An introduction to decomposition

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Abstract We review work on 'decomposition,' a property of two-dimensional theories with 1-form symmetries and, more generally, d-dimensional theories with $(d-1)$-form symmetries. Decomposition is the observation that such quantum field theories are equivalent to ('decompose into') disjoint unions of other QFTs, known in this context as "universes." Examples include two-dimensional gauge theories and orbifolds with matter invariant under a subgroup of the gauge group. Decomposition explains and relates several physical properties of these theories – for example, restrictions on allowed instantons arise as a "multiverse interference effect" between contributions from constituent universes. First worked out in 2006 as part of efforts to resolve technical questions in string propagation on stacks, decomposition has been the driver of a number of developments since. We give a general overview of decomposition, describe features of decomposition arising in gauge theories, then dive into specifics for orbifolds. We conclude with a discussion of the recent application to anomaly resolution of Wang-Wen-Witten in two-dimensional orbifolds. This is a contribution to the proceedings of the conference *Two-dimensional supersymmetric theories and related topics* (Matrix Institute, Australia, January 2022), giving an overview of a talk given there and elsewhere.

1 Introduction

Briefly, decomposition is the observation that some QFTs with a local action are secretly equivalent to sums (disjoint unions) of other QFTs, known in this context as 'universes.'

When this happens, we say that the QFT decomposes (into its constitutent univeres). Decomposition of the QFT can be applied to give insight into its properties.

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Decomposition was first described in 2006 in [26], where it arose as part of efforts to understand string compactifications on generalizations of spaces known as stacks and gerbes, and resolved some of the apparent physical inconsistencies of those theories. It has since been developed in numerous other papers, see for example [4–8, 13, 14, 16–18, 20, 27, 29, 32, 33, 38, 39, 43–46, 51, 53, 54, 57–59, 66] and discussed in review articles including [48–50, 52].

For one QFT to be a sum (or disjoint union[1]) of other QFTs means that, for example, there exist projection operators: a set of topological operators Π_i that commute with all operators in the theory

$$[\Pi_i, \mathscr{O}] = 0, \tag{1}$$

and with the properties that

$$\Pi_i \Pi_j = \delta_{ij} \Pi_j, \; \sum_i \Pi_i = 1. \tag{2}$$

From (1), the projectors are mutually commuting, so they can be simultaneously diagonalized: the Fock space can be diagonalized into eigenmodes of the projectors Π_i, which are the states of the constituent universes. From (2), one can show that correlation functions are a sum of correlation functions in the constituent theories. Using the properties above, we can write

$$\langle \mathscr{O}_1 \cdots \mathscr{O}_m \rangle = \sum_i \langle \Pi_i \mathscr{O}_1 \cdots \mathscr{O}_m \rangle, \tag{3}$$

$$= \sum_i \langle (\Pi_i \mathscr{O}_1) \cdots (\Pi_i \mathscr{O}_m) \rangle, \tag{4}$$

$$= \sum_i \langle \tilde{\mathscr{O}}_1 \cdots \tilde{\mathscr{O}}_m \rangle_i, \tag{5}$$

where the $\tilde{\mathscr{O}}$ are the projections of the operators $\mathscr{O}$ into the universes.

In practice, beyond exhibiting projection operators, another property will be used frequently in this article in describing and checking decompositions, namely that (on a connected spacetime) the partition function of a disjoint union is the sum of the partition functions of the constituent universes:

$$Z = \sum_i Z_i. \tag{6}$$

In essence, this is because the state space of the disjoint union is a sum of the state spaces of the constituent theories. Formally, if we write

$$Z = \sum_{\text{states}} \exp(-\beta H), \tag{7}$$

then using the fact that the whole state (Fock) space is a sum over state spaces of the constituent universes, we have immediately that

[1] We will use the terms ‘sum’ and ‘disjoint union’ interchangeably in this article.

$$Z = \sum_i \sum_{\text{states in i}} \exp(-\beta H) = \sum_i Z_i. \tag{8}$$

Let us take a moment to distinguish universes from superselection sectors arising in spontaneous symmetry breaking. Briefly, the idea of a superselection sector is that it is a sector of the theory characterized by a vacuum that cannot be shifted perturbatively by local operators, but can be changed by adding energy (broadly proportional to volume). A prototypical example of superselection sectors is the orientation of magnetization (microscopically, spins) in a bar magnet. In that example, local operators cannot change the direction of magnetization; however, by adding energy (broadly proportional to the volume), one can randomize the spins, then as the magnet cools, the spins may align in a new direction. Here, however, it should be noted that the different superselection sectors (magnetizations) are linked to one another – except in deep IR or infinite volume limits, there is a continuous path in field space connecting them. By contrast, in decomposition, one has disjoint QFTs at every energy scale, meaning that there is no continuous path linking states in different universes. A more detailed discussion can be found in [58].

To be clear, disjoint unions can also emerge as deep IR / infinite volume limits in spontaneous symmetry breaking, and also from limits of KK reductions as in [3]. However, once one has a disjoint union of QFTs, to 'rejoin' the theories at higher energies, one must deform by a (typically non-local) operator that bridges the disjoint union. (At the fixed point, if such an operator were local, it would be an irrelevant operator.) The coupling of that operator vanishes in the IR limit, but at higher energies, could result in a single theory. However, without such a deformation, a disjoint union remains disjoint at all energy scales. In any event, we usually reserve the term 'decomposition' for the special case of a disjoint union of quantum field theories described by a local action.

Later we will work through examples of decomposition in detail, but for the moment, let us outline some known examples.

