\gamma(1) \in R_1$ from R_0 to R_1.

With this definition, we state the corresponding notion of nondegeneracy

Definition 14.5. We say a Reeb chord (γ, T) of (R_0, R_1) is nondegenerate if the linearization map $d\phi^T(p) : \xi_p \to \xi_p$ satisfies

$$d\phi^T(p)(T_{\gamma(0)}R_0) \pitchfork T_{\gamma(1)}R_1 \quad \text{in } \xi_{\gamma(1)}$$

or equivalently

$$d\phi^T(p)(T_{\gamma(0)}R_0) \pitchfork T_{\gamma(1)}Z_{R_1} \quad \text{in}\, T_{\gamma(1)}Q.$$

Here $\phi^t_{R_\lambda}$ is the flow generated by the Reeb vector field R_λ.

More generally, we consider the following situation. We recall the definition of *Reeb trace* Z_R of a Legendrian submanifold R, which is defined to be

$$Z_R := \bigcup_{t \in \mathbb{R}} \phi^t_{R_\lambda}(R).$$

(See [Ohb, Appendix B] for detailed discussion on its genericity.)

Definition 14.6 (Nondegeneracy of Legendrian links)**.** Let $\vec{R} = (R_1, \cdots, R_k)$ be a chain of Legendrian submanifolds, which we call a (ordered) *Legendrian link*. We assume that we have

$$Z_{R_i} \pitchfork R_j$$

for all $i,\, j = 1, \ldots, k$ and $i \neq j$.

We denote by

$$\mathscr{L}eg(Q, \xi)$$

the set of Legendrian submanifold and by $\mathscr{L}eg(Q, \xi; R)$ its connected component containing $R \in \mathscr{L}eg(Q, \xi)$, i.e, the set of Legendrian submanifolds Legendrian isotopic to R. We denote by

$$\mathscr{P}(\mathscr{L}eg(Q, \xi))$$

the monoid of Legendrian isotopies $[0,1] \to \mathscr{L}eg(Q, \xi)$. We have natural evaluation maps

$$\mathrm{ev}_0, \mathrm{ev}_1 : \mathscr{P}(\mathscr{L}eg(Q, \xi)) \to \mathscr{L}eg(Q, \xi)$$

and denote by

$$\mathscr{P}(\mathscr{L}eg(Q, \xi), R) = \mathrm{ev}_0^{-1}(R) \subset \mathscr{P}(\mathscr{L}eg(Q, \xi))$$

and

$$\mathscr{P}(\mathscr{L}eg(Q, \xi), (R_0, R_1)) = (\mathrm{ev}_0 \times \mathrm{ev}_1)^{-1}(R_0, R_1) \subset \mathscr{P}(\mathscr{L}eg(Q, \xi)).$$

We now provide the off-shell framework for the proof of nondegeneracy in general. Denote by $\mathscr{P}(Q; R_0, R_1)$ the space of paths

$$\gamma : ([0,1], \{0,1\}) \to (Q; R_0, R_1).$$

We consider the assignment

$$\Phi : (T, \gamma, \lambda) \mapsto \dot\gamma - T R_\lambda(\gamma) \tag{3}$$

as a section of the Banach vector bundle over

$$(0,\infty) \times \mathscr{P}^{1,2}(Q; R_0, R_1) \times \mathscr{C}(Q, \xi)$$

where $\mathscr{P}^{1,2}(Q; R_0, R_1)$ is the $W^{1,2}$-completion of $\mathscr{P}(Q; R_0, R_1)$. We have

$$\dot\gamma - T R_\lambda(\gamma) \in \Gamma(\gamma^* TQ; T_{\gamma(0)} R_0, T_{\gamma(1)} R_1).$$

We define the vector bundle

$$\mathscr{L}^2(Q; R_0, R_1) \to (0,\infty) \times \mathscr{P}^{1,2}(Q; R_0, R_1) \times \mathscr{C}(Q, \xi)$$

whose fiber at (T, γ, λ) is $L^2(\gamma^* TQ)$. We denote by π_i, $i = 1, 2, 3$ the corresponding projections.

We denote $\mathfrak{Reeb}(M, \lambda; R_0, R_1) = \Phi_\lambda^{-1}(0)$, where

$$\Phi_\lambda := \Phi|_{(0,\infty) \times \mathscr{L}^{1,2}(Q; R_0, R_1) \times \{\lambda\}}.$$

Then we have

$$\mathfrak{Reeb}(\lambda; R_0, R_1) = \Phi_\lambda^{-1}(0) = \mathfrak{Reeb}(Q, \xi) \cap \pi_3^{-1}(\lambda).$$

The following relative version of Theorem 14.3 is proved in [Ohb, Appendix B].

Theorem 14.7 (Perturbation of contact forms; Theorem B.3 [Ohb])**.** *Let (Q, ξ) be a contact manifold. Let (R_0, R_1) be a pair of Legendrian submanifolds allowing the case $R_0 = R_1$. There exists a residual subset $\mathscr{C}_1^{\text{reg}}(Q, \xi) \subset \mathscr{C}(Q, \xi)$ such that for any $\lambda \in \mathscr{C}_1^{\text{reg}}(Q, \xi)$ all Reeb chords from R_0 to R_1 are nondegenerate for $T > 0$ and Bott-Morse nondegenerate when $T = 0$.*

The following theorem is also proved in [Ohb].

Theorem 14.8 (Perturbation of boundaries; Theorem B.10 [Ohb])**.** *Let (Q, ξ) be a contact manifold. Let (R_0, R_1) be a pair of Legendrian submanifolds allowing the case $R_0 = R_1$. For a given contact form λ and R_1, there exists a residual subset*

$$R_0 \in \mathscr{L}eg^{\text{reg}}(Q, \xi) \subset \mathscr{L}eg(Q, \xi)$$

of Legendrian submanifolds such that for all $R_0 \in \mathscr{L}eg(Q, \xi)$ all Reeb chords from R_0 to R_1 are nondegenerate for $T > 0$ and Morse-Bott nondegenerate when $T = 0$.

We refer readers to [Ohb, Appendix B] for the proofs of these results.

15 Subsequence convergence

In this section, we study the asymptotic behavior of contact instantons on the Riemann surface $(\dot\Sigma, j)$ associated with a metric h with *cylinder-like ends* for the closed string context and with *strip-like ends* for the open string context.

15.1 Closed string case

We assume there exists a compact set $K_\Sigma \subset \dot\Sigma$, such that $\dot\Sigma - \mathrm{Int}(K_\Sigma)$ is a disjoint union of punctured disks each of which is isometric to the half cylinder $[0,\infty)\times S^1$ or the half strip $(-\infty,0]\times[0,1]$, where the choice of positive or negative strips depends on the choice of analytic coordinates at the punctures. We denote by $\{p_i^+\}_{i=1,\cdots,l^+}$ the positive punctures, and by $\{p_j^-\}_{j=1,\cdots,l^-}$ the negative punctures. Here $l = l^+ + l^-$. Denote by $\phi_i^\pm$ such cylinder-like coordinates. We first state our assumptions for the study of the behavior of boundary punctures. (The case of interior punctures is treated in [OW18a, Section 6].)

Definition 15.1. Let $\dot\Sigma$ be a boundary-punctured Riemann surface of genus zero with punctures $\{p_i^+\}_{i=1,\cdots,l^+}\cup\{p_j^-\}_{j=1,\cdots,l^-}$ equipped with a metric h with *strip-like ends* outside a compact subset K_Σ. Let $w:\dot\Sigma\to Q$ be any smooth map with Legendrian boundary condition. We define the total π-harmonic energy $E^\pi(w)$ by

$$E^\pi(w) = E^\pi_{(\lambda,J;\dot\Sigma,h)}(w) = \frac{1}{2}\int_{\dot\Sigma}|d^\pi w|^2 \tag{1}$$

where the norm is taken in terms of the given metric h on $\dot\Sigma$ and the triad metric on M.

Throughout this section, we work locally near one interior puncture p, i.e., on a punctured semi-disc $D^\delta(p)\setminus\{p\}$. By taking the associated conformal coordinates $\phi^+ = (\tau,t): D^\delta(p)\setminus\{p\}\to[0,\infty)\times[0,1]$ such that $h = d\tau^2+dt^2$, we need only look at a map w defined on the half cylinder $[0,\infty)\times S^1\to Q$ without loss of generality.

We put the following hypotheses in our asymptotic study of the finite energy contact instanton maps w as in [OW18a]:

Hypothesis 15.2. Let h be the metric on $\dot\Sigma$ given above. Assume $w:\dot\Sigma\to Q$ satisfies the contact instanton equations (4), and

1. $E^\pi_{(\lambda,J;\dot\Sigma,h)}(w)<\infty$ (finite π-energy);
2. $\|dw\|_{C^0(\dot\Sigma)}<\infty$.
3. Image $w\subset \mathsf{K}\subset Q$ for some compact subset K.

The above finite π-energy and C^0 bound hypotheses imply

$$\int_{[0,\infty)\times S^1}|d^\pi w|^2\,d\tau\,dt<\infty,\quad \|dw\|_{C^0([0,\infty)\times S^1)}<\infty \tag{2}$$

in these coordinates.

Definition 15.3 (Asymptotic action and charge)**.** Assume that the limit of $w(\tau, \dot{)}$ as $\tau \to \infty$ exists. Then we can associate two natural asymptotic invariants at each puncture defined as

$$T := \lim_{r\to\infty} \int_{\{r\}\times S^1} (w|_{\{0\}\times S^1})^* \lambda$$
$$Q := \lim_{r\to\infty} \int_{\{r\}\times S^1} ((w|_{\{0\}\times[0,1]})^* \lambda \circ j).$$

(Here we only look at positive punctures. The case of negative punctures is similar.) We call T the *asymptotic contact action* and Q the *asymptotic contact charge* of the contact instanton w at the given puncture.

The proof of the following subsequence convergence result is proved in [OW18a, Theorem 6.4].

Theorem 15.4 (Subsequence Convergence, Theorem 6.4 [OW18a])**.** *Let* $w : [0,\infty) \times S^1 \to Q$ *satisfy the contact instanton equations* (4) *and Hypothesis* (2)*. Then for any sequence* $s_k \to \infty$*, there exists a subsequence, still denoted by* s_k*, and a* massless instanton $w_\infty(\tau,t)$ *(i.e.,* $E^\pi(w_\infty) = 0$*) on the cylinder* $\mathbb{R} \times S^1$ *that satisfies the following:*

1. $\overline{\partial}^\pi w_\infty = 0$ *and*
$$\lim_{k\to\infty} w(s_k + \tau, t) = w_\infty(\tau, t)$$
in the $C^l(K \times S^1, Q)$ *sense for any* l*, where* $K \subset [0,\infty)$ *is an arbitrary compact set.*
2. $w_\infty^* \lambda = -Q\, d\tau + T\, dt$

In general $Q = 0$ does not necessarily hold for the closed string case. When this $Q \neq 0$ combined with $T = 0$, we say w has the bad limit of *appearance of spiraling instantons along the Reeb core*. It is also proven in [Oha] that If $Q = 0 = T$, then the puncture is removable.

When $Q = 0$, which is always the case when contact instanton is exact such as those arising from the symplectization case, we have the following asymptotic convergence result.

Corollary 15.5. *Assume that* λ *is nondegenerate. Suppose that* w_τ *converges as* $|\tau| \to \infty$ *and its massless limit instanton has* $Q = 0$ *but* $T \neq 0$*, then the* w_τ *converges to a Reeb orbit of period* $|T|$ *exponentially fast.*

15.2 Open string case

In this section, we study the asymptotic behavior of contact instantons on bordered Riemann surface $(\dot\Sigma, j)$ associated with a metric h with *strip-like ends*. To be precise, we assume there exists a compact set $K_\Sigma \subset \dot\Sigma$, such that $\dot\Sigma - \mathrm{Int}(K_\Sigma)$ is a disjoint union of punctured semi-disks each of which is isometric to the half strip $[0,\infty) \times [0,1]$ or $(-\infty,0] \times [0,1]$, where the choice of positive or negative strips depends on the choice of analytic coordinates at the punctures.

Again under the assumption that the limit $\lim_{\tau\to\infty} w(\tau,\cdot)$ exists, we can define the asymptotic action T and charge Q at each puncture in the way as in the closed string case by replacing S^1 by $[0,1]$.

The following subsequence convergence and charge vanishing results are proved in [Oh21a], [OY22]. One may say that *the presence of Legendrian barrier prevents the instanton from spiraling.*

Theorem 15.6 (Subsequence convergence and Charge vanishing). *Let $w : [0,\infty) \times [0,1] \to Q$ satisfy the contact instanton equations* (4) *and converges to an isospeed Reeb chord (γ, T) as $|\tau| \to \infty$. Then for any sequence $s_k \to \infty$, there exists a subsequence, still denoted by s_k, and a massless instanton $w_\infty(\tau,t)$ (i.e., $E^\pi(w_\infty) = 0$) on the cylinder $\mathbb{R} \times [0,1]$ that satisfies the following:*

1. $\overline{\partial}^\pi w_\infty = 0$ *and*
$$\lim_{k\to\infty} w(s_k+\tau,t) = w_\infty(\tau,t)$$
in the $C^l(K \times [0,1], Q)$ sense for any l, where $K \subset [0,\infty)$ is an arbitrary compact set.
2. *w_∞ has vanishing asymptotic charge $Q = 0$ and satisfies $w_\infty(\tau,t) \equiv \gamma(t)$ for some Reeb chord γ is some Reeb chord joining R_0 and R_1 with period T at each puncture.*
3. *$T \neq 0$ at each puncture with the associated pair (R,R') of boundary condition with $R \cap R' = \emptyset$.*

Remark 15.7. This charge vanishing $Q = 0$ of the massless instanton is a huge advantage over the closed string case studied in [OW18a, OW18b, Oha]. We refer to [Oha, Section 8.1] for the classification of massless instantons on the cylinder $\mathbb{R} \times S^1$ for which the kind of massless instantons with $Q \neq 0$ but $T_w = 0$ appear for the closed string case. The *appearance of spiraling instantons along the Reeb core* is the only obstacle towards the Fredholm theory and the compactification of the moduli space of finite energy contact instantons for the general closed string context. (See [OSar] for a way of taking care of this phenomenton by 'quantizing' the charge through the $\mathfrak{lcs}$*-fication*.)

The above theorem automatically removes this obstacle for the open string case of Legendrian boundary condition. One could say that the presence of the Legendrian obstacle blocks this spiraling phenomenon of the contact instantons.

From the previous theorem, we immediately get the following corollary as in [OW18a, Section 8].

Corollary 15.8 (Corollary 5.11 [OY22]). *Assume that the pair $(\lambda, \vec{R})$ is nondegenerate in the sense of Definition 14.6. Let $w : \dot\Sigma \to Q$ satisfy the contact instanton equation* (4) *and Hypothesis* (2)*. Then on each strip-like end with strip-like coordinates $(\tau,t) \in [0,\infty) \times [0,1]$ near a puncture*

$$\lim_{s\to\infty}\left|\pi\frac{\partial w}{\partial \tau}(s+\tau,t)\right| = 0, \quad \lim_{s\to\infty}\left|\pi\frac{\partial w}{\partial t}(s+\tau,t)\right| = 0$$
$$\lim_{s\to\infty}\lambda(\frac{\partial w}{\partial \tau})(s+\tau,t) = 0, \quad \lim_{s\to\infty}\lambda(\frac{\partial w}{\partial t})(s+\tau,t) = T$$

and

$$\lim_{s\to\infty}|\nabla^l dw(s+\tau,t)| = 0 \quad \textit{for any} \quad l \geq 1.$$

All the limits are uniform for (τ,t) *in* $K\times[0,1]$ *with compact* $K\subset\mathbb{R}$.

