A conjecture on spectral sparsification with respect to hyperbolicity cones

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Abstract Spectral sparsification of sums of positive semidefinite matrices is both a remarkable fact about positive semidefinite matrices and a key ingredient in fast algorithms for graph problems. This note presents a conjectural generalization of spectral sparsification to the setting of hyperbolicity cones.

1 Spectral sparsification

Let $G = (\mathscr{V}, \mathscr{E}, \mu)$ be an undirected simple graph with vertex set $\mathscr{V}$, edge set $\mathscr{E}$, and positive edge weights μ_e for $e \in \mathscr{E}$. We can associate, with G, a matrix L_G called the *weighted graph Laplacian*. This is a $|\mathscr{V}| \times |\mathscr{V}|$ matrix of the form

$$L_G = \sum_{\{i,j\} \in \mathscr{E}} \mu_{\{i,j\}} (e_i - e_j)(e_i - e_j)^T \tag{1}$$

where e_i is the ith standard basis vector in $\mathbb{R}^{|\mathscr{V}|}$. Note that L_G is positive semidefinite, as it is the sum of positive semidefinite (rank one) matrices. Many graph parameters, such as the maximum weight of a cut in the graph, can be expressed in terms of the quadratic form associated with L_G, i.e.,

$$x^T L_G x = \sum_{\{i,j\} \in \mathscr{E}} \mu_{\{i,j\}} (x_i - x_j)^2.$$

Suppose that $0 < \varepsilon < 1$ and $H = (\mathscr{V}, \mathscr{E}', \mu')$ is a weighted graph with the same vertex set as G and with $\mathscr{E}' \subseteq \mathscr{E}$. We say that H is an ε-*spectral sparsifier* of G if

$$(1-\varepsilon) L_G \preceq L_H \preceq (1+\varepsilon) L_G$$

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where $X \preceq Y$ indicates that $Y - X$ is a positive semidefinite matrix. This notion of approximation ensures that the quadratic form defined by L_H is uniformly close, in the relative error sense, to that of L_G. The idea behind this definition is that if we can find a sparsifier H with many fewer non-zero edges than G, it would likely be much more efficient to work with algorithmically.

The main existence result in this direction is the following.

Theorem 1 (Spectral sparsification [1]). *Let X be a $d \times d$ positive definite matrix with a decomposition as $X = \sum_{i=1}^{m} X_i$ where each X_i is positive semidefinite. Let $0 < \varepsilon < 1$. Then there exists $S \subseteq \{1,2,\ldots,m\}$ with $|S| \leq \lceil 4d/\varepsilon^2 \rceil$, and positive scalars w_i for $i \in S$, such that*

$$(1-\varepsilon)X \preceq \sum_{i\in S} w_i X_i \preceq (1+\varepsilon)X.$$

This result is remarkable because the number of terms in the 'sparsified' sum is not only independent of m, the number of terms in the original decomposition, but is also linear in d. Applying this to the decomposition of L_G given in (1) (or rather of the restriction of L_G to the orthogonal complement of $(1,1,\ldots,1)$), shows that it is possible to find ε-spectral sparsifiers of dense graphs with $O(|\mathscr{V}|/\varepsilon^2)$ edges. The initial proof of this result (appearing in [1]) focused on the case where the X_i are all of rank one. A generalization to the case of X_i having any rank appears, for instance, in [4].

2 Hyperbolic polynomials and hyperbolicity cones

Let V denote a finite dimensional real vector space. We denote by $\mathbb{R}[V]_d$, polynomials on V with real coefficients that are homogeneous of degree d.

Definition 1. Let V be a finite dimensional real vector space and let $e \in V$. We say that $p \in \mathbb{R}[V]_d$ is *hyperbolic with respect to e* if $p(e) > 0$ and for all $x \in V$, the univariate polynomial $t \mapsto p(te - x)$ has only real zeros.

Associated with a hyperbolic polynomial is the following geometric object.

Definition 2. Let V be a finite dimensional vector space, and let $p \in \mathbb{R}[V]_d$ be hyperbolic with respect to $e \in V$. The *(open) hyperbolicity cone* associated with (p,e) is

$$\Lambda_{++}(p,e) = \{x \in V \;:\; \text{all zeros of } t \mapsto p(te-x) \text{ are positive}\}.$$

We denote by $\Lambda_+(p,e)$ the closure of $\Lambda_{++}(p,e)$. A foundational result of Gårding [2] is that $\Lambda_{++}(p,e)$ is a convex cone (and so $\Lambda_+(p,e)$ is a closed convex cone).

The prototypical example of a hyperbolic polynomial is $p(X) = \det(X)$, the determinant restricted to real symmetric matrices. This polynomial is hyperbolic with respect to $e = I$, the identity matrix, because $\det(tI - X)$, the characteristic polynomial of X, has only real zeros. The associated hyperbolicity cone is the cone of symmetric positive semidefinite matrices.

Generalizing the notation for the positive semidefinite order, given $x, y \in V$ we write $x \succeq_{\Lambda_+} y$ to mean that $x - y \in \Lambda_+(p, e)$.

3 Hyperbolic sparsification conjecture

The natural analogue of spectral sparsification in the setting of hyperbolicity cones, would be the following.

Conjecture 1 (Hyperbolic spectral sparsification). Let V be a finite dimensional real vector space and let $p \in \mathbb{R}[V]_d$ be hyperbolic with respect to $e \in V$. Let $x \in \Lambda_{++}(p, e)$ have a decomposition as $x = \sum_{i=1}^{m} x_i$ where $x_i \in \Lambda_+(p, e)$ for $i = 1, 2, \ldots, m$. Let $0 < \varepsilon < 1$. Then there exists $S \subseteq \{1, 2, \ldots, m\}$ with $|S| \leq \lceil 4d/\varepsilon^2 \rceil$, and positive scalars w_i for $i \in S$, such that

$$(1-\varepsilon)x \preceq_{\Lambda_+} \sum_{i \in S} w_i x_i \preceq_{\Lambda_+} (1+\varepsilon)x.$$

If true, this would directly generalize Theorem 1, and would imply that spectral sparsification depends only on the degree of the hyperbolic polynomial, and is completely independent of the number of terms in the decomposition of x and the dimension of the cone $\Lambda_+(p, e)$.

Spectral sparsification of sums of positive semidefinite matrices has a number of interesting applications to the simplification of semidefinite optimization problems, as discussed in [4]. It would be interesting to investigate what implications Conjecture 1 (if it were true) would have for hyperbolic optimization [3].

References

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