- Orbifolds. Broadly speaking, orbifolds in which a subgroup of the gauge group acts trivially decompose, see e.g. [26, 43–46]. There is a longer story behind why orbifolds with trivially-acting subgroups differ from ordinary orbifolds, see [40–42] for early work, but in any event, much of this review will be devoted to orbifolds, so we will see detailed examples shortly.
- Two-dimensional gauge theories in which a subgroup of the gauge group acts trivially also decompose. Such gauge theories, and their differences from ordinary gauge theories, were discussed in [40–42]. Examples of their decomposition include the following:
  - A two-dimensional $U(1)$ gauge theory with nonminimal charges is equivalent to a sum of $U(1)$ theories with minimal charges [26],

  - A two-dimensional G gauge theory with center-invariant matter is equivalent to a union of $G/Z(G)$ gauge theories (for $Z(G)$ the center of G) with discrete theta angles [51],
  - Two-dimensional pure Yang-Mills theory for gauge group G is equivalent to a sum of invertible field theories, indexed by irreducible representations of G [38, 39] (see also [13] for the abelian case).

- Four-dimensional Yang-Mills theory with a restriction to instantons of degree divisible by $k > 1$ is equivalent to a disjoint union of k ordinary four-dimensional Yang-Mills theories with different theta angles [58]. (This also, correctly, suggests a subtlety involving cluster decomposition, to which we shall return shortly.)
- Unitary two-dimensional topological field theories (with semisimple local operator algebras). It has been known for many years that these are equivalent to disjoint unions of invertible field theories, see e.g. [15, 37], and by utilizing non-invertible higher-form symmetries, it was argued in [32, 33] that these are also examples of decomposition.
- Finally, sigma models on gerbes. Gerbes are examples of stacks, generalizations of spaces which admit metrics, spinors, gauge fields, and everything else one would require to make sense of a sigma model. Sigma models with target stacks were studied in [40–42], in the hope of discovering new string compactifications, new (2,2) SCFTs, and amongst the technical challenges that arose (construction of an action, presentation dependence, naively inconsistent moduli) was, in the case of gerbes, a violation of cluster decomposition. The original motivation for decomposition [26] was to resolve this problem. The resolution observed that a sigma model on a gerbe is equivalent to a disjoint union of sigma models on spaces, solving the issue with cluster decomposition, but also clarifying that one could not construct new (2,2) SCFTs in this fashion.

So far we have outlined a number of rather diverse-looking examples of decomposition, in both two and four dimensions. The reader may well ask, what do these examples have in common?

Briefly, in d spacetime dimensions, a theory decomposes when it has a $(d-1)$-form symmetry. (In two dimensions, this was the point of [26], and it was generalized to higher dimensions in [13, 58].) Thus, decomposition and higher-form symmetries go hand-in-hand.

In this review, we will primarily focus on the case $d = 2$, for which one will have a decomposition if a $(d-1) = 1$-form symmetry is present.

To that end, let us take a moment to briefly review one-form symmetries. For this review, intuitively, a one-form symmetry group is (something like) a group that exchanges nonperturbative sectors.

For example, consider a G gauge theory or orbifold in which the matter/fields are invariant under a subgroup $K \subset G$. For simplicity, let us assume that K is abelian, and in fact lies within the center of G. Then, in this case, there is a permutation symmetry amongst the nonperturbative sectors. Schematically, the path integral is invariant under

$$(G\text{-bundle}) \mapsto (G\text{-bundle}) \otimes (K\text{-bundle}), \tag{9}$$

or if the reader prefers, in terms of gauge fields,

$$A \mapsto A + A', \tag{10}$$

where A is a G-instanton and A' is a K-instanton.

Instead of an action of elements of groups, we have an action of bundles of groups. This structure is almost a group, except that associativity of multiplication only holds up to multiplication. Technically, this is known as a 2-group.

The 2-group whose elements are K-bundles, is denoted either $K^{(1)}$ (recently in physics) or BK (in math). The latter notation has been used for decades, so we will use the notation BK to denote a 2-group of K bundles.

One-form symmetries can also been seen in the algebra of topological local operators, where they are often realized nonlinearly. This is how the decomposition story connects to two-dimensional topological field theories [32, 33], but is beyond the scope of this article.

There are several descriptions of the two-dimensional quantum field theories which we will discuss in this article:

- A gauge theory or orbifold with a trivially-acting subgroup (i.e. a non-complete charge spectrum),
- A theory with a restriction on instantons,
- Sigma models on gerbes,
- A theory with multiple topological local operators.

Decomposition often relates these different pictures.

- For one example, we will see that restrictions on instantons are implemented as a "multiverse interference effect" between the different universes of a decomposition.
- The one-form symmetry of the quantum field theory can be understood in the sigma model language. A 'gerbe' is a fiber bundle whose fibers are 2-groups BK of one-form symmetries. In any case in which the target space of a sigma model is a fiber bundle, the sigma model possesses a global symmetry corresponding to translations along the fibers. For a sigma model whose target is a gerbe, since the fibers are copies of BK, the sigma model has a BK symmetry.
- This is described by gauge theories with trivially-acting subgroups because $BK = [\text{point}/K]$. (In ordinary geometry, a quotient of a point by any group is the same point back again, but the pertinent mathematics keeps track of automorphisms, and so this is different from a point.) Utlimately, a sigma model whose target is a gerbe involves fibering a trivially-acting K gauge theory over an ordinary theory, hence gauge theories with trivially-acting subgroups.