16 Off-shell energy of contact instantons

Now following [Oha], [Oh21c], we explain the definition of off-shell energy that governs the global convergence behavior of finite energy contact instantons.

Assume that λ for the closed string case (or $(\lambda,\vec{R})$ for the open string case) is nondegenerate. We also assume that the asymptotic charges vanish so that in cylindrical (or in strip-like) coordinates $w^*\lambda \to T\,dt$ and $w^*\lambda\circ j \to T\,d\tau$ exponentially fast.

We start with the π-energy

Definition 16.1 (The π-energy of contact instanton)**.** Let $w:\mathbb{R}\times[0,1]\to J^1B$ be any smooth map, where J^1B is the 1-jet bundle of B. We define

$$E^\pi(w) := \frac{1}{2}\int |d^\pi w|_J^2.$$

Next we borrow the presentation of λ-energy from [Oha, Section 5], [Oh21c, Section 11]. We follow the procedure exercised in [Oha] for the closed string case. We introduce the following class of test functions. Especially the automatic charge vanishing in our current circumstance also enables us to define the vertical part of energy, called the λ-energy whose definition is in order. We will focus on the more nontrivial open-string case below.

Definition 16.2. We define

$$\mathscr{C} = \left\{\varphi:\mathbb{R}\to\mathbb{R}_{\geq 0}\ \middle|\ \operatorname{supp}\varphi \text{ is compact}, \int_{\mathbb{R}}\varphi = 1\right\} \tag{1}$$

Definition 16.3 (Contact instanton potential)**.** We call the above function f the *contact instanton potential* of the contact instanton charge form $w^*\lambda\circ j = df$ on $\dot\Sigma$.

Such a function exists modulo addition by a constant. *Using the assumption of vanishing of asymptotic charge*, we can explicitly write the potential as

$$f(z) = \int_{+\infty}^{z} w^*\lambda\circ j \tag{2}$$

on the strip-like regions of $\dot\Sigma$ where the integral is over any path from any puncture denote by ∞ to z along a path $\dot\Sigma$. By the closedness of $w^*\lambda\circ j$ on $\dot\Sigma$, the function f is well-defined and satisfies $w^*\lambda\circ j = df$ on each of the strip-like regions. (Compare this with [Oha, Formula above (5.5)] where the general case with nontrivial charge is considered.) By the vanishing theorem, Theorem 15.6 of the asymptotic charge of bordered contact instantons, the local version of potential function is always well-defined locally near punctures for the bordered contact instantons

We denote by ψ the function determined by

$$\psi' = \varphi, \quad \psi(-\infty) = 0,\, \psi(\infty) = 1. \tag{3}$$

Definition 16.4. Let w satisfy $d(w^*\lambda\circ j) = 0$. Then we define

$$\begin{aligned} E_{\mathscr{C}}(j,w;p) &= \sup_{\varphi\in\mathscr{C}} \int_{D_\delta(p)\setminus\{p\}} df\circ j\wedge d(\psi(f)) \\ &= \sup_{\varphi\in\mathscr{C}} \int_{D_\delta(p)\setminus\{p\}} (-w^*\lambda)\wedge d(\psi(f)). \end{aligned}$$

We note that

$$df\circ j\wedge d(\psi(f)) = \psi'(f)df\circ j\wedge df = \varphi(f)df\circ j\wedge df \geq 0$$

since

$$df\circ j\wedge df = |df|^2\, d\tau\wedge dt.$$

Therefore we can rewrite $E_{\mathscr{C}}(j,w;p)$ into

$$E_{\mathscr{C}}(j,w;p) = \sup_{\varphi\in\mathscr{C}} \int_{D_\delta(p)\setminus\{p\}} \varphi(f)df\circ j\wedge df.$$

The following proposition shows that the definition of $E_{\mathscr{C}}(j,w;p)$ does not depend on the constant shift in the choice of f.

Proposition 16.5 (Proposition 11.6 [Oh21c])**.** *For a given smooth map w satisfying $d(w^*\lambda\circ j) = 0$, we have $E_{\mathscr{C};f}(w) = E_{\mathscr{C},g}(w)$ for any pair (f,g) with*

$$df = w^*\lambda\circ j = dg$$

on $D^2_\delta(p)\setminus\{p\}$.

This proposition enables us to introduce the following vertical energy on each strip-like region, where we write $E^\lambda_\pm := E^\lambda_{\pm\infty}$ on $\mathbb{R}\times[0,1]\cong D^2\setminus\{\pm1\}$.

Definition 16.6 (Vertical energy)**.** We define the *vertical energy*, denoted by $E^\perp(w)$, to be the sum

$$E^\perp(w) = E^\lambda_+(w) + E^\lambda_-(w)$$

Now we define the final form of the off-shell energy.

Definition 16.7 (Total energy)**.** Let $w : \dot{\Sigma} \to Q$ be any smooth map. We define the *total energy* to be the sum

$$E(w) = E^{\pi}(w) + E^{\perp}(w). \tag{4}$$

Remark 16.8 (Uniform C^1 bound)**.** The upshot is that the Sachs-Uhlenbeck [SU81], Gromov [Gro85] and Hofer [Hof93] style bubbling-off analysis can be carried out with this choice of energy. (See [Oha, Oh21c] for the details of this bubbling-off analysis.) In particular *all moduli spaces of finite energy perturbed contact instantons we consider in the present paper will have uniform C^1-bounds inside each given moduli spaces.*

17 Exponential C^∞ convergence

In this section, we outline the main steps for proving the exponential convergence result from [OW18b, Section 11] for the closed string case and from [OY22, Section 6] for the open string case, respectively.

Under the nondegeneracy hypothesis from Definition 14.6 we can improve the subsequence convergence to the exponential C^∞ convergence under the transversality hypothesis. Suppose that the tuple $\vec{R} = (R_0, \ldots, R_k)$ are transversal in the sense all pairwise Reeb chords are nondegenerate. In particular we assume that the tuples are pairwise disjoint. The proof is divided into several steps.

17.1 L^2-exponential decay of the Reeb component of dw

We will prove the exponential decay of the Reeb component $w^*\lambda$. We focus on a punctured neighborhood around a puncture $z_i \in \partial\Sigma$ equipped with strip-like coordinates $(\tau, t) \in [0, \infty) \times [0, 1]$.

We again consider a complex-valued function α given in (2). Then by the Legendrian boundary condition, we know $\alpha(\tau, i) \in \mathbb{R}$, i.e.

$$\operatorname{Im} \alpha = 0$$

for $i = 0, 1$.

The following lemma was proved in the closed string case in [OW18b] (in the more general context of Morse-Bott case). For readers' convenience, we provide some details by indicating how we adapt the argument with the presence of boundary condition.

Lemma 17.1 (Lemma 6.1 [OY22]; Compare with Lemma 11.20 [OW18b])**.** *Suppose the complex-valued functions α and ν defined on $[0, \infty) \times [0, 1]$ satisfy*

$$\begin{cases} \overline{\partial}\alpha = \nu, \\ \alpha(\tau,i) \in \mathbb{R} \text{ for } i = 0,1, \\ \|\nu\|_{L^2([0,1])} + \|\nabla\nu\|_{L^2([0,1])} \leq Ce^{-\delta\tau} \quad \text{for some constants } C, \delta > 0 \\ \lim_{\tau\to+\infty}\alpha(\tau,t) = T \end{cases}$$

then $\|\alpha - T\|_{L^2(S^1)} \leq \overline{C}e^{-\delta\tau}$ *for some constant* $\overline{C}$.

Outline of proof. Notice that from previous section we have already established the $W^{1,2}$-exponential decay of $\nu = \frac{1}{2}|\zeta|^2$. Once this is established, the proof of this L^2-exponential decay result is proved by the standard three-interval method. See Theorem 24.11 for a general abstract framework which is in Appendix 24 of the present paper. We also refer to the Appendix of the arXiv version of [OW18a] for friendly details for the current nondegenerate case. □

(See Remark 13.1 for the difference of the definitions of α here and in Lemma 11.20 [OW18b].)

17.2 C^0 exponential convergence

Now the C^0-exponential convergence of $w(\tau,\cdot)$ to some Reeb chord as $\tau \to \infty$ can be proved from the L^2-exponential estimates presented in previous sections by the verbatim same argument as the proof of [OW18b, Proposition 11.21] with S^1 replaced by $[0,1]$ here. Therefore we omit its proof.

Proposition 17.2 (Proposition 11.21 [OW18b], Proposition 6.3 [OY22])**.** *Under Hypothesis 15.2, for any contact instanton w satisfying the Legendrian boundary condition, there exists a unique Reeb orbit γ such that the curve $z(\cdot) = \gamma(T\cdot)$: $[0,1] \to Q$ satisfies*

$$\|d(w(\tau,\cdot), z(\cdot))\|_{C^0([0,1])} \to 0,$$

as $\tau \to +\infty$, where d denotes the distance on Q defined by the triad metric. Here $T = T_\gamma$ is action of γ given by $T_\gamma = \int \gamma^*\lambda$.

Then the following C^0-exponential convergence is also proved.

Proposition 17.3 (Proposition 11.23 [OW18b], Proposition 6.5 [OY22])**.** *There exist some constants $C > 0$, $\delta > 0$ and τ_0 large such that for any $\tau > \tau_0$,*

$$\|d\,(w(\tau,\cdot), z(\cdot))\,\|_{C^0([0,1])} \leq C\,e^{-\delta\tau}$$

17.3 C^∞-exponential decay of $dw - R_\lambda(w)\,d\tau$

So far, we have established the following:

- $W^{1,2}$-exponential decay of w,
- C^0-exponential convergence of $w(\tau,\cdot) \to z(\cdot)$ as $\tau \to \infty$ for some Reeb chord z between two Legendrians R, R'.

Now we are ready to complete the proof of C^∞-exponential convergence $w(\tau,\cdot) \to z$ by establishing the C^∞-exponential decay of $dw - R_\lambda(w)\,dt$. The proof of the latter decay is now in order which will be carried out by the bootstrapping arguments applied to the system (4).

Combining the above three, we have obtained L^2-exponential estimates of the full derivative dw. By the bootstrapping argument using the local uniform a priori estimates on the strip-like region as in the proof of Lemma 17.1, we obtain higher order $C^{k,\alpha}$-exponential decays of the term

$$\frac{\partial w}{\partial t} - TR_\lambda(z), \quad \frac{\partial w}{\partial \tau}$$

for all $k \geq 0$, where $w(\tau,\cdot)$ converges to z as $\tau \to \infty$ in C^0 sense. Combining this, Lemma 17.1 and elliptic $C^{k,\alpha}$-estimates given in Theorem 13.3, we complete the proof of C^∞-convergence of $w(\tau,\cdot) \to z$ as $\tau \to \infty$.

Part V
Compactification, Fredholm theory and asymptotic analysis

In this section, we develop the Fredholm theory of moduli space of pseudoholomorphic curves on the symplectization as a special case of the theory of pseudoholomorphic curves on the $\mathfrak{lcs}$-fication of contact manifolds as done in by the first author with Savelyev [OSar]. The symplectization is just the zero-temperature limit of the $\mathfrak{lcs}$-fications. Contact triad connection and its symplectification interact very well with the decomposition

$$TQ = \xi \oplus \mathbb{R}\langle R_\lambda \rangle$$
$$TM \cong \xi \oplus \mathbb{R}\langle R_\lambda \rangle \oplus \mathbb{R}\left\langle \frac{\partial}{\partial s} \right\rangle$$

which enable us to derive a tensorial formula for the linearized operator in a precise matrix form all whose summands carry natural geometric meaning. This enables us to do completely coordinate free asymptotic analysis. On the other hand, [HWZ96b, HWZ96a, HWZ02] rely on the choice of special coordinates in 3 dimension. Such a coordinate approach, especially in higher dimensions, leads to many complicated tedious expressions in the asymptotic convergence results. (See [Bou02] for example.) This was the starting point of Wang and the first author of the present survey for them to develop the notion of contact instantons and its tensorial study [OW18a, OW18b], and to have discovered the notion of *contact triad connection* [OW14] which well suits the purpose. This approach especially leads to an simple transparent formulae for the linearized operators both for the contact instantons and for the pseudoholomorphic curves in symplectization.

Leaving the case of contact instantons to [Oha] for the closed string case and to [Ohb] for the open string case, we now focus on the case of pseudoholomorphic curves in symplectization.

We start with the following completion of exponential convergence result for the finite energy punctured pseudoholomorphic curves.

18 Exponential convergence in symplectization

In this section, we consider the symplectization

$$M = Q \times \mathbb{R}, \quad \omega = d(e^s \pi^* \lambda) = e^s(ds \wedge \pi^* \lambda + d\pi^* \lambda)$$

of the contact manifold (Q, ξ) equipped with contact form λ.

On Q, the Reeb vector field R_λ associated to the contact form λ is the unique vector field satisfying

$$R_\lambda \rfloor \lambda = 1, \quad R_\lambda \rfloor d\lambda = 0. \tag{1}$$

We call (y, s) the cylindrical coordinates. On the cylinder $[0, \infty) \times Q \subset (-\infty, \infty) \times Q$, we have the natural splitting

$$TM \cong TQ \oplus \mathbb{R} \cdot \frac{\partial}{\partial s} \cong \xi \oplus \operatorname{span}\left\{\widetilde{R}_\lambda, \frac{\partial}{\partial s}\right\} \cong \xi \oplus \mathbb{R}^2.$$

We denote by $\widetilde{R}_\lambda$ the unique vector field on $[0,\infty) \times Q$ which is invariant under the translation, tangent to the level sets of s and projected to R_λ. When there is no danger of confusion, we will sometimes just denote it by R_λ.

Now we describe a special family of almost complex structure adapted to the given cylindrical structure of M.

Definition 18.1. An almost complex structure $\widetilde{J}$ on $Q \times \mathbb{R}$ is called λ*-adapted* if it is split into

$$\widetilde{J} = J \oplus J_0 : TM \cong \xi \oplus \mathbb{R}^2 \to TM \cong \xi \oplus \mathbb{R}^2$$

where J is compatible to $d\lambda|_\xi$ and $J_0 : \mathbb{R}^2 \to \mathbb{R}^2$ maps $\frac{\partial}{\partial s}$ to R_λ.

For our purpose, we will need to consider a family of symplectic forms to which the given $\widetilde{J}$ is compatible and their associated metrics. For any λ-adapted $\widetilde{J}$, the $\widetilde{J}$-compatible metric associated to ω is expressed as

$$g_{(\omega,\widetilde{J})} = e^s(ds^2 + g_Y) \tag{2}$$

on $Q \times \mathbb{R}$.

Now we regard the triple $(\omega, \widetilde{J}, g_{(\omega,\widetilde{J})})$ be an almost Kähler manifold near the level surface $s = 1$. We then fix the canonical connection ∇ associated to $(\omega, \widetilde{J}, g_{(\omega,\widetilde{J})})$. The following is a general property of the canonical connection.