2 Generalities on gauge theories

Suppose we have a two-dimensional G gauge theory, where G is semisimple, and a subgroup K of the center of G acts trivially on all the matter.

As outlined above, this theory has a global BK one-form symmetry, and so one expects that it should decompose.

The projection operators are, schematically, twist fields / Gukov-Witten operators [23, 24] corresponding to elements of the center of the group algebra $\mathbb{C}[K]$. Existence of projectors (idempotents), forming a basis for the center, is a consequence of Wedderburn's theorem (see e.g. [35, section XVII.3]).

In particular, judging from the projectors of Wedderburn's theorem, universes are in one-to-one correspondence with irreducible representations of K.

Now, abstractly, that is a formal argument for a decomposition, but it does not specify the form of the decomposition. In this particular case, decomposition takes the form (see e.g. [51, section 2])

$$\mathrm{QFT}\left(G - \text{gauge theory}\right) = \coprod_{\theta\in\hat{K}} \mathrm{QFT}\left(G/K - \text{gauge theory with discrete theta angle } \theta\right). \tag{11}$$

For example, $SU(2)$ gauge theory with center invariant matter is the disjoint union of a pair of $SO(3)$ theories, schematically

$$SU(2) = SO(3)_+ + SO(3)_-, \tag{12}$$

where the $\pm$ denotes the ($\mathbb{Z}_2$-valued) discrete theta angle coupling to the second Stiefel-Whitney class $w_2 \in H^2(\mathbb{Z}_2)$.

Perturbatively, the $SU(2)$ and $SO(3)_\pm$ theories are identical, but nonperturbatively, they differ. Specifically, there are more $SO(3)$ instantons (bundles) than $SU(2)$ instantons (bundles). The effect of the discrete theta angle is to weight the non-$SU(2)$ $SO(3)$ instantons by a sign, so that when the partition functions for the $SO(3)_\pm$ theories are added, contributions from non-$SU(2)$ $SO(3)$ instantons cancel out between the two theories, leaving only $SU(2)$ instantons – consistent with decomposition.

We can describe this more formally as follows. Write the partition function of the disjoint union as

$$Z = \sum_{\theta\in\hat{K}} \int [DA] \exp(-S) \exp\left(\theta \int w_2(A)\right), \tag{13}$$

where we use $w_2 \in H^2(K)$ to denote the degree-two characteristic class of G/K bundles. Now, moving the summation inside the path integral, we have

$$Z = \int [DA] \exp(-S) \left(\sum_{\theta\in\hat{K}} \exp\left(\theta \int w_2(A)\right)\right). \tag{14}$$

However,

$$\sum_{\theta \in \hat{K}} \exp\left(\theta \int w_2(A)\right) \tag{15}$$

is proportional[2] to a projection operator, projecting out instantons (bundles) for which $w_2 \neq 0$, leaving only those bundles which exist in the G gauge theory.

In effect, an interference effect between the universes of decomposition – a "multiverse interference effect" – has cancelled out some of the nonperturbative sectors.

As a quick consistency check, let us compare to pure $SU(2)$ Yang-Mills theory in two dimensions, for which in essence everything is computable [36, 47, 64]. We will check a decomposition due to the $B\mathbb{Z}_2$ center symmetry [51, section 2.4]; a more extreme decomposition (to invertible field theories, utilizing noninvertible higher-form symmetries) was discussed in [38, 39].

Consider partition functions in pure Yang-Mills theory in two dimensions. From [36, 47, 64], the partition functions of the pure $SU(2)$ and the pure $SO(3)$ theory without a discrete theta angle (denoted $SO(3)_+$), the partition functions are of the form

$$Z(G) = \sum_R (\dim R)^{2-2g} \exp\left(-AC_2(R)\right), \tag{16}$$

where G is the gauge group ($SU(2)$ or $SO(3)$ here), g is the genus of the two-dimensional spacetime, A its area, $C_2(R)$ the second Casimir, and the sum is over all irreducible representations of G. We also need the partition function of the $SO(3)_-$ theory; this was computed in [55], and takes the same form as above, namely

$$Z(SO(3)_-) = \sum_R (\dim R)^{2-2g} \exp\left(-AC_2(R)\right), \tag{17}$$

where the sum is now over representations of $SU(2)$ that are not representations of $SO(3)$, a complementary sum to that appearing in the $SO(3)_+$ partition function. Assembling these pieces, we see immediately that adding a sum over $SO(3)$ representations to a sum over $SU(2)$ representations minus $SO(3)$ representations, we get

$$Z(SU(2)) = Z(SO(3)_+) + Z(SO(3)_-), \tag{18}$$

We have discussed a number of theories with properties such as a restriction on instantons, and multiple identity operators, properties which are often taken to signal a violation of cluster decomposition (see e.g. [63, section 23.6]). Especially as cluster decomposition is sometimes used interchangeably with locality, this is often taken to indicate a sickness or inconsistency of the theory.

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2 The proportionality factor reflects that fact that the G gauge theory has more gauge transformations than the G/K gauge theory, so the path integrals have, in principle, slightly different normalization factors.

However, that is an oversimplification. For example, cluster decomposition can also be violated in spontaneous symmetry breaking, in infinite-volume limits. This does not imply an inconsistency of the theory, but rather is just a statement about the vacuum. Perhaps cluster decomposition is best thought of as a property of vacua, and locality a property of the theory, and as these examples illustrate, these are distinct notions, not interchangeable with one another.

In any event, the theories we are describing are manifestly local, in that they have local Lagrangians, and the separate universes are perfectly consistent.

3 Consistency tests and applications

Since 2005, decomposition has been checked in a wide variety of examples, in a wide variety of different kinds of examples, and in many ways. We list a few examples below:

- Gauged linear sigma models (GLSMs): Decomposition has been checked in gauged linear sigma models via mirror symmetry and in quantum cohomology rings (the latter through Coulomb branch computations). For abelian GLSMs, this was described in [40, 41] using Hori-Vafa mirrors [31]; for nonabelian GLSMs, this was checked in the papers describing nonabelian mirror constructions [11, 19, 21, 22].
- In orbifolds, decomposition has been checked extensively [26, 43–46] in, for example, partition functions and massless spectra, as we will outline later in this article.
- Decomposition has also been checked in open strings and K theory [26]. Briefly, in a gauge theory in which a subgroup K of the gauge group acts trivially on bulk degrees of freedom, K can still act nontrivially on boundary degrees of freedom, which therefore organize according to irreducible representations of K, precisely matching the description of universes earlier. In this picture, from gauge invariance, open string states can only exist on open strings connecting the same irreducible representations of K – meaning, that there are no open strings connecting different universes. This also can be understood in terms of K theory [65]. As discussed in [26], K theory on gerbes is equivalent to (twisted) K theory on a disjoint union of spaces, following the same pattern as decomposition.
- In supersymmetric gauge theories in two dimensions, supersymmetric localization can be applied to give further tests of decomposition, as discussed in [51].
- In nonsupersymmetric pure Yang-Mills in two dimensions, decomposition can also be checked. We have previously outlined tests of decomposition along center one-form symmetries, which are described in greater detail in [51]. In addition, there exists a more extreme decomposition of nonsupersymmetric pure

Yang-Mills to a disjoint union of invertible field theories, indexed by irreducible representations of the gauge group [38, 39].
- Decomposition in adjoint QCD_2 was studied in [33].
- Decomposition has been checked numerically in lattice gauge theory [29].
- Finally, lest we give a different impression, decomposition is not restricted to two-dimensional theories, but has also been studied in other dimensions, see e.g. [13, 14, 58].

We should also mention that decomposition has a number of applications:

- The original application was to understand and resolve certain technical issues in making sense of sigma models whose targets are generalized spaces known as 'stacks' [40–42]. This was part of a program of trying to construct new string compactifications, new conformal field theories. After one understands basic issues such as the construction of an action to describe such a sigma model, potential presentation-dependence issues, and moduli mismatches, more subtle issues remain. For example, in the special cases of stacks known as 'gerbes' (fiber bundles whose fibers are one-form symmetry groups), the sigma models violated cluster decomposition. Understanding this issue was the original motivation for work on decomposition, and the resolution was that such sigma models are equivalent to disjoint unions of sigma models on ordinary spaces. As a result, we were not able to construct new (2,2) supersymmetric SCFTs, though we did learn about decomposition. (In the more general case of (0,2) SCFTs for gerbes, it is still an open question of whether new string compactifications exist, see e.g. [4].)
- Decomposition makes predictions for Gromov-Witten theory, specifically the Gromov-Witten theory of stacks and gerbes [1, 9, 10]. From decomposition, the Gromov-Witten theory of a gerbe must match that of a disjoint union of spaces, and this was checked rigorously and discussed in e.g. [5–7, 17, 57, 59].
- In gauged linear sigma models, decomposition was used to provide a novel nonperturbative construction of branched double covers in [8], giving examples of GLSMs with nonbirational phases, realizing examples of Kuznetsov's homological projective duality [34]. This construction has been utilized in the GLSM community in a number of examples since, see e.g. [12, 25, 30] for a few examples.
- Applications to computing elliptic genera of pure gauge theory, studying IR limits of pure supersymmetric gauge theories in two dimensions [2], were discussed in [16].
- Recently decomposition has been applied [44–46] to understand and simplify the Wang-Wen-Witten anomaly resolution proposal [62], as we shall discuss in section 7.

4 Multiverse interference, portals, and wormholes

Next, let us summarize some of the more entertaining features of decomposition:

- *Multiverse interference effects.* We have already seen that summing over universes has the effect of projecting out some nonperturbative contribution, hence a "multiverse interference effect." Our primary exmaples was of a two-dimensional $SU(2)$ gauge theory with center-invariant matter, for which, schematically,

$$\mathrm{QFT}\,(SU(2)) \;=\; \mathrm{QFT}\,(SO(3)_+) \coprod \mathrm{QFT}\,(SO(3)_-)\,. \tag{19}$$

- *Fundamentally-charged Wilson lines are defects bridging universes.* Consider for example two-dimensional abelian BF theory at level k. This theory decomposes into a disjoint union of k invertible field theories. The projectors are

$$\Pi_m \;=\; \frac{1}{k}\sum_{n=0}^{k-1} \xi^{nm}\mathscr{O}_n, \tag{20}$$

  where $\xi = \exp(2\pi i/k)$ and

$$\mathscr{O}_n \;=: \exp\,(nB) : . \tag{21}$$

  These local operators have clock-shift commutation relations with the Wilson lines (see e.g. [27])

$$\mathscr{O}_p W_q \;=\; \xi^{pq} W_q \mathscr{O}_p, \tag{22}$$

  which algebraically are equivalent to

$$\Pi_m W_p \;=\; W_p \Pi_{m+p \mod k}. \tag{23}$$

  Thus, moving a projector past a Wilson line changes the projector, and so Wilson lines in abelian BF theory act as (nondynamical) defects bridging different universes.
- *Wormholes between universes.* This is how GLSMs realize branched double covers nonperturbatively [8]. Consider for example a GLSM with gauge group $U(1)$, two chiral superfields p_a of charge $+2$ and four chiral superfields ϕ_i of charge -1, with a superpotential

$$W \;=\; \sum_{ij} \phi_i \phi_j A^{ij}(p). \tag{24}$$

  In the phase $r \ll 0$, this describes a branched double cover of $\mathbb{P}^1$, where the sheets of the cover are (approximate) universes, spanned by the p fields (of nonminimal charge), and the branch locus is the region where the mass matrix A^{ij} develops zero eigenvalues, forming a Euclidean wormhole.