Proposition 18.2. *Let $(W, \omega, \widetilde{J})$ be an almost Kähler manifold and ∇ be the canonical connection. Denote by T be its torsion tensor. Then*

$$T(\widetilde{J}Y, Y) = 0 \tag{3}$$

for all vector fields Y on W.

Consider the decomposition

$$TM \cong \xi \oplus \mathbb{R}\{R_\lambda\} \oplus \mathbb{R}\left\{\frac{\partial}{\partial r}\right\}$$

and the canonical connection $\widetilde{\nabla}$ on $Q \times \mathbb{R}$, which in particular is $\widetilde{J}$-linear.

Now denote $f = s \circ u$ and $w = \pi \circ u$, i.e., $u = (w, f)$ in $Q \times \mathbb{R}$ be $\widetilde{J}$-holomorphic. Then w is automatically an exact contact instanton for which we have already shown the exponential convergence of $w(\tau, \cdot)$ to an isospeed Reeb orbit (γ, T) such that $w(\infty, t) = \gamma(t)$. For the convergence, we use the equation

$$w^*\lambda \circ j = df$$

which implies

$$w^*\lambda = -df \circ j$$

By taking differential, we obtain

$$-d(df \circ j) = w^* d\lambda = \frac{1}{2}|d^\pi w|^2\, d\tau \wedge dt.$$

This first shows that f is a subharmonic function and satisfies

$$\frac{\partial^2 f}{\partial \tau^2} + \frac{\partial^2 f}{\partial t^2} - \frac{1}{2}|d^\pi w|^2 = 0 \tag{4}$$

where we know from Theorem 15.4 that the convergence $\frac{1}{2}|d^\pi w|^2 \to T^2$ is exponentially fast. This immediately gives rise to the following exponential convergence of the radial component which will complete the study of asymptotic convergence property of finite energy pseudoholomorphic planes in symplectization by combining that of w. (See Theorem 24.11 for the general abstract framework of establishing exponential convergence.)

Proposition 18.3 (Exponential convergence of radial component)**.** *Let $u = (w, f)$ be a finite energy $\widetilde{J}$-holomorphic plane in $Q \times \mathbb{R}$. Then we have convergence*

$$df \to T\, d\tau$$

exponentially fast.

Proof. We have already established $w^*\lambda \to T\, dt$ before. By composing by j, the statement follows. ☐

19 The moduli spaces of contact instantons and of pseudoholomorphic curves

In this section, we recall the definitions of the moduli spaces of contact instantons and of pseudoholomorphic curves and compare their compactifications.

19.1 Moudli space of pseudoholomorphic curves on symplectization

Let $(\dot\Sigma, j)$ be a punctured Riemann surface and let

$$p_1, \cdots, p_{s^+}, q_1, \cdots, q_{s^-}$$

be the positive and negative punctures. For each p_i (resp. q_j), we associate the isothermal coordinates $(\tau, t) \in [0, \infty) \times S^1$ (resp. $(\tau, t) \in (-\infty, 0] \times S^1$) on the punctured disc $D_{e^{-2\pi R_0}}(p_i) \setminus \{p_i\}$ (resp. on $D_{e^{-2\pi R_0}}(q_i) \setminus \{q_i\}$) for some sufficiently large $R_0 > 0$.

Following [Hof93], [BEHZ03], we define the associated energy

$$E_\eta(u) = E^\pi(u) + E^\perp_\eta(u)$$

for each smooth map $u = (w, f)$ in class η, i.e., $[u]_{S^1} := f^* d\theta = \eta$.

Definition 19.1. We define

$$\widetilde{\mathscr{M}}_{k,\ell}(\dot\Sigma, M; \widetilde{J}), \quad k, \ell \geq 0, k+\ell \geq 1$$

to be the moduli space of pseudoholomorphic curve $u = (w, f)$ with $E_\eta(u) < \infty$.

Then we have a decomposition

$$\widetilde{\mathscr{M}}_{k,\ell}(\dot\Sigma, M; \widetilde{J}) = \bigcup_{\vec\gamma^+, \vec\gamma^-} \widetilde{\mathscr{M}}_{k,\ell}\left(\dot\Sigma, M; \widetilde{J}; (\vec\gamma^+, \vec\gamma^-)\right)$$

by Theorem 15.4 where

$$\begin{aligned}\widetilde{\mathscr{M}}_{k,\ell}(\dot\Sigma, M; \widetilde{J}; (\vec\gamma^+, \vec\gamma^-)) = \{u = (w, f) \mid u \,&\text{is an lcs instanton with} \\ &E_\eta(u) < \infty, w(-\infty_j) = \gamma_j^-, w(\infty_i) = \gamma_i\}.\end{aligned}$$

Here we have the collections of Reeb orbits γ_i^+ and γ_j^- and of points p_i, q_j for $i = 1, \cdots, s^+$ and for $j = 1, \cdots, s^-$ respectively such that

$$\lim_{\tau\to\infty} w((\tau,t)_i) = \gamma_i^+(T_i(t+t_i)), \quad \lim_{\tau\to-\infty} w((\tau,t)_j) = \gamma_j^-(T_j(t-t_j)) \tag{1}$$

for some $t_i, t_j \in S^1$, where

$$T_i = \int_{S^1} (\gamma_i^+)^*\lambda, \, T_j = \int_{S^1} (\gamma_j^-)^*\lambda.$$

Here t_i, t_j depends on the given analytic coordinate and the parameterization of the Reeb orbits.

Due to the $\mathbb{R}$-equivariance of the equation (3) under the $\mathbb{R}$ action of translations, this action induces a free action on $\widetilde{\mathscr{M}}_{k,\ell}(\dot\Sigma, M; \widetilde{J})$. Then we denote

$$\mathscr{M}_{k,\ell}(\dot\Sigma, M; \widetilde{J}) = \widetilde{\mathscr{M}}_{k,\ell}(\dot\Sigma, M; \widetilde{J})/\mathbb{R}. \tag{2}$$

We also have the decomposition

$$\widetilde{\mathscr{M}}_{k,\ell}(\dot\Sigma, M; \widetilde{J}) = \bigcup_{\vec\gamma^\pm} \widetilde{\mathscr{M}}_{k,\ell}(\dot\Sigma, M; \widetilde{J}; (\vec\gamma^+, \vec\gamma^-)).$$

Here we denote $\vec\gamma^+ = (\gamma_i^+), \quad \vec\gamma^- = (\gamma_j^-)$ By quotienting the above out by the $\mathbb{R}$-action of $Q \times \mathbb{R}$, we obtain the moduli space of J-holomorphic curves

$$\mathscr{M}_{k,\ell}(\dot\Sigma, M; \widetilde J) = \bigcup_{\vec\gamma^\pm} \mathscr{M}_{k,\ell}(\dot\Sigma, M; \widetilde J; (\vec\gamma^+, \vec\gamma^-)).$$

We now introduce a uniform energy bound for $u = (w, f)$ with given asymptotic condition at its punctures. Recall that they satisfy $w^*\lambda \circ j = df$.

The following proposition is the analog to [BEHZ03, Lemma 5.15] and [Oha, Proposition 9.2] whose proof is also similar.

Proposition 19.2. *Let $u = (w, f) \in \mathscr{M}_{k,\ell}(\dot\Sigma, M; \widetilde J; (\vec\gamma^+, \vec\gamma^-))$. Suppose that $E^\pi(w) < \infty$ and the function $f : \dot\Sigma \to \mathbb{R}$ is proper. Then $E(w) < \infty$.*

The following a priori energy bounds for $u = (w, f)$ is proved in [BEHZ03].

Proposition 19.3 (Lemma 5.15, [BEHZ03])**.** *Let $u = (w, f) \in \mathscr{M}_{k,\ell}(\dot\Sigma, M; \widetilde J; (\vec\gamma^+, \vec\gamma^-))$. Let $\widetilde f : \dot\Sigma \to \mathbb{R}$ be the function satisfying $w^*\lambda \circ j = d\widetilde f$. Suppose the function $\widetilde f$ is proper. Then we have*

$$\begin{aligned} E^\pi(u) = E^\pi(w) &= \sum_{i=1}^k \mathscr{A}_\lambda(\gamma_i^+) - \sum_{j=1}^\ell \mathscr{A}_\lambda(\gamma_j^-) \\ E^\perp_\eta(u) &= \sum_{j=1}^k \mathscr{A}_\lambda(\gamma_i^+) \\ E(u) &= 2\sum_{i=1}^k \mathscr{A}_\lambda(\gamma_i^+) - \sum_{j=1}^\ell \mathscr{A}_\lambda(\gamma_j^-). \end{aligned}$$

19.2 Moduli space of contact instantons with prescribed charge

We first consider the closed string case. In this case, the asymptotic charge $Q(p)$ at some interior puncture p may not vanish in general although they always vanish at the boundary punctures. In [OSar], the authors introduced the notion of *charge class* $\eta \in H^1(\dot\Sigma, \mathbb{Z})$ and defined the moduli space of pseudoholomorphic curves on the $\mathfrak{lcs}$-fication

$$(Q \times S^1, \omega_\lambda), \quad \omega_\lambda = d\lambda + d\theta \wedge \lambda$$

for the canonical angular form $d\theta$ on S^1 which is the generator of $H^1(S^1, \mathbb{Z})$ similarly as in the case of symplectization or more precisely in the zero-temperature limit of $\mathfrak{lcs}$-fication

$$(Q \times \mathbb{R}, \omega_\lambda), \quad \omega_\lambda = d\lambda + ds \wedge \lambda.$$

They lift a λ-adapted CR almost complex structure to an almost complex structure on $Q \times S^1$ by requiring

$$\widetilde J|_\xi = J|_\xi, \quad \widetilde J\left(\frac{\partial}{\partial\theta}\right) = R_\lambda.$$

Similarly as in the case of symplectization, they showed that a $\widetilde{J}$-holomorphic curve $u = (w, f)$ satisfies

$$\overline{\partial}^{\pi} w = 0, \quad w^*\lambda \circ j = df \tag{3}$$

where $f : \dot{\Sigma} \to S^1$ given by $f := \theta \circ u$. By definition, each lcs instanton carries a cohomology class $\eta_w := [w^*\lambda \circ j] \in H^1(\dot{\Sigma}, \mathbb{Z})$.

Recalling the isomorphism

$$[\dot{\Sigma}, S^1] \cong H^1(\dot{\Sigma}, \mathbb{Z})$$

we may also regard the cohomology class $[u]_{S^1}$ as an element in $[\dot{\Sigma}, S^1]$. This enables us to define an element in the set of homotopy classes $[\dot{\Sigma}, S^1]$, which we also denote by $\eta = \eta_u$. In fact, the isomorphism $[\dot{\Sigma}, S^1] \cong H^1(\dot{\Sigma}; \mathbb{Z})$ is directly induced by the period map

$$[f] \mapsto [f^* d\theta].$$

Definition 19.4 (Period map and the charge class; Definition 5.5 [OSar])**.** Let $u = (w, f) : \dot{\Sigma} \to Q \times S^1$ be a smooth map.

1. We call the map

$$C^\infty(\dot{\Sigma}, S^1) \to H^1(\dot{\Sigma}, \mathbb{Z}); \quad f \mapsto [f^* d\theta]$$

the *period map* and call the cohomology class $[f^* d\theta]$ the *charge class* of the map f.

2. For an lcs instanton $u = (w, f) : \dot{\Sigma} \to Q \times S^1$, we call the cohomology class

$$[f^* d\theta] \in H^1(\dot{\Sigma}, \mathbb{Z})$$

the *charge class* of u and write

$$[u]_{S^1} := [f^* d\theta].$$

Now we consider the maps $u = (w, f)$ with a fixed charge class $\eta = \eta_u$ and define the moduli space

$$\begin{aligned}\widetilde{\mathscr{M}}^{\eta}_{k,\ell}(\dot{\Sigma}, Q; J; (\vec{\gamma}^+, \vec{\gamma}^-)) = \{u = (w, f) \mid u \text{ is an lcs instanton with} \\ E_\eta(u) < \infty, w(-\infty_j) = \gamma_j^-, w(\infty_i) = \gamma_i\}.\end{aligned}$$

For the open string case, as proven in Theorem 15.6 the (local) charge near every boundary puncture vanishes. This shows that the closed one-form $w^*\lambda \circ j$ defines a cohomology class

$$[w^*\lambda \circ j] \in H^1((\overline{\Sigma}, \partial_\infty \dot{\Sigma}); \mathbb{Z}). \tag{4}$$

Definition 19.5 (The charge class; open string case)**.** Let $w : (\dot{\Sigma}, \partial \dot{\Sigma}) \to (Q, \vec{R})$ be a bordered contact instanton. We call the cohomology class $[w^*\lambda \circ j]$ given in (4) the charge class of w.

The following is an immediate consequence of this definition and Theorem 15.6.

Corollary 19.6. *Suppose Σ is an open Riemann surface of genus 0. Then for any bordered contact instanton on $\dot\Sigma = \Sigma \setminus \{z_0, \cdots z_k\}$, the charge class vanishes.*

19.3 Comparison of compactifications of the two moduli spaces

Now we consider contact instantons w arising from a pseudoholomorphic curves on symplectization (w, f). In particular all such w has its charge class $[w^*\lambda \circ j] = 0$ in $H^1(\dot\Sigma, \mathbb{Z})$ but can be lifted to $H^1(\overline{\Sigma}, \partial_\infty \dot\Sigma)$ where $\overline{\Sigma}$ is the real blow-up of $\dot\Sigma$ along the punctures. We denote the moduli space of such contact instantons of finite energy by
$$\widetilde{\mathscr{M}}^{\text{exact}}(\dot\Sigma, Q, J; \vec\gamma^-, \vec\gamma^+)$$
and
$$\mathscr{M}^{\text{exact}}(\dot\Sigma, Q, J; \vec\gamma^-, \vec\gamma^+) := \widetilde{\mathscr{M}}^{\text{exact}}(\dot\Sigma, Q, J; \vec\gamma^-, \vec\gamma^+)/\mathrm{Aut}(\dot\Sigma)$$
the set of isomorphism classes thereof. We have natural forgetful map $(w, f) \mapsto w$ which descends to
$$\mathfrak{forget} : \mathscr{M}(\dot\Sigma, M, \widetilde{J}; \vec\gamma^-, \vec\gamma^+) \to \mathscr{M}^{\text{exact}}(\dot\Sigma, Q, J; \vec\gamma^-, \vec\gamma^+).$$
By definition of the equivalence relation on $\widetilde{\mathscr{M}}(\dot\Sigma, M, \widetilde{J}; \vec\gamma^-, \vec\gamma^+)$, it follows that this forgetful map is a bijective correspondence, *provided $\dot\Sigma$ is connected*.

However when one considers the SFT compactification as in [EGH00], [BEHZ03], one needs to consider the case of pseudoholomorphic curves with disconnected domain. So let us consider such cases. Suppose we have the union
$$\dot\Sigma = \bigsqcup_{i=1}^k \dot\Sigma_i$$
of connected components with $k \geq 2$. We denote by
$$\overline{\mathscr{M}}(\dot\Sigma, M, \widetilde{J}; \vec\gamma^-, \vec\gamma^+)$$
and
$$\overline{\mathscr{M}}^{\text{exact}}(\dot\Sigma, Q, J; \vec\gamma^-, \vec\gamma^+)$$
the stable map compactification respectively. The following proposition shows the precise relationship between the two. We know that each story carries at least one non-cylindrical component.