5 Specifics on orbifolds

Now, let us turn to examples of decomposition in two-dimensional orbifolds.

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially, and let $G = \Gamma/K$. We can write Γ as

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1. \tag{25}$$

Decomposition is known for general extensions [26], but for simplicity in this overview, let us assume for the moment that this is a central extension, so that K is a subset of the center of Γ. Let $[\omega] \in H^2(G,K)$ denote the element of group cohomology classifying the extension.

In this case, if K acts trivially on X and lies within the center of Γ, then this orbifold decomposes as

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\coprod_{\rho \in \hat{K}} [X/G]_{\hat{\omega}(\rho)}\right), \tag{26}$$

where $\hat{K}$ denotes the set of isomorphism classes of irreducible representations of K, and $\hat{\omega}(\rho)$ represent discrete torsion phases, essentially finite-group analogues of theta angles, which is the image of the extension class under ρ:

$$H^2(G,K) \stackrel{\rho}{\longrightarrow} H^2(G,U(1)),$$
$$\omega \mapsto \omega \circ \rho = \hat{\omega}(\rho).$$

This will be a finite-group analogue of the $SU(2)$ decomposition described earlier, with $[X/\Gamma]$ playing the analogue of the $SU(2)$ theory and the $[X/G]_{\hat{\omega}(\rho)}$ theories playing the analogue of the $SO(3)_{\pm}$ theories.

To justify this decomposition, we must first provide projectors. Corresponding to any irreducible representation $R \in \hat{K}$, the projector is [54]

$$\Pi_R = \sum_i \frac{\dim R_i}{|K|} \sum_{k \in K} \chi_{R_i}\left(k^{-1}\right) \tau_k, \tag{27}$$

where τ_k is a twist field for the trivially-acting element $k \in K$, $\chi_R(g)$ denotes the character of g in representation R, and the R_i are a set of representatives of the irreducible representations. (For a detailed examination of why trivially-acting group elements have associated twist fields, and the unitarity violations that ensue if one assumes otherwise, see [40].) It can be shown that these projectors have the expected properties, namely

$$\Pi_R \Pi_S = \delta_{R,S} \Pi_R, \ \sum_R \Pi_R = 1. \tag{28}$$

As the twist fields in this case can be understood formally as the center of the group algebra, this expression for the projector is a formal consequence of Wedderburn's theorem in mathematics (see e.g. [35, section XVII.3]).

To make this more concrete, let us examine all the details in one particular example (taken from [26, section 5.2]). Take $\Gamma = D_4$, the eight-element dihedral group, with center $K = \mathbb{Z}_2$, which we will assume acts trivially on X. In this case, decomposition (26) predicts

$$\mathrm{QFT}\left([X/D_4]\right) \,=\, \mathrm{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\mathrm{w/o\,d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\mathrm{d.t.}}\right), \tag{29}$$

a disjoint union of two $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifolds, one with discrete torsion, and the other without.

In passing, note that this is a very precise analogue of the earlier example of an $SU(2)$ gauge theory (compare $[X/D_4]$) decomposing into a pair of $SO(3)$ theories (compare $[X/\mathbb{Z}_2\times\mathbb{Z}_2]$).

Now, let us check this statement. First, we consider projection operators. If let let $z\in D_4$ denote the generator of the (trivially-acting) $\mathbb{Z}_2$ center, and $\hat{z}$ the corresponding twist field, then $\hat{z}^2=1$, and the projectors are

$$\Pi_\pm \,=\, \frac{1}{2}\left(1\pm\hat{z}\right), \tag{30}$$

which obey

$$\Pi_\pm^2 \,=\, \Pi_\pm,\;\; \Pi_\pm\Pi_\mp \,=\, 0,\;\; \Pi_+ + \Pi_- \,=\, 1. \tag{31}$$

These projectors are in principle the specialization of (27) to this case, but in fact here are sufficiently simple that they can be seen by inspection, as indeed was the case in [26].

So far we have produced a pair of local projection operators, which tell us that the theory breaks into two pieces, but to verify the decomposition 29, we need more information about the pieces. To that end, we will next compute the partition function of this orbifold.

To compute the partition function, we need to describe the dihedral group more explicitly. Briefly, it is generated by elements z (generating the center), a, and b, whose products we list below:

$$D_4 \,=\, \{1,z,a,b,az,bz,ab,ba=abz\}. \tag{32}$$

Let us compute the partition function on T^2. For any orbifold $[X/\Gamma]$, the partition function on T^2 is

$$Z_{T^2}\left([X/\Gamma]\right) \,=\, \frac{1}{|\Gamma|}\sum_{gh=hg} Z_{g,h}, \tag{33}$$

where each $Z_{g,h}$ represents the path integral contribution from the Γ nonperturbative sector ("twisted sector") defined by the commuting pair $g,h\in G$. (Since Γ is finite, there is no perturbative contribution to the Γ gauge theory, only nonperturbative contributions.) Schematically, each $Z_{g,h}$ is a path integral sum over maps $T^2\to X$ with branch cuts defined by g,h:

$$Z_{g,h} \,=\, \left({}_{g}\underset{h}{\square} \longrightarrow X \right) \tag{34}$$

(In order for the corners of the square to close, one only sums over commuting $g,h\in G$.)

We will argue that

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{d.t.}}\right) \tag{35}$$

verifying decomposition (29).

To that end, first note that since z acts trivially and each $Z_{g,h}$ only depends upon boundary conditions, we immediately have that

$$Z_{g,h} = {}_{g}\underset{h}{\square} = {}_{gz}\underset{h}{\square} = {}_{g}\underset{hz}{\square} = {}_{gz}\underset{hz}{\square} = Z_{gz,hz}. \tag{36}$$

Each square ${}_{g}\square_{h}$ can be associated (modulo automorphisms) with a Γ bundle, so this is a symmetry amongst the nonperturbative sectors of the Γ orbifold. Furthermore, those sectors are related by tensoring in a $B\mathbb{Z}_2$ bundle:

$$Z_{g,h} = {}_{g}\underset{h}{\square} \xrightarrow{{}_{z}\blacksquare_{1}} {}_{gz}\underset{h}{\square} \xrightarrow{{}_{z}\blacksquare_{z}} {}_{g}\underset{hz}{\square} \xrightarrow{{}_{z}\blacksquare_{1}} {}_{gz}\underset{hz}{\square} = Z_{gz,hz} \tag{37}$$

This is the $B\mathbb{Z}_2$ one-form symmetry, explicitly.

Next, to help clarify, write the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 = D_4/\mathbb{Z}_2$ as

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{a}\overline{b}\}, \tag{38}$$

where $\overline{a}$ is the projection of $\{a, az\}$, and $\overline{b}$ is the projection of $\{b, bz\}$. Then, we see that each D_4 twisted sector ($Z_{g,h}$) that appears is the same as a $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sector, *except* for the sectors

$${}_{\overline{a}}\underset{\overline{b}}{\square}, \; {}_{\overline{a}}\underset{\overline{a}\overline{b}}{\square}, \; {}_{\overline{b}}\underset{\overline{a}\overline{b}}{\square}, \tag{39}$$

which do not appear, because their lifts do not commute in D_4. (These form a modular orbit – modular invariance is ensured at every step.)

This is a restriction on the nonperturbative sectors.

So far, we have argued that

$$\begin{aligned} Z_{T^2}\left([X/D_4]\right) &= \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - (\text{some twisted sectors})\right), \\ &= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - (\text{some twisted sectors})\right). \end{aligned} \tag{40}$$

In particular, despite the fact that the $\mathbb{Z}_2$ acts trivially, this is a different theory than the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. *Physics knows when we gauge even a trivially-acting group.*

We can simplify the expression above via the use of discrete torsion [60], which is a set of modular-invariant phases that one can add to partition functions – in essence, theta angles for the finite gauge theory. The new partition function in a G orbifold on T^2, for example, has the form

$$Z_{T^2}([X/G]) = \frac{1}{|G|} \sum_{gh=hg} \varepsilon(g,h) Z_{g,h}, \tag{41}$$

where $\varepsilon(g,h)$ represent the discrete torsion phases.

Now, in a G orbifold, possible choices of discrete torsion phases are classified by $H^2(G,U(1))$. In the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$,

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2, \tag{42}$$

and the twisted sectors that get a phase (specifically, a sign) are

$$\overline{a} \mathop{\square}_{\overline{b}}, \quad \overline{a} \mathop{\square}_{\overline{a}\overline{b}}, \quad \overline{b} \mathop{\square}_{\overline{a}\overline{b}}, \tag{43}$$

the same sectors that are omitted from the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold partition functino in the description of the D_4 orbifold partition function in (40). Thus, we see that the D_4 partition function on T^2 can be rewritten as

$$Z_{T^2}([X/D_4]) = Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) + Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\rm d.t.}). \tag{44}$$

This matches the prediction of decomposition (29) in this case.

In particular, adding the universes has the effect of cancelling out some of the nonperturbative sectors, namely those listed in (39). This is an example of a *multiverse interference effect.*

So far, we have verified that partition functions on T^2 reproduce decomposition. Analogous computations on higher-genus Riemann surfaces also reproduce decomposition, though the combinatorics is more complex. See [26, section 5.2] for details.

Now, let us turn to massless spectrum computations. (In principle, this is all implicit in the partition function computations, but we find it instructive to explicitly study this particular facet.) For the case $X = T^6$, with a standard $\mathbb{Z}_2 \times \mathbb{Z}_2$ action [61], the massless spectrum of $[T^6/D_4]$ is easily computed and given by

$$\begin{array}{ccccccc} & & & 2 & & & \\ & & 0 & & 0 & & \\ & 0 & & 54 & & 0 & \\ 2 & & 54 & & 54 & & 2 \\ & 0 & & 54 & & 0 & \\ & & 0 & & 0 & & \\ & & & 2 & & & \end{array} \tag{45}$$

This result is problematic – the 2's in the corners signal a violation of cluster decomposition, which ordinarily would be reason to believe that the result is incorrect. However, as we have seen, cluster decomposition arises in any theory describing a disjoint union of QFTs, and so this is not unexpected. Hand-in-hand, the 2's can also be interpreted to mean that the theory has two components. In particular, the massless spectra of the $[T^6/\mathbb{Z}_2 \times \mathbb{Z}_2]$ orbifolds, with and without discrete torsion, are given by [61]

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 51 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 51 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} + \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 51 & & 51 & & 1 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{46}$$

It is easy to see that the sum of the two $[T^6/\mathbb{Z}_2 \times \mathbb{Z}_2]$ orbifold spectra matches that of $[T^6/D_4]$, verifying decomposition (29).