Proposition 19.7. *Let $1 \leq \ell \leq k$ be the number of connected components which are not cylinderical. The forgetful map $\mathfrak{forget}$ is a principle $\mathbb{R}^{\ell-1}$ fibration.*

Proof. Recall the equivalence relation on $\widetilde{\mathscr{M}}(\dot\Sigma, M, \widetilde{J}; \vec\gamma^-, \vec\gamma^+)$: We say two elements $(u_1, \cdots, u_k) \sim (u'_1, \cdots, u'_k)$ if there is a $s_0 \in \mathbb{R}$ and a reparameterization $\varphi = (\varphi_1, \cdots, \varphi_k)$ of $\dot\Sigma$ such that

$$(w_i', f_i') = (w \circ \varphi_i, f_i \circ \varphi_i - s_0)$$

for all $i = 1, \cdots, k$. We define the map

$$T_{\vec{s}} : \widetilde{\mathscr{M}}(\dot{\Sigma}, M, \widetilde{J}; \vec{\gamma}^-, \vec{\gamma}^+) \to \widetilde{\mathscr{M}}(\dot{\Sigma}, M, \widetilde{J}; \vec{\gamma}^-, \vec{\gamma}^+)$$

by

$$T_{\vec{s}}(u_1, \cdots, u_k) = (T_{s_1} u_1, \cdots, T_{s_k} u_k)$$

where $T_{s_i} u_i = T_{s_i}(w_i, f_i) := (w_i, f_i - s_i)$. Note that this action is trivial for each trivial component.

Then all the following maps

$$T_{\vec{s}}(u) = (T_{s_1} u_1, \cdots, T_{s_k} u_k)$$

project down to the same element $[(w_1, \cdots, w_k)] \in \mathscr{M}^{\text{exact}}(\dot{\Sigma}, Q, J; \vec{\gamma}^-, \vec{\gamma}^+)$. The abelian group $\mathbb{R}^{\ell-1}$ admits a free transitive action of $\mathbb{R}^{\ell-1}$ on each fiber

$$\mathfrak{forget}^{-1}([(w_1, \cdots, w_k)])$$

given by

$$([(u_1, \cdots, u_k)], (s_1, \cdots s_k)) \mapsto ([(T_{s_1} u_1, \cdots, T_{s_k} u_k)]$$

where we realize

$$\mathbb{R}^{\ell-1} \cong \{(s_1, \cdots, s_\ell) \mid s_1 + \cdots + s_\ell = 0\}$$

This finishes the proof. □

20 Fredholm theory and index calculations

In this section, we work out the Fredholm theories of pseudoholomorphic curves on symplectization. We will adapt the exposition given in [Oha] [OSar] for the case of contact instantons to that of pseudoholomorphic curves thereon as the zero-temperature limit lcs instantons considered in [OSar] just by incorporating the presence of the $\mathbb{R}$-factor in the product $M = Q^{2n-1} \times \mathbb{R}$.

We divide our discussion into the closed case and the punctured case.

20.1 Calculation of the linearization map

Let Σ be a closed Riemann surface and $\dot{\Sigma}$ be its associated punctured Riemann surface. We allow the set of whose punctures to be empty, i.e., $\dot{\Sigma} = \Sigma$. We would like to regard the assignment $u \mapsto \overline{\partial}_J u$ which can be decomposed into

$$u = (w,f) \mapsto \left(\overline{\partial}^\pi w, w^*\lambda \circ j - f^* ds\right)$$

for a map $w : \dot\Sigma \to Q$ as a section of the (infinite dimensional) vector bundle over the space of maps of w. In this section, we lay out the precise relevant off-shell framework of functional analysis.

Let $(\dot\Sigma, j)$ be a punctured Riemann surface, the set of whose punctures may be empty, i.e., $\dot\Sigma = \Sigma$ is either a closed or a punctured Riemann surface. We will fix j and its associated Kähler metric h.

We consider the map

$$\Upsilon(w,f) = \left(\overline{\partial}^\pi w, w^*\lambda \circ j - f^* ds\right)$$

which defines a section of the vector bundle

$$\mathscr{H} \to \mathscr{F} = C^\infty(\Sigma, Q)$$

whose fiber at $u \in C^\infty(\Sigma, Q \times \mathbb{R})$ is given by

$$\mathscr{H}_u := \Omega^{(0,1)}(u^*\xi) \oplus \Omega^{(0,1)}(u^*\mathscr{V}).$$

Recalling $\mathscr{V}_{(q,s)} = \operatorname{span}_{\mathbb{R}}\{R_\lambda, \frac{\partial}{\partial s}\}$, we have a natural isomorphism

$$\Omega^{(0,1)}(u^*\mathscr{V}) \cong \Omega^1(\Sigma) = \Gamma(T^*\mathbb{R})$$

via the map

$$\alpha \in \Gamma(T^*\mathbb{R}) \mapsto \frac{1}{2}\left(\alpha \otimes \frac{\partial}{\partial s} + \alpha \circ j \otimes R_\lambda\right).$$

Utilizing this isomorphism, we decompose $\Upsilon = (\Upsilon_1, \Upsilon_2)$ where

$$\Upsilon_1 : \Omega^0(w^*TQ) \to \Omega^{(0,1)}(w^*\xi); \quad \Upsilon_1(w) = \overline{\partial}^\pi(w) \tag{1}$$

and

$$\Upsilon_2 : \Omega^0(w^*TQ) \to \Omega^1(\Sigma); \quad \Upsilon_2(w) = w^*\lambda \circ j - f^* ds. \tag{2}$$

We first compute the linearization map which defines a linear map

$$D\Upsilon(u) : \Omega^0(w^*TQ) \oplus \Omega^0(f^*T\mathbb{R}) \to \Omega^{(0,1)}(w^*\xi) \oplus \Omega^1(\dot\Sigma)$$

where we have

$$T_u\mathscr{F} = \Omega^0(w^*TQ) \oplus \Omega^0(f^*T\mathbb{R}).$$

For the optimal expression of the linearization map and its relevant calculations, we use the $\mathfrak{lcs}$-fication connection ∇ of $(Q \times \mathbb{R}, \lambda, J)$ which is the lcs-lifting of contact triad connection introduced in [OW18a]. We refer readers to [OW18a], [Oha], [OSar] for the unexplained notations appearing in our tensor calculations during the proof.

We define the covariant differential

$$\overline{\partial}^{\nabla^\pi} := \frac{1}{2}\left(\nabla^\pi + J\nabla^\pi \circ j\right).$$

Theorem 20.1 (Theorem 10.1 [OSar]). *We decompose $dw = d^\pi w + w^*\lambda \otimes R_\lambda$ and $Y = Y^\pi + \lambda(Y)R_\lambda$, and $X = (Y,v) \in \Omega^0(w^*T(Q\times\mathbb{R}))$. Denote $\kappa = \lambda(Y)$ and $\upsilon = ds(v)$. Then we have*

$$\begin{aligned} D\Upsilon_1(w)(Y,v) &= \overline{\partial}^{\nabla^\pi} Y^\pi + B^{(0,1)}(Y^\pi) + T^{\pi,(0,1)}_{dw}(Y^\pi) \\ &\quad + \frac{1}{2}\kappa \cdot \big((\mathscr{L}_{R_\lambda} J)J(\partial^\pi w)\big) \qquad (3) \\ D\Upsilon_2(u)(Y,v) &= w^*(\mathscr{L}_Y\lambda)\circ j - \mathscr{L}_v ds = d\kappa \circ j - d\upsilon + w^*(Y\rfloor d\lambda)\circ j \qquad (4) \end{aligned}$$

where $B^{(0,1)}$ and $T^{\pi,(0,1)}_{dw}$ are the $(0,1)$-components of B and of T^π_{dw}, where B, T^π_{dw} : $\Omega^0(w^*TQ) \to \Omega^1(w^*\xi)$ are zero-order differential operators given by

$$B(Y) = -\frac{1}{2} w^*\lambda\left((\mathscr{L}_{R_\lambda} J)JY\right)$$

and

$$T^\pi_{dw}(Y) = \pi T(Y, dw)$$

respectively.

We often omit w^* from the Lie derivative $\mathscr{L}_Y\lambda$ and from the interior product $Y\rfloor d\lambda$ regarding them 'over the map w' when Y is already a vector field along the map w below.

We can also express the operator $D\Upsilon(u)$ in the following matrix form

$$D\Upsilon(u) = \begin{pmatrix} \overline{\partial}^{\nabla^\pi} + B^{(0,1)} + T^{\pi,(0,1)}_{dw} & , \ \frac{1}{2}(\cdot)\cdot\big((\mathscr{L}_{R_\lambda} J)J(\partial^\pi w)\big) \\ ((\cdot)^\pi \rfloor d\lambda)\circ j & , \qquad \overline{\partial} \end{pmatrix} \qquad (5)$$

with respect to the decomposition

$$\left(Y, \upsilon\frac{\partial}{\partial s}\right) = \left(Y^\pi + \kappa R_\lambda, \upsilon\frac{\partial}{\partial s}\right) \cong (Y^\pi, \upsilon + i\kappa)$$

in terms of the splitting

$$T(Q\times\mathbb{R}) = \xi \oplus (\mathrm{span}\{R_\lambda\} \oplus T\mathbb{R}) \cong \xi \oplus \mathbb{C}.$$

Now we evaluate $D\Upsilon_1(w)$ more explicitly. We have the expression of $B^{(0,1)}(Y)$

$$B^{(0,1)}(Y) = -\frac{1}{4}\left(w^*\lambda \otimes \pi((\mathscr{L}_{R_\lambda} J)JY) + (w^*\lambda\circ j)\otimes \pi((\mathscr{L}_{R_\lambda} J)Y)\right).$$

Remark 20.2. Abstractly the linearization of the equation is well-defined on shell which does not depend on the choice of connections. The upshot of making a good choice of connection is to have a good formula that makes it easier to extract its consequence from it.

20.2 The punctured case

We consider some choice of weighted Sobolov spaces

$$\mathscr{W}^{k,p}_{\delta;\eta}\left(\dot\Sigma, Q\times\mathbb{R}; \vec\gamma^+, \vec\gamma^-\right)$$

as the off-shell function space and linearize the map

$$(w,\widetilde f)\mapsto\left(\overline\partial^\pi w, d\widetilde f\right).$$

This linearization operator then becomes cylindrical in cylindrical coordinates near the punctures.

The local model of the tangent space of $\mathscr{W}^{k,p}_\delta(\dot\Sigma, Q; J;\gamma^+,\gamma^-)$ at $w\in C^\infty_\delta(\dot\Sigma,Q)\subset W^{k,p}_\delta(\dot\Sigma,Q)$ is given by the sum

$$\Gamma_{s^+,s^-}\bigoplus W^{k,p}_\delta(w^*TQ) \tag{6}$$

where $W^{k,p}_\delta(w^*TQ)$ is the Banach space

$$\begin{aligned}&\{Y=(Y^\pi,\lambda(Y)R_\lambda)\mid e^{\frac{\delta}{p}|\tau|}Y^\pi\in W^{k,p}(\dot\Sigma,w^*\xi),\,\lambda(Y)\in W^{k,p}(\dot\Sigma,\mathbb{R})\}\\ \cong\ & W^{k,p}(\dot\Sigma,\mathbb{R})\otimes R_\lambda(w)\bigoplus W^{k,p}(\dot\Sigma,w^*\xi).\end{aligned}$$

Here we measure the various norms in terms of the triad metric of the triad (Q,λ,J).

To describe the choice of $\delta>0$, we need to recall the *covariant linearization* of the map $D\Phi_{\lambda,T}: W^{1,2}(z^*\xi)\to L^2(z^*\xi)$ of the map

$$\Phi_{\lambda,T}: z\mapsto\dot z - T\,R_\lambda(z)$$

for a given T-periodic Reeb orbit (T,z). The operator has the expression

$$D\Phi_{\lambda,T}=\nabla^\pi_t-\frac{T}{2}(\mathscr{L}_{R_\lambda}J)J \tag{7}$$

where ∇^π_t is the covariant derivative with respect to the pull-back connection $z^*\nabla^\pi$ along the Reeb orbit z and $(\mathscr{L}_{R_\lambda}J)J$ is a (pointwise) symmetric operator with respect to the triad metric. (See Lemma 3.4 [OW14].)

Remark 20.3. Again this *covariant* linearization map can be defined along any smooth curve and does not depend on the choice of connection *along the Reeb chords*.

We choose $\delta > 0$ so that $0 < \delta/p < 1$ is smaller than the spectral gap

$$\operatorname{gap}(\gamma^+,\gamma^-) := \min_{i,j}\{d_{\mathrm{H}}(,Q,\operatorname{Spec}A_{(T_i,z_i)},0),\, d_{\mathrm{H}}(,Q,\operatorname{Spec}A_{(T_j,z_j)},0)\}. \tag{8}$$

We now provide details of the Fredholm theory and the index calculation. Fix an elongation function $\rho : \mathbb{R} \to [0,1]$ so that

$$\rho(\tau) = \begin{cases} 1 & \tau \geq 1 \\ 0 & \tau \leq 0 \end{cases}$$
$$0 \leq \rho'(\tau) \leq 2.$$

Then we consider sections of w^*TQ by

$$\overline{Y}_i = \rho(\tau - R_0)R_\lambda(\gamma_k^+(t)), \quad \underline{Y}_j = \rho(\tau + R_0)R_\lambda(\gamma_k^+(t)) \tag{9}$$

and denote by $\Gamma_{s^+,s^-} \subset \Gamma(w^*TQ)$ the subspace defined by

$$\Gamma_{s^+,s^-} = \bigoplus_{i=1}^{s^+}\mathbb{R}\{\overline{Y}_i\} \oplus \bigoplus_{j=1}^{s^-}\mathbb{R}\{\underline{Y}_j\}.$$

Let $k \geq 2$ and $p > 2$. We denote by

$$\mathscr{W}_\delta^{k,p}(\dot{\Sigma},Q;J;\gamma^+,\gamma^-), \quad k \geq 2$$

the Banach manifold such that

$$\lim_{\tau\to\infty} w((\tau,t)_i) = \gamma_i^+(T_i(t+t_i)), \quad \lim_{\tau\to-\infty} w((\tau,t)_j) = \gamma_j^-(T_j(t-t_j))$$

for some $t_i, t_j \in S^1$, where

$$T_i = \int_{S^1}(\gamma_i^+)^*\lambda,\, T_j = \int_{S^1}(\gamma_j^-)^*\lambda.$$

Here t_i, t_j depends on the given analytic coordinate and the parameterization of the Reeb orbits.

Now for each given $w \in \mathscr{W}_\delta^{k,p} := \mathscr{W}_\delta^{k,p}(\dot{\Sigma},Q;J;\gamma^+,\gamma^-)$, we consider the Banach space

$$\Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi)$$

the $W^{k-1,p}_\delta$-completion of $\Omega^{(0,1)}(w^*\xi)$ and form the bundle

$$\mathscr{H}^{(0,1)}_{k-1,p;\delta}(\xi) = \bigcup_{w \in \mathscr{W}^{k,p}_\delta} \Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi)$$

over $\mathscr{W}^{k,p}_\delta$. Then we can regard the assignment

$$\Upsilon_1 : (w, f) \mapsto \overline{\partial}^\pi w$$

as a smooth section of the bundle $\mathscr{H}^{(0,1)}_{k-1,p;\delta}(\xi) \to \mathscr{W}^{k,p}_\delta$.