This example was not a one-off, but in fact verifies the general prediction of [26] for orbifolds with trivially-acting subgroups.

In most of this review we have focused on cases of gauge theories and orbifolds in which the trivially-acting subgroup is in the center, but more general examples exist and have been studied. Decomposition in the more general case is the statement [26]

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right), \tag{47}$$

where the discrete torsion on universes $\hat{\omega}$ is described in [26]. In the general case, where the trivially-acting $K \subset \Gamma$ need not be central, $G = \Gamma/K$ can act nontrivially on the set of isomorphism classes of irreducible representations $\hat{K}$, and the universes are identified with orbits of G in $\hat{K}$. In the special case of central extensions, the G action on $\hat{K}$ is trivial, the orbits are single elements of $\hat{K}$, and the decomposition reduces to a disjoint union of copies of $[X/G]_{\hat{\omega}}$, indexed by $\hat{K}$, as described earlier in (26).

For example, consider the orbifold $[X/\mathbb{H}]$, where $\mathbb{H}$ is the eight-element group of unit quaternions, and where $K = \langle i \rangle \subset \mathbb{H}$ acts trivially. In this case, the center of $\mathbb{H}$ is $\mathbb{Z}_2$, which is contained within $K \cong \mathbb{Z}_4$, but K also has elements that are not central. On the set $\hat{K}$, G leaves two elements invariant but exchanges two elements, so that there are a total of three G orbits, and three universes.

As is discussed in detail in [26, section 5.4], in this case decomposition predicts

$$\mathrm{QFT}\left([X/\mathbb{H}]\right) \,=\, \mathrm{QFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2]\right), \tag{48}$$

which is checked by exhibiting projection operators, computing partition functions at arbitrary genus, and comparing massless spectra. In particular, the universes in this example are not all just the same orbifold with different choices of discrete torsion, but rather they are qualitatively different from one another.

So far we have outlined decomposition in ordinary orbifolds with trivially-acting subgroups. Next we outline decomposition in such orbifolds in the presence of discrete torsion, following [43].

Consider an orbifold $[X/\Gamma]_\omega$, where $K \subset \Gamma$ acts trivially, $\omega \in H^2(\Gamma, U(1))$ is an element of discrete torsion, and we define $G = \Gamma/K$, so that

$$1 \longrightarrow K \stackrel{\iota}{\longrightarrow} \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1. \tag{49}$$

For simplicity, we assume that this is a central extension (that K maps to a subgroup of the center of Γ). It will be helpful to utilize the maps below:

$$H^2(G,U(1)) \stackrel{\pi^*}{\longrightarrow} \left(\mathrm{Ker}\,\iota^* \subset H^2(\Gamma,U(1))\right) \stackrel{\beta}{\longrightarrow} H^1(G,H^1(K,U(1))) = \mathrm{Hom}(G,\hat{K}). \tag{50}$$

Then, we can describe decomposition of $[X/\Gamma]_\omega$ in terms of the following cases:

1. If $\iota^*\omega \neq 0$, then

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) \,=\, \mathrm{QFT}\left(\left[\frac{X \times \hat{K}_{\iota^*\omega}}{G}\right]_{\hat{\omega}}\right). \tag{51}$$

2. If $\iota^*\omega = 0$ and $\beta(\omega) \neq 0$, then

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) \,=\, \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\,\beta(\omega)}}{\mathrm{Ker}\,\beta(\omega)}\right]_{\hat{\omega}}\right). \tag{52}$$

3. If $\iota^*\omega = 0$ and $\beta(\omega) = 0$, then $\omega = \pi^*\overline{\omega}$ for some $\overline{\omega} \in H^2(G,U(1))$ and

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) \,=\, \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G}\right]_{\hat{\omega}+\overline{\omega}}\right), \tag{53}$$

   essentially the same decomposition as in the case of no discrete torsion, but with the discrete torsion on the universes shifted by $\overline{\omega}$.

This description was developed more systematically in [43], which also checked the results in numerous examples.

6 Quantum symmetries in noneffective orbifolds

As one final prerequisite before describing the Wang-Wen-Witten anomaly resolution procedure in orbifolds [62], we describe some novel modular-invariant phases that one can add to noneffective orbifolds, which were deemed ‘quantum symmetries’ in [46] (see also [56]) as they generalize the notion of quantum symmetries in orbifolds from the late 1980s.

Briefly, these phases describe actions on twisted sector states. Consider an orbifold $[X/\Gamma]$ in which a subgroup $K \subset \Gamma$ acts trivially, and define $G = \Gamma/K$, as before. For simplicity, assume that K is central in Γ, so that

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \tag{54}$$

is a central extension. In this case, the quantum symmetries (as the term is used in [46]) are classified by $H^1(G, H^1(K, U(1))) = \mathrm{Hom}(G, \hat{K})$, and describe an action of K on G-twisted sector states. Schematically, in terms of path integral data,

$${}_{gz}\square_{h} = B(\pi(h), z) \left({}_{g}\square_{h} \right), \tag{55}$$

for $B \in H^1(G, H^1(K, U(1)))$, $z \in K$, and $g, h \in \Gamma$.