Furthermore the assignment

$$\Upsilon_2 : (w, f) \mapsto w^*\lambda \circ j - f^*ds$$

defines a smooth section of the bundle

$$\Omega^1_{k-1,p}(u^*\mathscr{V}) \to \mathscr{W}^{k,p}_\delta.$$

We have already computed the linearization of each of these maps in the previous section.

With these preparations, the following is a corollary of exponential estimates established in [OW18a].

Proposition 20.4 (Corollary 6.5 [OW18a])**.** *Assume λ is nondegenerate. Let $w : \dot\Sigma \to Q$ be a contact instanton and let $w^*\lambda = a_1\, d\tau + a_2\, dt$. Suppose*

$$\begin{aligned} \lim_{\tau\to\infty} a_{1,i} = -Q(p_i), \quad \lim_{\tau\to\infty} a_{2,i} = T(p_i) \\ \lim_{\tau\to-\infty} a_{1,j} = -Q(q_j), \quad \lim_{\tau\to-\infty} a_{2,j} = T(q_j) \end{aligned} \tag{10}$$

at each puncture p_i and q_j. Then $w \in \mathscr{W}^{k,p}_\delta(\dot\Sigma, Q; J; \gamma^+, \gamma^-)$.

Now we are ready to describe the moduli space of lcs instantons with prescribed asymptotic condition as the zero set

$$\mathscr{M}(\dot\Sigma, Q; J; \gamma^+, \gamma^-) = \left(\mathscr{W}^{k,p}_\delta(\dot\Sigma, Q; J; \gamma^+, \gamma^-) \bigoplus \mathscr{W}^{k,p}_\delta(\dot\Sigma, \mathbb{R})\right) \cap \Upsilon^{-1}(0) \tag{11}$$

whose definition does not depend on the choice of k, p or δ as long as $k \geq 2$, $p > 2$ and $\delta > 0$ is sufficiently small. One can also vary λ and J and define the universal moduli space whose detailed discussion is postponed.

In the rest of this section, we establish the Fredholm property of the linearization map

$$D\Upsilon_{(\lambda,T)}(u) : \Omega^0_{k,p;\delta}(u^*T(Q\times\mathbb{R}); J; \gamma^+, \gamma^-) \to \Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi) \oplus \Omega^{(0,1)}_{k-1,p}(f^*T\mathbb{R})$$

and compute its index. Here we also denote

$$\Omega^0_{k-1,p;\delta}(u^*T(Q\times\mathbb{R}); J; \gamma^+, \gamma^-) = W^{k-1,p}_\delta(u^*T(Q\times\mathbb{R}); J; \gamma^+, \gamma^-)$$

for the semantic reason.

For this purpose, we remark that as long as the set of punctures is non-empty, the symplectic vector bundle $w^*\xi \to \dot\Sigma$ is trivial. We recall that $\overline{\Sigma}$ stands for the real blow-up of the boundary punctured Riemann surface $\dot\Sigma$. We denote by $\Phi : E \to \overline{\Sigma} \times \mathbb{R}^{2n}$ a trivialization of $E \to \overline{\Sigma}$ and by

$$\Phi_i^+ := \Phi|_{\partial_i^+ \overline{\Sigma}}, \quad \Phi_j^- = \Phi|_{\partial_j^- \overline{\Sigma}}$$

its restrictions on the corresponding boundary components of $\partial\overline{\Sigma}$. Using the cylindrical structure near the punctures, we can extend the bundle to the bundle $E \to \overline{\Sigma}$. We then consider the following set

$$\begin{aligned}\mathscr{S} := \{A : [0,1] \to Sp(2n,\mathbb{R}) \mid 1 \notin \operatorname{Spec}(A(1)),\\ A(0) = id, \dot A(0)A(0)^{-1} = \dot A(1)A(1)^{-1}\}\end{aligned}$$

of regular paths in $Sp(2n,\mathbb{R})$ and denote by $\mu_{CZ}(A)$ the Conley-Zehnder index of the paths following [RS93]. Recall that for each closed Reeb orbit γ with a fixed trivialization of ξ, the covariant linearization $A_{(T,z)}$ of the Reeb flow along γ determines an element $A_\gamma \in \mathscr{S}$. We denote by Ψ_i^+ and Ψ_j^- the corresponding paths induced from the trivializations Φ_i^+ and Φ_j^- respectively.

We have the decomposition

$$\Omega^0_{k,p;\delta}(w^*T(Q\times\mathbb{R});J;\gamma^+,\gamma^-) = \Omega^0_{k,p;\delta}(w^*\xi) \oplus \Omega^0_{k,p;\delta}(u^*\mathscr{V}),$$

and decomposition of the operator

$$D\Upsilon_{(\lambda,T)}(u) : \Omega^0_{k,p;\delta}(w^*T(Q\times\mathbb{R});J;\gamma^+,\gamma^-) \to \Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi) \oplus \Omega^{(0,1)}_{k-1,p;\delta}(u^*\mathscr{V}) \tag{12}$$

into

$$D\Upsilon_1(u)(Y,v) \oplus D\Upsilon_2(u)(\kappa)$$

where the summands are given as in (3) and (4) respectively. We see therefrom that $D\Upsilon_{(\lambda,T)}$ is the first-order differential operator whose first-order part is given by the direct sum operator

$$(Y^\pi,(\kappa,\upsilon)) \mapsto \overline{\partial}^{\nabla^\pi} Y^\pi \oplus (d\kappa\circ j - d\upsilon)$$

where we write $(Y,v) = \left(Y^\pi + \kappa R_\lambda, \upsilon\frac{\partial}{\partial s}\right)$ for $\kappa = \lambda(Y)$, $\upsilon = ds(v)$. Here we have

$$\overline{\partial}^{\nabla^\pi} : \Omega^0_{k,p;\delta}(w^*\xi;J;\gamma^+,\gamma^-) \to \Omega^{(0,1)}_{k-1,p;\delta}(w^*\xi)$$

and the second summand can be written as the standard Cauchy-Riemann operator

$$\overline{\partial} : W^{k,p}(\dot\Sigma;\mathbb{C}) \to \Omega^{(0,1)}_{k-1,p}(\dot\Sigma,\mathbb{C}); \quad \upsilon + i\kappa =: \varphi \mapsto \overline{\partial}\varphi.$$

The following proposition can be derived from the arguments used by Lockhart and McOwen [LM85]. However before applying their general theory, one needs to pay some preliminary measure to handle the fact that the order of the operators $D\Upsilon(w)$ are different depending on the direction of ξ or on that of R_λ.

Proposition 20.5. *Suppose $\delta > 0$ satisfies the inequality*

$$0 < \delta < \min\left\{\frac{\operatorname{gap}(\gamma^+,\gamma^-)}{p}, \frac{2}{p}\right\}$$

where $\operatorname{gap}(\gamma^+,\gamma^-)$ is the spectral gap, given in (8)*, of the asymptotic operators $A_{(T_j,z_j)}$ or $A_{(T_i,z_i)}$ associated to the corresponding punctures. Then the operator* (12) *is Fredholm.*

Proof. We first note that the operators $\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}$ and $\overline{\partial}$ are Fredholm: The relevant a priori coercive $W^{k,2}$-estimates for any integer $k \geq 1$ for the derivative dw on the punctured Riemann surface $\dot\Sigma$ with cylindrical metric near the punctures are established in [OW18a] for the operator $\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}$ and the one for $\overline{\partial}$ is standard. From this, the standard interpolation inequality establishes the $W^{k,p}$-estimates for $D\Upsilon(w)$ for all $k \geq 2$ and $p \geq 2$.

Proposition 20.6. *The off-diagonal terms decay exponentially fast as $|\tau| \to \infty$.*

Proof. For the $(1,2)$-term, we derive from $\overline{\partial}^\pi w = 0$

$$\left(\frac{\partial w}{\partial \tau}\right)^\pi + J\left(\frac{\partial w}{\partial t}\right)^\pi = 0.$$

Therefore we have

$$\partial^\pi w\left(\frac{\partial}{\partial \tau}\right) = \frac{1}{2}\left(\left(\frac{\partial w}{\partial \tau}\right)^\pi - J\left(\frac{\partial w}{\partial t}\right)^\pi\right) = -J\left(\frac{\partial w}{\partial t}\right)^\pi.$$

By the exponential convergence $\frac{\partial w}{\partial t} \to TR_\lambda(\gamma_\infty(t))$, we derive

$$J\partial^\pi w\left(\frac{\partial}{\partial \tau}\right) = \left(\frac{\partial w}{\partial t}\right)^\pi \to 0$$

since $\frac{\partial w}{\partial t} \to TR_\lambda$. Therefore the off-diagonal term converges to the zero operator exponentially fast.

For the $(2,1)$-term, we evaluate

$$(Y^\pi \rfloor d\lambda) \circ j\left(\frac{\partial}{\partial \tau}\right) = d\lambda\left(Y, \frac{\partial w}{\partial t}\right)$$
$$(Y^\pi \rfloor d\lambda) \circ j\left(\frac{\partial}{\partial t}\right) = -d\lambda\left(Y, \frac{\partial w}{\partial \tau}\right).$$

Therefore we have derived

$$(Y^\pi \rfloor d\lambda)\circ j = d\lambda\left(Y, \frac{\partial w}{\partial t}\right) d\tau - d\lambda\left(Y, \frac{\partial w}{\partial \tau}\right) dt.$$

Therefore we have shown

$$((\cdot)^\pi \rfloor d\lambda)\circ j \to d\lambda\left(\cdot, \frac{\partial w}{\partial t}\right) d\tau - d\lambda\left(\cdot, \frac{\partial w}{\partial \tau}\right) dt.$$

Since $\frac{\partial w}{\partial t} \to TR_\lambda$, the first term converges to zero, and the second term converges to

$$-d\lambda(\cdot, JTR_\lambda) = Td\lambda\left(\cdot, \frac{\partial}{\partial s}\right) = 0.$$

This finishes the proof. □

Therefore it can be homotoped to the block-diagonal form, i.e., into the direct sum operator

$$\left(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}\right) \oplus \overline{\partial}$$

via a continuous path of Fredholm operators given by

$$s \in [0,1] \mapsto \begin{pmatrix} \overline{\partial}^{\nabla^\pi} + B^{(0,1)} + T^{\pi,(0,1)}_{dw} & , \frac{s}{2}(\cdot)\cdot\left((\mathscr{L}_{R_\lambda}J)J(\partial^\pi w)\right) \\ s\,((\cdot)^\pi \rfloor d\lambda)\circ j & , \qquad \overline{\partial} \end{pmatrix}$$

from $s = 1$ to $s = 0$. The Fredholm property of this path follows from the fact that the off-diagonal terms are 0-th order linear operators. □

Then by the continuous invariance of the Fredholm index, we obtain

$$\operatorname{Index} D\Upsilon_{(\lambda,T)}(w) = \operatorname{Index}\left(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}\right) + \operatorname{Index}(\overline{\partial}). \tag{13}$$

Therefore it remains to compute the latter two indices.

We denote by $m(\gamma)$ the multiplicity of the Reeb orbit in general. Then we have the following index formula.

Theorem 20.7. *We fix a trivialization $\Phi : E \to \overline{\Sigma}$ and denote by Ψ_i^+ (resp. Ψ_j^-) the induced symplectic paths associated to the trivializations Φ_i^+ (resp. Φ_j^-) along the Reeb orbits γ_i^+ (resp. γ_j^-) at the punctures p_i (resp. q_j) respectively. Then we have*

$$\begin{aligned} &\operatorname{Index}(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}) \\ &= n(2-2g-s^+-s^-) + 2c_1(w^*\xi) + (s^+ + s^-) \\ &\quad + \sum_{i=1}^{s^+} \mu_{CZ}(\Psi_i^+) - \sum_{j=1}^{s^-} \mu_{CZ}(\Psi_j^-) \end{aligned} \tag{14}$$

$$\mathrm{Index}(\overline{\partial}) = 2\sum_{i=1}^{s^+} m(\gamma_i^+) + 2\sum_{j=1}^{s^-} m(\gamma_j^-) - 2g. \tag{15}$$

In particular,

$$\begin{aligned}
&\mathrm{Index}\, D\Upsilon_{(\lambda,T)}(u)\\
&= n(2-2g-s^+-s^-) + 2c_1(w^*\xi)\\
&\quad + \sum_{i=1}^{s^+} \mu_{CZ}(\Psi_i^+) - \sum_{j=1}^{s^-} \mu_{CZ}(\Psi_j^-)\\
&\quad + \sum_{i=1}^{s^+} (2m(\gamma_i^+)+1) + \sum_{j=1}^{s^-} (2m(\gamma_j^-)+1) - 2g.
\end{aligned} \tag{16}$$

Proof. The formula (14) can be immediately derived from the general formula given in the top of p. 52 of Bourgeois's thesis [Bou02]: The summand $(s^+ + s^-)$ comes from the factor Γ_{s^+,s^-} in the decomposition (6) which has dimension $s^+ + s^-$.

So it remains to compute the index (15). To compute the (real) index of $\overline{\partial}$, we consider the Dolbeault complex

$$0 \to \Omega^0(\Sigma; D) \to \Omega^1(\Sigma; D) \to 0$$

where $D = D^+ + D^-$ is the divisor associated to the set of punctures

$$D^+ = \sum_{i=1}^{s^+} m(\gamma_i^+) p_i, \quad D^- = \sum_{j=1}^{s^-} m(\gamma_j^-) q_j$$

where $m(\gamma_i^+)$ (resp. $m(\gamma_j^-)$) is the multiplicity of the Reeb orbit γ_i^+ (resp. γ_j^-). The standard Riemann-Roch formula then gives rise to the formula for the Euler characteristic

$$\begin{aligned}
\chi(D) &= 2\dim_{\mathbb{C}} H^0(D) - 2\dim_{\mathbb{C}} H^1(D) = 2\deg(D) - 2g\\
&= \sum_{i=1}^{s^+} 2m(\gamma_i^+) + \sum_{j=1}^{s^-} 2m(\gamma_j^-) - 2g.
\end{aligned}$$

This finishes the proof. □

Remark 20.8. We can also symplectify the Fredholm theory from [Ohb] and the index calculation given in [OY22] in the similar way to give rise to the relevant theory for the pseudoholomorphic curves on the symplectization which we leave to the readers as an exercise.

21 Exponential asymptotic analysis

Recalling that for any $\widetilde{J}$-holomorphic curve (w,f), w is a contact instanton for J on Q. Furthermore we have

$$(w,f)^*T(Q\times\mathbb{R}) = w^*TQ\oplus f^*T\mathbb{R} = w^*\xi\oplus\mathrm{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial s}, R_\lambda\right\}.$$

The last splitting is respected to the *canonical connection* of the almost Hermitian manifold

$$(Q\times\mathbb{R}, d\lambda + ds\wedge\lambda, \widetilde{J}).$$

Indeed, we have

$$\nabla^{\mathrm{can}} = \nabla^\pi\oplus\nabla_0$$

where $\nabla^\pi = \nabla|_\xi$ and ∇_0 is the trivial connection on $\mathrm{span}_{\mathbb{R}}\{\frac{\partial}{\partial s}, R_\lambda\}$. (Recall Proposition 0.3.)

Remark 21.1. Here again we would like to emphasize that the usage of the canonical connection of the almost Hermitian manifold, not the Levi-Civita connection, admits this splitting.

21.1 Definition of asymptotic operators and their formulae

Now we study a finer analysis of the asymptotic behavior along the Reeb orbit. Our discussion thereof is close to the one given in [OW18b, Section 11.2 &11.5] where the more general Morse-Bott case is studied.

For this purpose, we evaluate the linearization operator $D\Upsilon$ against $\frac{\partial}{\partial\tau}$. We have already checked in Proposition 20.6 that the off-diagonal terms of the matrix representation of $D\Upsilon(w)$ decays exponentially fast in the direction τ in the previous section and so we have only to examine the diagonal terms $D\Upsilon_1(w)$ and $D\Upsilon_2(w)$.

First we consider $D\Upsilon_2$ and rewrite

$$D\Upsilon_2 = \overline{\partial} = \frac{1}{2}(\partial_\tau + i\partial_t).$$

Therefore we have the asymptotic operator

$$A^\perp_{(\lambda,J,\nabla)} := i\partial_t \tag{1}$$

which does not depend on the choice of $J\in\mathscr{J}_\lambda(Q,\xi)$. The eigenfunction expansions for this operator is nothing but the standard Fourier series for $f\in L^2(S^1,Q)$.

This being said, we now focus on the Q-component $D\Upsilon_1$ of the asymptotic operator and compute

$$D\Upsilon_1(w)\left(\frac{\partial}{\partial\tau}\right) = \frac{1}{2}(\nabla^\pi_\tau + J\nabla^\pi_t) + T^{\pi,(0,1)}_{dw}\left(\frac{\partial}{\partial\tau}\right) + B^{(0,1)}\left(\frac{\partial}{\partial\tau}\right). \tag{2}$$

In fact, this is nothing but the left hand side of (4). We write

$$2D\Upsilon(w)\left(\frac{\partial}{\partial\tau}\right) = \nabla^\pi_\tau + A^\pi_{(\lambda,J,\nabla)}.$$

and define the family of operators

$$A^\tau_{(\lambda,J,\nabla)} : \Gamma(w_\tau^*\xi) \to \Gamma(w_\tau^*\xi)$$

Thanks to the exponential convergence of $w_\tau \to \gamma_\pm$ as $\tau \to \pm\infty$, we can take the limit of the conjugate operators

$$\Pi_\tau^\infty A^\tau_{(\lambda,J,\nabla)}(\Pi_\tau^\infty)^{-1} : \Gamma(\gamma_\pm^*\xi) \to \Gamma(\gamma_\pm^*\xi) \tag{3}$$

as $\tau \to \pm\infty$ respectively, where Π_τ^∞ is the parallel transport along the short geodesics from $w(\tau,t)$ to $w(\infty,t)$. This conjugate is defined for all sufficiently large $|\tau|$.

Since the discussion at $\tau = -\infty$ will be the same, we will focus our discussion on the case at $\tau = +\infty$ from now on.

Definition 21.2 (Asymptotic operator)**.** Let (τ,t) be the cylindrical (or strip-like) coordinate, and let ∇^π be the almost Hermitian connection on $w^*\xi$ induced by the contact triad connection ∇ of (Q,λ,J). We define the *asymptotic operator* of a contact instanton w to be the limit operator

$$A^\pi_{(\lambda,J,\nabla)} := \lim_{\tau\to+\infty} \Pi_\tau^\infty A^\tau_{(\lambda,J,\nabla)}(\Pi_\tau^\infty)^{-1}. \tag{4}$$

Obviously we can define the asymptotic operator at negative punctures in the similar way.

Remark 21.3. The upshot of our definition lies in its naturality depending only on the given adapted pair (λ,J) and its associated triad connection. Other literature definition of the asymptotic operators is given differently in the way how the dependence on the given J of the final formula is hard to analyze. This definition can be given any connection ∇ on Q that has the property that $\nabla(\xi) \subset \xi$ and $\nabla^\pi(J|_\xi) = 0$. Using the exponential convergence, one can check that the definition does not depend on the choice of such connections. (See Proposition 5.3 for such a formula.)

Therefore it is conceivable that a good choice of connection will facilitate the study asymptotic operators, which is precisely what is happening by our choice of contact triad connection.

Now we find the formula for this limit operator *with respect to the contact triad connection*. Since $T(R_\lambda,\cdot) = 0$, $\left(\frac{\partial w}{\partial\tau}\right)^\pi = -J\left(\frac{\partial w}{\partial t}\right)^\pi$ and $\frac{\partial w}{\partial t}(\tau,\cdot) \mapsto TR_\lambda$ exponentially fact, we obtain

$$2T^{\pi,(0,1)}_{dw}(\frac{\partial}{\partial \tau}) = T^\pi\left(\frac{\partial w}{\partial \tau},\cdot\right) \to 0.$$

On the other hand, we have

$$2B^{(0,1)}\left(\frac{\partial}{\partial \tau}\right) = -\frac{1}{2}\lambda\left(\frac{\partial w}{\partial \tau}\right)\mathscr{L}_{R_\lambda}J - \frac{1}{2}\lambda\left(\frac{\partial w}{\partial t}\right)J\mathscr{L}_{R_\lambda}J.$$

We note that the right hand side converges to $\frac{T}{2}J\nabla_{R_\lambda}$ since $w^*\lambda \to T\,dt$ as $\tau \to \infty$. This convergence proves

Proposition 21.4. *Let ∇ be the contact triad connection associated to any adapted pair (λ, J). Then the asymptotic operator $A^\pi_{(\lambda,J,\nabla)}$ is given by*

$$A^\pi_{(\lambda,J,\nabla)} = -J\nabla_t + \frac{T}{2}\mathscr{L}_{R_\lambda}JJ.$$

Proof. We recall $T(R_\lambda,\cdot) = 0$ and the identity (2). On the other hand we have $\frac{\partial w}{\partial t}(\infty,t) = TR_\lambda$. Combining the two and above convergence of $B^{(0,1)}(\partial_\tau)$, we have finished the proof. ☐

Now one may define the full asymptotic operator $A^\pi_{(\lambda,J,\nabla)}$ of the pseudoholomorphic curves to be the operator

$$A^\pi_{(\lambda,J,\nabla)} : \gamma^*\xi \oplus \mathbb{C} \to \gamma^*\xi \oplus \mathbb{C}$$

defined by

$$A_{(\lambda,J,\nabla)} = A^\pi_{(\lambda,J,\nabla)} \oplus A^\perp_{(\lambda,J,\nabla)}.$$

Here $\mathbb{C}$ stands for the pull-back bundle

$$(\pi \circ u_\tau)^*\left(\mathbb{R}\left\{\frac{\partial}{\partial s}, R_\lambda\right\}\right) = w_\tau^*\left(\mathbb{R}\left\{\frac{\partial}{\partial s}, R_\lambda\right\}\right) \cong \mathbb{R}\left\{\frac{\partial}{\partial s}, R_\lambda\right\}$$

which is canonically trivialized, and hence may be regarded as a vector bundle over the curves w_τ on Q. (Compare this with [Par19, Definition 2.28].)

21.2 Asymptotic operator and the Levi-Civita connection

Up until now, we have emphasized the usage of triad connection which give rise to an optimal form of tensorial expression. The main reason behind this switch is the following surprising property of Levi-Civita connection of the triad metric.

We recall that while $\nabla_Y J = 0$ for all $Y \in \xi$, $\nabla_{R_\lambda} J \neq 0$ for the contact triad connection in general. (In fact, the latter holds if and only if R_λ is a Killing vector field, i.e., $\nabla R_\lambda = 0$ with respect to ∇. See [OW18a, Remark 2.4].) However while $\nabla^{\mathrm{LC}}_Y J \neq 0$

for the Levi-Civita connection in general, the Levi-Civita connection carries the following useful property for the study of asymptotic operators.

Proposition 21.5 (Lemma. 6.1 [Bla10], Proposition 4 [OW14]). $\nabla^{\mathrm{LC}}_{R_\lambda} J = 0$.

To utilize this property in our study of eigenfunction analysis for the linearized operator $A_{(\lambda,K,\nabla)} = -\frac{1}{2}J\nabla_t$, we convert it in terms of the Levi-Civita connection. For this purpose, the following lemma is crucial.

Lemma 21.6 (Lemma 6.2 [Bla10], Lemma 9 [OW14]). *For any $Y \in \xi$, we have*

$$\nabla^{\mathrm{LC}}_Y R_\lambda = \frac{1}{2}JY + \frac{1}{2}\mathscr{L}_{R_\lambda} JJY.$$

Therefore combining Proposition 21.4 and this lemma, we can rewrite the linearized operator in terms of the Levi-Civita connection as follows. It reveals a rather remarkable property of the asymptotic operator *computed in terms of the triad connection* that it becomes a Hermitian operator for any choice of adapted pair (λ, J) for the contact manifold (Q, ξ).

Corollary 21.7. *Let (λ, J) be any adapted pair and let ∇^{LC} be the Levi-Civita connection of the triad metric of (Q, λ, J). For given contact instanton w with its action $\int \gamma^*\lambda = T$ at a puncture, let $A_{(\lambda,J,\nabla)}$ be the asymptotic operator of w at the with cylindrical coordinate (τ, t). Then*

1. $[\nabla^{\mathrm{LC}}_t, J] (= \nabla^{\mathrm{LC}}_t J) = 0$,
2. $A^\pi_{(\lambda,J,\nabla)} = -J\nabla_t + \frac{T}{2}\mathscr{L}_{R_\lambda} JJ = -J\nabla^{\mathrm{LC}}_t - \frac{T}{2}Id + \frac{T}{2}\mathscr{L}_{R_\lambda} JJ$.

In particular, it induces a J-Hermitian operator on $(\xi, g|_\xi)$ with respect to the triad metric g.

The same kind of property also holds for the open string case for the Legendrian pair (R_0, R_1). This explicit formula for the asymptotic operator enables us to prove the following series of perturbation results in [KO23] on the eigenfunctions and eigenvalues of the asymptotic operators *under the perturbation of J's* inside the set $\mathscr{J}_\lambda$ of λ-adapted CR almost complex structures J. The above explicit form of asymptotic operator on J and λ enable us to prove the following genericity in terms of the choice of contact triads (Q, λ, J) of contact manifold (Q, ξ).

Theorem 21.8 (Simpleness of eigenvalues; [KO23]). *Let (Q, ξ) be a contact manifold. Assume that λ is nondegenerate. For a generic choice of adapted pair (λ, J), all eigenvalues μ_i of the asymptotic operator are simple for all closed Reeb orbits.*

21.3 Finer asymptotic behavior

Once these theorems at our disposal, we can further proceed with a finer asymptotic convergence result for the pseudoholomorphic curves on symplectization in a canonical covariant tensorial way. Similar study is given in [HWZ96b, HWZ96a, HWZ02] in 3 dimension using special coordinates followed by adjusting the almost complex structure along the given Reeb orbit.

Let $u = (w, f)$ be any finite energy $\widetilde{J}$-pseudoholomorphic curve from a punctured Riemann surface $\dot{\Sigma}$ to the symplectization of $M = Q \times \mathbb{R}$. Then w is a contact instanton of the triad (Q, λ, J). Consider a puncture and let

$$w : [\tau_0, \infty) \times S^1 \to Q$$

be the representation of w in the cylindrical coordinate for some large $\tau_0 > 0$ and denote $w_\tau := w(\tau, \cdot)$.

We have shown that there exists a isospeed Reeb trajectory (γ, T) such that

$$w(\tau, t) = w_\tau(t) \to w_\infty = \gamma(T(\cdot)) \text{ as } \tau \to \infty.$$

Denote by $\gamma_T : S^1 \to Q$ the curve given by $\gamma_T(t) := \gamma(Tt)$.

At each τ we have the operator

$$A^{\tau,\pi}_{(\lambda,J,\nabla)} : \Gamma(w_\tau^* \xi) \to \Gamma(w_\tau^* \xi)$$

defined by

$$A^{\tau,\pi}_{(\lambda,J,\nabla)} := J\nabla^\pi_t + T^{\pi,(0,1)}_{dw}\left(\frac{\partial}{\partial \tau}\right) + B^{(0,1)}\left(\frac{\partial}{\partial \tau}\right). \tag{5}$$

We have shown that as $\tau \to \infty$ the operator

$$\Pi^\infty_\tau \circ A^{\tau,\pi}_{(\lambda,J,\nabla)} \circ (\Pi^\infty_\tau)^{-1} : \Gamma(\gamma_T^* \xi) \to \Gamma(\gamma_T^* \xi),$$

converges to the asymptotic operator

$$A^{\pi}_{(\lambda,J,\nabla)} : \Gamma(\gamma_T^* \xi) \to \Gamma(\gamma_T^* \xi), \tag{6}$$

We have shown that there exists an iso-speed Reeb trajectory (γ, T) such that

$$w(\tau, t) = w_\tau(t) \to w_\infty = \gamma(\cdot) \text{ as } \tau \to \infty.$$

At each τ we have the operator

$$A^{\tau,\pi}_{(\lambda,J,\nabla)} : \Gamma(w_\tau^* \xi) \to \Gamma(w_\tau^* \xi)$$

defined by

$$A^{\tau,\pi}_{(\lambda,J,\nabla)} := J\nabla^\pi_t + T^{\pi,(0,1)}_{dw}\left(\frac{\partial}{\partial \tau}\right) + B^{(0,1)}\left(\frac{\partial}{\partial \tau}\right). \tag{7}$$

We have shown that as $\tau \to \infty$ the operator

$$\Pi^\infty_\tau \circ A^{\tau,\pi}_{(\lambda,J,\nabla)} \circ (\Pi^\infty_\tau)^{-1} : \Gamma(\gamma_T^* \xi) \to \Gamma(\gamma_T^* \xi),$$

converges to the asymptotic operator

$$A^{\pi}_{(\lambda,J,\nabla)} := A^{\pi}_{(\lambda,J,\nabla)}(\gamma) : \Gamma(\gamma_T^*\xi) \to \Gamma(\gamma_T^*\xi), \tag{8}$$

given by $A^{\pi}_{(\lambda,J,\nabla)}(\gamma)$.

Then the following theorem describes the asymptotic behavior of finite energy contact instanon w as $|\tau| \to \infty$, which is the analog to [HWZ96b, Theorem 1.4].

Theorem 21.9 (Asymptotic behavior; Theorem 1.10 [KO23])**.** *Assume that* (γ, T) *is nondegenerate. Consider t*

$$\nu(\tau,t) := \frac{\zeta(\tau,t)}{\|\zeta(\tau)\|_{L^2(w_\tau^*\xi}}, \quad \alpha(\tau) = \frac{d}{d\tau}\log\|\zeta\|_{L^2(w_\tau^*\xi)}$$

Then

1. *Either* $\zeta(\tau,t) = 0$ *for all* $(\tau,t) \in [\tau_0,\infty)$
2. *or otherwise we have the following:*
   a. *There exists an eigenvector e of* $A^{\pi}_{(\lambda,J,\nabla)}$ *of eigenvalue* μ *such that* $\nu_\tau \to e$ *as* $\tau \to \infty$ *and*
   $$\zeta(\tau,t) = e^{\int_{\tau_0}^{\tau}\alpha(s)\,ds}(e(t) + \widetilde{r}(\tau,t))$$
   b. *There exist constants* $\delta > 0$*, and* C_β *for all multi-indices* $\beta = (\beta_1,\beta_2) \in \mathbb{N}\times\mathbb{N}$ *such that*
   $$\sup_{(\tau,t)} |(\nabla^\beta \nu)(\tau,t)| \leq C_\beta$$
   for all $\tau \geq \tau_0$ *and* $t \in S^1$.

Note that the above representation formula of ζ implies the following convergence of the tangent plane.

Corollary 21.10 (Convergence of tangent plane; Corollary 1.11 [KO23])**.** *Assume the second alternative in Theorem 8.2 and denote*

$$P(\tau,t) := \mathrm{Image}\, dw(\tau,t) \in \mathrm{Gr}_2(\xi_{w(\tau,t)})$$

where $\mathrm{Gr}_2(\xi_x)$ *is the set of 2 dimensional subspaces of the contact hyperplane* $\xi_x \subset T_xQ$*. Then* $P(\tau,t) \to \mathrm{span}_{\mathbb{R}}\{e(t), Je(t) + TR_\lambda(\gamma(t))\}$ *exponentially fast in* C^∞ *topology uniformly in* $t \in S^1$.

Finally we would like to just mention that the same asymptotic study can be made by now in a straightforward way by incorporating the boundary condition as done in [OY22], [Ohb], [Oh22b]. The exponential convergence statement can be derived by a boot-strap argument using the exponential convergence result on ζ as $\tau \to \infty$ given in Section 17. (See [OW18b], [OY22] for the details of this exponential convergence and the boot-strap argument.)

Combining all the above, we have finished the proof of Theorem 21.9.

Now we prove the following two theorems which are analogs to Theorem 5.1 and Theorem 5.2 respectively whose proofs follow those in [HWZ01, Section 5] after translating their coordinate exposition into that of covariant tensorial exposition, and so omitted.

Theorem 21.11 (Compare with Theorem 5.1 [HWZ01])**.** *Consider a nondegenerate finite energy contact instanton w on cylindrical coordinates* $(\tau,t) \in [0,\infty) \times S^1$. *Then there exists some* $\tau_0 > 0$ *such that the sets*

$$\{(\tau,t) \mid w(\tau,z) \in \operatorname{Image}\gamma,\, \tau \geq \tau_0\}$$

$$\{(\tau,z) \mid \zeta(z) = 0,\, \tau \geq \tau_0\}$$

consist of finitely many points.

The following naturally occurs in the study of asymptotic behavior of bubbles which provides a strong embdding control of such planes in 3 dimension.

Theorem 21.12 (Compare with Theorem 5.2 [HWZ01])**.** *Consider a nondegenerate finite energy contact instanton w on* $\mathbb{C} \cong S^2 \setminus \{pt\}$. *Then each of the sets*

$$\{z \in \mathbb{C} \mid w(z) \in \operatorname{Image}\gamma\}, \quad \{z \in \mathbb{C} \mid d^\pi w(z) = 0\}$$

consists of finitely many points.

Significance of this theorem is not as large as in 3 dimension in higher dimension because they are expected to be empty for a generic choice of J by the transversality argument by the dimensional reason.

22 Wedge products of vector-valued forms

In this section, we continue with the setting from Appendix 23. To be specific, we assume (P,h) is a Riemannian manifold of dimension n with metric h, and denote by D the Levi–Civita connection. $E \to P$ is a vector bundle with inner product $\langle\cdot,\cdot\rangle$ and ∇ is a connection of E which is compatible with $\langle\cdot,\cdot\rangle$.

We remark that we include this section for the sake of completeness of our treatment of vector valued forms, and the content of this appendix is not used in any section of this article. Actually one can derive exponential decay using the differential inequality method from the formulas we provide here. We leave the proof to interested reader or to the earlier arXiv version of [OW18a].

The wedge product of forms can be extended to E-valued forms by defining

$$\begin{aligned} \wedge \ : \ \Omega^{k_1}(E) \times \Omega^{k_2}(E) &\to \Omega^{k_1+k_2}(E) \\ \beta_1 \wedge \beta_2 &= \langle \zeta_1, \zeta_2 \rangle\, \alpha_1 \wedge \alpha_2, \end{aligned}$$

where $\beta_1 = \alpha_1 \otimes \zeta_1 \in \Omega^{k_1}(E)$ and $\beta_2 = \alpha_2 \otimes \zeta_2 \in \Omega^{k_2}(E)$ are E-valued forms.

Lemma 22.1. *For* $\beta_1, \beta_2 \in \Omega^k(E)$,

$$\langle \beta_1, \beta_2 \rangle = *(\beta_1 \wedge *\beta_2).$$

Proof. Write $\beta_1 = \alpha_1 \otimes \zeta_1$ and $\beta_2 = \alpha_2 \otimes \zeta_2$. Then

$$\begin{aligned}
*(\beta_1 \wedge *\beta_2) &= *\big((\alpha_1 \otimes \zeta_1) \wedge ((*\alpha_2) \otimes \zeta_2)\big) \\
&= *(\langle \zeta_1, \zeta_2 \rangle \, \alpha_1 \wedge *\alpha_2) \\
&= \langle \zeta_1, \zeta_2 \rangle * (\alpha_1 \wedge *\alpha_2) \\
&= \langle \zeta_1, \zeta_2 \rangle \, h(\alpha_1, \alpha_2) \\
&= \langle \beta_1, \beta_2 \rangle.
\end{aligned}$$

$\square$

The following lemmas exploit the compatibility of ∇ with the inner product $\langle \cdot, \cdot \rangle$.

Lemma 22.2.
$$d(\beta_1 \wedge \beta_2) = d^\nabla \beta_1 \wedge \beta_2 + (-1)^{k_1} \beta_1 \wedge d^\nabla \beta_2,$$
where $\beta_1 \in \Omega^{k_1}(E)$ and $\beta_2 \in \Omega^{k_2}(E)$ are E-valued forms.

Proof. We write $\beta_1 = \alpha_1 \otimes \zeta_1$ and $\beta_2 = \alpha_2 \otimes \zeta_2$ and calculate

$$\begin{aligned}
d(\beta_1 \wedge \beta_2) &= d(\langle \zeta_1, \zeta_2 \rangle \, \alpha_1 \wedge \alpha_2) \\
&= d\langle \zeta_1, \zeta_2 \rangle \wedge \alpha_1 \wedge \alpha_2 + \langle \zeta_1, \zeta_2 \rangle \, d(\alpha_1 \wedge \alpha_2) \\
&= \langle \nabla \zeta_1, \zeta_2 \rangle \wedge \alpha_1 \wedge \alpha_2 + \langle \zeta_1, \nabla \zeta_2 \rangle \wedge \alpha_1 \wedge \alpha_2 \\
&\quad + \langle \zeta_1, \zeta_2 \rangle \, d\alpha_1 \wedge \alpha_2 + (-1)^{k_1} \langle \zeta_1, \zeta_2 \rangle \, \alpha_1 \wedge d\alpha_2,
\end{aligned}$$

while

$$\begin{aligned}
d^\nabla \beta_1 \wedge \beta_2 &= d^\nabla (\alpha_1 \otimes \zeta_1) \wedge (\alpha_2 \otimes \zeta_2) \\
&= (d\alpha_1 \otimes \zeta_1 + (-1)^{k_1} \alpha_1 \wedge \nabla \zeta_1) \wedge (\alpha_2 \otimes \zeta_2) \\
&= \langle \zeta_1, \zeta_2 \rangle \, d\alpha_1 \wedge \alpha_2 + \langle \nabla \zeta_1, \zeta_2 \rangle \wedge \alpha_1 \wedge \alpha_2.
\end{aligned}$$

A similar calculation shows that

$$(-1)^{k_1} \beta_1 \wedge d^\nabla \beta_2 = (-1)^{k_1} \langle \zeta_1, \zeta_2 \rangle \, \alpha_1 \wedge d\alpha_2 + \langle \zeta_1, \nabla \zeta_2 \rangle \wedge \alpha_1 \wedge \alpha_2.$$

Summing these up, we get the equality we want. $\square$

Lemma 22.3. *Assume $\beta_0 \in \Omega^k(E)$ and $\beta_1 \in \Omega^{k+1}(E)$, then we have*

$$\langle d^\nabla \beta_0, \beta_1 \rangle - (-1)^{n(k+1)} \langle \beta_0, \delta^\nabla \beta_1 \rangle = *d(\beta_0 \wedge *\beta_1).$$

Proof. We calculate

$$\begin{aligned}
*d(\beta_0 \wedge *\beta_1) &= *\big(d^\nabla \beta_0 \wedge *\beta_1 + (-1)^k \beta_0 \wedge (d^\nabla * \beta_1)\big) \\
&= \langle d^\nabla \beta_0, \beta_1 \rangle + (-1)^n * \big(\beta_0 \wedge *(*d^\nabla * \beta_1\big) \\
&= \langle d^\nabla \beta_0, \beta_1 \rangle - (-1)^{n(k+1)} \langle \beta_0, \delta^\nabla \beta_1 \rangle.
\end{aligned}$$

□

23 The Weitzenböck formula for vector-valued forms

In this appendix, we recall the standard Weitzenböck formulas applied to our current circumstance. A good exposition on the general Weitzenböck formula is provided in the appendix of [FU84].

Assume (P,h) is a Riemannian manifold of dimension n with metric h, and D is the Levi–Civita connection. Let $E \to P$ be any vector bundle with inner product $\langle \cdot, \cdot \rangle$, and assume ∇ is a connection on E which is compatible with $\langle \cdot, \cdot \rangle$.

For any E-valued form s, calculating the (Hodge) Laplacian of the energy density of s, we get

$$-\frac{1}{2}\Delta|s|^2 = |\nabla s|^2 + \langle Tr\nabla^2 s, s\rangle,$$

where for $|\nabla s|$ we mean the induced norm in the vector bundle $T^*P \otimes E$, i.e., $|\nabla s|^2 = \sum_i |\nabla_{E_i} s|^2$ with $\{E_i\}$ an orthonormal frame of TP. $Tr\nabla^2$ denotes the connection Laplacian, which is defined as $Tr\nabla^2 = \sum_i \nabla^2_{E_i,E_i} s$, where $\nabla^2_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$.

Denote by $\Omega^k(E)$ the space of E-valued k-forms on P. The connection ∇ induces an exterior derivative by

$$\begin{aligned} d^\nabla \;:\; \Omega^k(E) &\to \Omega^{k+1}(E) \\ d^\nabla(\alpha \otimes \zeta) &= d\alpha \otimes \zeta + (-1)^k \alpha \wedge \nabla \zeta. \end{aligned}$$

It is not hard to check that for any 1-forms, equivalently one can write

$$d^\nabla \beta(v_1, v_2) = (\nabla_{v_1}\beta)(v_2) - (\nabla_{v_2}\beta)(v_1),$$

where $v_1, v_2 \in TP$.

We extend the Hodge star operator to E-valued forms by

$$\begin{aligned} * \;:\; \Omega^k(E) &\to \Omega^{n-k}(E) \\ *\beta &= *(\alpha \otimes \zeta) = (*\alpha) \otimes \zeta \end{aligned}$$

for $\beta = \alpha \otimes \zeta \in \Omega^k(E)$.

Define the Hodge Laplacian of the connection ∇ by

$$\Delta^\nabla := d^\nabla \delta^\nabla + \delta^\nabla d^\nabla,$$

where δ^∇ is defined by

$$\delta^\nabla := (-1)^{nk+n+1} * d^\nabla * .$$

The following lemma is important for the derivation of the Weitzenböck formula.

Lemma 23.1. *Assume $\{e_i\}$ is an orthonormal frame of P, and $\{\alpha^i\}$ is the dual frame. Then we have*

$$\begin{aligned} d^\nabla &= \sum_i \alpha^i \wedge \nabla_{e_i} \\ \delta^\nabla &= -\sum_i e_i \rfloor \nabla_{e_i}. \end{aligned}$$

Proof. Assume $\beta = \alpha \otimes \zeta \in \Omega^k(E)$. Then

$$\begin{aligned} d^\nabla(\alpha \otimes \zeta) &= (d\alpha) \otimes \zeta + (-1)^k \alpha \wedge \nabla \zeta \\ &= \sum_i \alpha^i \wedge \nabla_{e_i} \alpha \otimes \zeta + (-1)^k \alpha \wedge \nabla \zeta. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_i \alpha^i \wedge \nabla_{e_i}(\alpha \otimes \zeta) &= \sum_i \alpha^i \wedge \nabla_{e_i} \alpha \otimes \zeta + \alpha^i \wedge \alpha \otimes \nabla_{e_i} \zeta \\ &= \sum_i \alpha^i \wedge \nabla_{e_i} \alpha \otimes \zeta + (-1)^k \alpha \wedge \nabla \zeta, \end{aligned}$$

so we have proved the first statement.

For the second equality, we compute

$$\begin{aligned} \delta^\nabla(\alpha \otimes \zeta) &= (-1)^{nk+n+1} * d^\nabla * (\alpha \otimes \zeta) \\ &= (\delta \alpha) \otimes \zeta + (-1)^{nk+n+1} * (-1)^{n-k} (*\alpha) \wedge \nabla \zeta \\ &= -\sum_i e_i \rfloor \nabla_{e_i} \alpha \otimes \zeta + \sum_i (-1)^{nk-k+1} * ((*\alpha) \wedge \alpha^i) \otimes \nabla_{e_i} \zeta \\ &= -\sum_i e_i \rfloor \nabla_{e_i} \alpha \otimes \zeta - \sum_i e_i \rfloor \alpha \otimes \nabla_{e_i} \zeta \\ &= -\sum_i e_i \rfloor \nabla_{e_i} (\alpha \otimes \zeta). \end{aligned}$$

□

Theorem 23.2 (Weitzenböck Formula)**.** *Assume $\{e_i\}$ is an orthonormal frame of P, and $\{\alpha^i\}$ is the dual frame. Then when applied to E-valued forms*

$$\Delta^\nabla = -Tr\nabla^2 + \sum_{i,j} \alpha^j \wedge (e_i \rfloor R(e_i, e_j)(\cdot))$$

where R is the curvature tensor of the bundle E with respect to the connection ∇.

Proof. Since the right hand side of the equality is independent of the choice of orthonormal basis, and it is a pointwise formula, we can take the normal coordinates $\{e_i\}$ at a point $p \in P$ (and $\{\alpha^i\}$ the dual basis), i.e., $h_{ij} := h(e_i, e_j)(p) = \delta_{ij}$ and $dh_{i,j}(p) = 0$, and prove that the given formula holds at p for such coordinates. For the Levi–Civita connection, the condition $dh_{i,j}(p) = 0$ of the normal coordinate is equivalent to letting $\Gamma^k_{i,j}(p) := \alpha^k(D_{e_i}e_j)(p)$ be 0.

For $\beta \in \Omega^k(E)$, using Lemma 23.1 we calculate

$$\begin{aligned}\delta^\nabla d^\nabla \beta &= -\sum_{i,j} e_i \rfloor \nabla_{e_i}(\alpha^j \wedge \nabla_{e_j}\beta) \\ &= -\sum_{i,j} e_i \rfloor (D_{e_i}\alpha^j \wedge \nabla_{e_j}\beta + \alpha^j \wedge \nabla_{e_i}\nabla_{e_j}\beta).\end{aligned}$$

At the point p, the first term vanishes, and we get

$$\begin{aligned}\delta^\nabla d^\nabla \beta(p) &= -\sum_{i,j} e_i \rfloor (\alpha^j \wedge \nabla_{e_i}\nabla_{e_j}\beta)(p) \\ &= -\sum_i \nabla_{e_i}\nabla_{e_i}\beta(p) + \sum_{i,j} \alpha^j \wedge (e_i \rfloor \nabla_{e_i}\nabla_{e_j}\beta)(p) \\ &= -\sum_i \nabla^2_{e_i,e_i}\beta(p) + \sum_{i,j} \alpha^j \wedge (e_i \rfloor \nabla_{e_i}\nabla_{e_j}\beta)(p).\end{aligned}$$

Also,

$$\begin{aligned}d^\nabla \delta^\nabla \beta &= -\sum_{i,j} \alpha^i \wedge \nabla_{e_i}(e_j \rfloor \nabla_{e_j}\beta) \\ &= -\sum_{i,j} \alpha^i \wedge (e_j \rfloor \nabla_{e_i}\nabla_{e_j}\beta) - \sum_{i,j} \alpha^i \wedge ((D_{e_i}e_j) \rfloor \nabla_{e_j}\beta),\end{aligned}$$

As before, at the point p, the second term vanishes.

Now we sum the two parts $d^\nabla \delta^\nabla$ and $\delta^\nabla d^\nabla$ and get

$$\Delta^\nabla \beta(p) = -\sum_i \nabla^2_{e_i,e_i}\beta(p) + \sum_{i,j} \alpha^j \wedge (e_i \rfloor R(e_i,e_j)\beta)(p).$$

□

In particular, when acting on zero forms, i.e., sections of E, the second term on the right hand side vanishes, and there is

$$\Delta^\nabla = -Tr\nabla^2.$$

When acting on full rank forms, the above also holds by easy checking.

When $\beta \in \Omega^1(E)$, which is the case we use in this article, there is the following

Corollary 23.3. *For $\beta = \alpha \otimes \zeta \in \Omega^1(E)$, the Weizenböck formula can be written as*

$$\Delta^\nabla \beta = -\sum_i \nabla^2_{e_i,e_i}\beta + \mathrm{Ric}^{D*}(\alpha) \otimes \zeta + \mathrm{Ric}^\nabla \beta,$$

where Ric^{D*} *denotes the adjoint of* Ric^D*, which acts on* 1*-forms.*

In particular, when P is a surface, we have

$$\Delta^{\nabla}\beta = -\sum_i \nabla^2_{e_i,e_i}\beta + K\cdot\beta + \mathrm{Ric}^{\nabla}(\beta)$$
$$-\frac{1}{2}\Delta|\beta|^2 = |\nabla\beta|^2 - \langle\Delta^{\nabla}\beta,\beta\rangle + K\cdot|\beta|^2 + \langle\mathrm{Ric}^{\nabla}(\beta),\beta\rangle. \tag{1}$$

where K is the Gaussian curvature of the surface P.

24 Abstract framework of the three-interval method

In this section, we provide the method in proving exponential decay using the abstract framework of the three-interval method from [OW18b] referring interested readers [OW18b, Section 11]. We remark that the method can deal with the case with any exponentially decaying perturbation too (see Theorem 24.11).

The three-interval method is based on the following analytic lemma.

Lemma 24.1 ([MiRT09] Lemma 9.4)**.** *For a sequence of nonnegative numbers $\{x_k\}_{k=0,1,\cdots,N}$, if there exists some constant $0<\gamma<\frac{1}{2}$ such that*

$$x_k \le \gamma(x_{k-1}+x_{k+1})$$

for every $1\le k\le N-1$, then it follows

$$x_k \le x_0\xi^{-k} + x_N\xi^{-(N-k)}, \quad k=0,1,\cdots,N,$$

where $\xi := \frac{1+\sqrt{1-4\gamma^2}}{2\gamma}$.

Remark 24.2. 1. If we write $\gamma=\gamma(c):=\frac{1}{e^c+e^{-c}}$ where $c>0$ is uniquely determined by γ, then the conclusion can be written into the exponential form

$$x_k \le x_0 e^{-ck} + x_N e^{-c(N-k)}.$$

2. For an infinite nonnegative sequence $\{x_k\}_{k=0,1,\cdots}$, if we have a uniform bound of in addition, then the exponential decay follows as

$$x_k \le x_0 e^{-ck}.$$

The analysis of proving the exponential decay will be carried on a Banach bundle $\mathscr{E}\to[0,\infty)$ modelled by the Banach space $\mathbb{E}$, for which we mean every fiber $\mathbb{E}_\tau$ is identified with the Banach space $\mathbb{E}$ smoothly depending on τ. We omit this identification if there is no way of confusion.

First we emphasize the base $[0,\infty)$ is non-compact and carries a natural translation map for any positive number r, which is $\sigma_r:\tau\mapsto\tau+r$. We introduce the following definition which ensures us to study the sections in local trivialization after taking a subsequence.

Definition 24.3. Let $\mathscr{E}$ be a Banach bundle modelled with a Banach space $\mathbb{E}$ over $[0,\infty)$. Let $[a,b] \subset [0,\infty)$ be any given bounded interval and let $s_k \to \infty$ be any given sequence. A *tame family of trivialization* over $[a,b]$ relative to the sequence s_k is defined to be a sequence of trivializations $\{\Phi_k\} : \mathscr{E}|_{[a,b]} \to [a,b] \times \mathbb{E}$

$$\Phi_k : \sigma_{s.}^* \mathscr{E}|_{[a+s_k, b+s_k]} \to [a,b] \times \mathbb{E}$$

for $k \geq 0$ satisfying the following: There exists a sufficiently large $k_0 > 0$ such that for any $k \geq k_0$ the bundle map

$$\Phi_{k_0+k} \circ \Phi_{k_0}^{-1} : [a,b] \times \mathbb{E} \to [a,b] \times \mathbb{E}$$

satisfies

$$\|\nabla_\tau^l (\Phi_{k_0+k} \circ \Phi_{k_0}^{-1})\|_{\mathscr{L}(\mathbb{E},\mathbb{E})} \leq C_l < \infty \tag{1}$$

for constants $C_l = C_l(|b-a|)$ depending only on $|b-a|$, $l = 0, 1, \cdots$.

We call $\mathscr{E}$ *uniformly locally tame*, if it carries a tame family of trivializations over $[a,b]$ relative to the sequence s_k for any given bounded interval $[a,b] \subset [0,\infty)$ and a sequence $s_k \to \infty$.

Definition 24.4. Suppose $\mathscr{E}$ is uniformly locally tame. We say a connection ∇ on $\mathscr{E}$ is *uniformly locally tame* if the push-forward $(\Phi_k)_* \nabla_\tau$ can be written as

$$(\Phi_k)_* \nabla_\tau = \frac{d}{d\tau} + \Gamma_k(\tau)$$

for any tame family $\{\Phi_k\}$ so that $\sup_{\tau \in [a,b]} \|\Gamma_k(\tau)\|_{\mathscr{L}(\mathbb{E},\mathbb{E})} < C$ for some $C > 0$ independent of k's.

Definition 24.5. Consider a pair $\mathscr{E}_2 \subset \mathscr{E}_1$ of uniformly locally tame bundles, and a bundle map $B : \mathscr{E}_2 \to \mathscr{E}_1$. We say B is *uniformly locally bounded*, if for any compact set $[a,b] \subset [0,\infty)$ and any sequence $s_k \to \infty$, there exists a subsequence, still denoted by s_k, a sufficiently large $k_0 > 0$ and tame families $\Phi_{1,k}$, $\Phi_{2,k}$ such that for any $k \geq 0$

$$\sup_{\tau \in [a,b]} \|\Phi_{i,k_0+k} \circ B \circ \Phi_{i,k_0}^{-1}\|_{\mathscr{L}(\mathbb{E}_2,\mathbb{E}_1)} \leq C \tag{2}$$

where C is independent of k.

For a given locally tame pair $\mathscr{E}_2 \subset \mathscr{E}_1$, we denote by $\mathscr{L}(\mathscr{E}_2, \mathscr{E}_1)$ the set of bundle homomorphisms which are uniformly locally bounded.

Lemma 24.6. *If $\mathscr{E}_1, \mathscr{E}_2$ are uniformly locally tame, then so is $\mathscr{L}(\mathscr{E}_2, \mathscr{E}_1)$.*

Definition 24.7. Let $\mathscr{E}_2 \subset \mathscr{E}_1$ be as above and let $B \in \mathscr{L}(\mathscr{E}_2, \mathscr{E}_1)$. We say B is *precompact* on $[0,\infty)$ if for any locally tame families Φ_1, Φ_2, there exists a further subsequence such that $\Phi_{1,k_0+k} \circ B \circ \Phi_{1,k_0}^{-1}$ converges to some $B_{\Phi_1\Phi_2;\infty} \in \mathscr{L}(\Gamma([a,b] \times \mathbb{E}_2), \Gamma([a,b] \times \mathbb{E}_1))$.

Assume B is a bundle map from $\mathscr{E}_2$ to $\mathscr{E}_1$ which is uniformly locally bounded, where $\mathscr{E}_1 \supset \mathscr{E}_2$ are uniformly locally tame with tame families $\Phi_{1,k}$, $\Phi_{2,k}$. We can write

$$\Phi_{2,k_0+k} \circ (\nabla_\tau + B) \circ \Phi_{1,k_0}^{-1} = \frac{\partial}{\partial \tau} + B_{\Phi_1 \Phi_2,k}$$

as a linear map from $\Gamma([a,b] \times \mathbb{E}_2)$ to $\Gamma([a,b] \times \mathbb{E}_1)$, since ∇ is uniformly locally tame.

Next we introduce the following notion of coerciveness.

Definition 24.8. Let $\mathscr{E}_1, \mathscr{E}_2$ be as above and $B : \mathscr{E}_2 \to \mathscr{E}_1$ be a uniformly locally bounded bundle map. We say the operator

$$\nabla_\tau + B : \Gamma(\mathscr{E}_2) \to \Gamma(\mathscr{E}_1)$$

is *uniformly locally coercive*, if the following holds:

1. For any pair of bounded closed intervals I, I' with $I \subset \operatorname{Int} I'$,

$$\|\zeta\|_{L^2(I,\mathscr{E}_2)} \le C(I,I') \|\nabla_\tau \zeta + B\zeta\|_{L^2(I,\mathscr{E}_1)}. \tag{3}$$

2. if for given bounded sequence $\zeta_k \in \Gamma(\mathscr{E}_2)$ satisfying

$$\nabla_\tau \zeta_k + B\zeta_k = L_k$$

with $\|L_k\|_{\mathscr{E}_1}$ bounded on a given compact subset $K \subset [0,\infty)$, there exists a subsequence, still denoted by ζ_k, that uniformly converges in $\mathscr{E}_2$.

Remark 24.9. Let $E \to [0,\infty) \times S$ be a (finite dimensional) vector bundle and denote by $W^{k,2}(E)$ the set of $W^{k,2}$-section of E and $L^2(E)$ the set of L^2-sections. Let $D : L^2(E) \to L^2(E)$ be a first order elliptic operator with cylindrical end. Denote by $i_\tau : S \to [0,\infty) \times S$ the natural inclusion map. Then there is a natural pair of Banach bundles $\mathscr{E}_2 \subset \mathscr{E}_1$ over $[0,\infty)$ associated to E, whose fiber is given by $\mathscr{E}_{1,\tau} = L^2(i_\tau^* E)$, $\mathscr{E}_{2,\tau} = W^{1,2}(i_\tau^* E)$. Furthermore assume $\mathscr{E}_i$ for $i = 1,2$ is uniformly local tame if S is a compact manifold (without boundary). Then D is uniformly locally coercive, which follows from the elliptic bootstrapping and the Sobolev's embedding.

Finally we introduce the notion of asymptotically cylindrical operator B.

Definition 24.10. We call B *locally asymptotically cylindrical* if the following holds: Any subsequence limit $B_{\Phi_1 \Phi_2;\infty}$ appearing in Definition 24.7 is a *constant* section, and $\|B_{\Phi_1 \Phi_2,k} - \Phi_{2,k_0+k} \circ B \circ \Phi_{1,k_0}^{-1}\|_{\mathscr{L}(\mathbb{E}_i,\mathbb{E}_i)}$ converges to zero as $k \to \infty$ for both $i = 1,2$.

Now we specialize to the case of Hilbert bundles $\mathscr{E}_2 \subset \mathscr{E}_1$ over $[0,\infty)$ and assume that $\mathscr{E}_1$ carries a connection which is compatible with the Hilbert inner product of $\mathscr{E}_1$. We denote by ∇_τ the associated covariant derivative. We assume that ∇_τ is uniformly locally tame.

Denote by $L^2([a,b];\mathscr{E}_i)$ the space of L^2-sections ζ of $\mathscr{E}_i$ over $[a,b]$, i.e., those satisfying

$$\int_a^b |\zeta(\tau)|^2_{\mathscr{E}_i}\, dt < \infty.$$

where $|\zeta(\tau)|_{\mathscr{E}_i}$ is the norm with respect to the given Hilbert bundle structure of $\mathscr{E}_i$.

Theorem 24.11 (Three-Interval Method)**.** *Assume $\mathscr{E}_2 \subset \mathscr{E}_1$ is a pair of Hilbert bundles over $[0,\infty)$ with fibers $\mathbb{E}_2$ and $\mathbb{E}_1$, and $\mathbb{E}_2 \subset \mathbb{E}_1$ is dense. Let B be a section of the associated bundle $\mathscr{L}(\mathscr{E}_2,\mathscr{E}_1)$ and $L \in \Gamma(\mathscr{E}_1)$. We assume the following:*

1. *There exists a covariant derivative ∇_τ that preserves the Hilbert structure;*
2. *$\mathscr{E}_i$ for $i = 1, 2$ are uniformly locally tame;*
3. *B is precompact, uniformly locally coercive and asymptotically cylindrical;*
4. *Every subsequence limit B_∞ is a self-adjoint unbounded operator on $\mathbb{E}_1$ with its domain $\mathbb{E}_2$, and satisfies $\ker B_\infty = \{0\}$;*
5. *There exists some positive number δ such that any subsequence limiting operator B_∞ of the above mentioned pre-compact family has their first non-zero eigenvalues $|\lambda_1| > \delta$;*
6. *There exists some $R_0 > 0$, $C_0 > 0$ and $\delta_0 > \delta$ such that*

$$\|L(\tau)\|_{\mathscr{E}_{1,\tau}} \leq C_0 e^{-\delta_0 \tau}$$

   for all $\tau \geq R_0$.

Then for any (smooth) section $\zeta \in \Gamma(\mathscr{E}_2)$ with

$$\sup_{\tau\in[R_0,\infty)} \|\zeta(\tau,\cdot)\|_{\mathscr{E}_{2,\tau}} < \infty \tag{4}$$

and satisfying the equation

$$\nabla_\tau \zeta + B(\tau)\zeta(\tau) = L(\tau), \tag{5}$$

there exist some constants R, $C > 0$ such that for any $\tau > R$,

$$\|\zeta(\tau)\|_{\mathscr{E}_{1,\tau}} \leq Ce^{-\delta\tau}.$$

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