It will be useful to note that the gruop classifying quantum symmetries fits into an exact sequence [28]

$$\left(\mathrm{Ker}\, \iota^* \subset H^2(\Gamma, U(1))\right) \stackrel{\beta}{\longrightarrow} H^1(G, H^1(K, U(1))) \stackrel{d_2}{\longrightarrow} H^3(G, U(1)). \tag{56}$$

(Technically, this is part of a seven-term sequence slightly extending the inflation-restriction sequence.) The map d_2 is a differential in the Lyndon-Hochschild-Serre spectral sequence, and for any $\omega \in \mathrm{Ker}\, \iota^*$,

$$\beta(\omega)(\pi(g), z) = \frac{\omega(g, z)}{\omega(z, g)}, \tag{57}$$

for $g \in \Gamma$, $z \in K$. (This is the same map β that appeared in (50).)

The first term in the sequence above can be interpreted in terms of discrete torsion, and as in two-dimensional orbifolds $[X/G]$ gauge anomalies are counted by $H^3(G, U(1))$, the last term can be interpreted in terms of anomalies, so the sequence (56) can be represented schematically as

$$(\text{discrete torsion}) \stackrel{\beta}{\longrightarrow} (\text{quantum symmetries}) \stackrel{d_2}{\longrightarrow} (\text{anomalies})\,. \tag{58}$$

As this sequence suggests, those quantum symmetries in the image of β are equivalent to choices of discrete torsion, and in fact are equivalent to quantum symmetries in the older sense of the term.

For applications to Wang-Wen-Witten, we will need quantum symmetries B such that $d_2(B) \neq 0$. As the sequence above is exact, such quantum symmetries are necessarily not equivalent to discrete torsion.

Before going on, let us briefly describe decomposition in orbifolds with quantum symmetries. Briefly,

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \,=\, \mathrm{QFT}\left(\coprod_{\widehat{\mathrm{Coker}\,B}} [X/\mathrm{Ker}\,B]_{\hat{\omega}} \right), \tag{59}$$

where we interpret the quantum symmetry B as an element of $\mathrm{Hom}(G,\hat{K})$. Decomposition in this case is more or less uniquely dictated by results for decomposition in orbifolds with discrete torsion such that $\iota^*\omega = 0$ and $\beta(\omega) \neq 0$, as described earlier in (52). It was also checked in numerous examples in [46].

7 Wang-Wen-Witten anomaly resolution procedure

In [62], Wang, Wen, and Witten proposed an algorithm to resolve gauge anomalies in anomalous orbifolds $[X/G]$. In this section we will review that procedure, and then observe how decomposition gives an alternative interpretation of the result that clarifies why the procedure removes the anomaly.

We will use the fact that in two dimensions, gauge anomalies in finite G gauge theories are classified by elements of $H^3(G,U(1))$. In the anomalous orbifold $[X/G]$, we will let $\alpha \in H^3(G,U(1))$ denote the anomaly.

The Wang-Wen-Witten procedure has two steps:

1. We replace G by a larger group Γ,

$$1 \longrightarrow K \longrightarrow \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1, \tag{60}$$

   which we will assume is a central extension, and where K acts trivially.
   From decomposition, if all we did was to replace G by Γ, we would not have resolved the orbifold, as physically $[X/\Gamma]$ is equivalent to copies and covers of $[X/G]$, as we have seen previously.
2. The second step of the Wang-Wen-Witten procedure is to turn on a quantum symmetry phase $B \in H^1(G,H^1(K,U(1)))$, chosen so that $d_2B = \alpha$. This implies that $\pi^*\alpha \in H^3(\Gamma,U(1))$ is trivial.

These two choices together – an extension Γ plus a choice of suitable quantum symmetry B – resolve the anomaly.

From decomposition, we can see how the anomaly is resolved. Recall from (59 that

$$\mathrm{QFT}\left([X/\Gamma]_B\right) \,=\, \mathrm{QFT}\left(\coprod_{\widehat{\mathrm{Coker}\,B}} [X/\mathrm{Ker}\,B]_{\hat{\omega}} \right). \tag{61}$$

As we chose the quantum symmetry B so that $d_2 B = \alpha$, we have immediately that

$$\alpha|_{\mathrm{Ker} B} = 0. \tag{62}$$

Thus, each orbifold $[X/\mathrm{Ker} B]$ is automatically anomaly-free.

Put another way, the result of the Wang-Wen-Witten procedure – replacing $[X/G]$ by a larger orbifold $[X/\Gamma]_B$ – is equivalen to replacing $[X/G]$ by a collection of smaller orbifolds $[X/\mathrm{Ker} B]$, in which Ker $B \subset G$ is non-anomalous.

Let us see this explicitly in examples. We will consider several resolutions of orbifolds of the form $[X/G]$ for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. For this group, $H^3(G, U(1)) = (\mathbb{Z}_2)^3$, corresponding to the three $\mathbb{Z}_2$ subgroups, so if we write

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}, \tag{63}$$

then

$$H^3(G, U(1)) = (\mathbb{Z}_2)^3 = \langle a \rangle \times \langle b \rangle \times \langle ab \rangle. \tag{64}$$

To apply the Wang-Wen-Witten procedure, one must make two choices,

- a choice of larger gauge group Γ, and
- a choice of quantum symmetry.

We will list several examples of larger group Γ, and for each Γ, all possible choices of quantum symmetry and the resulting theories.

For our first resolution, we take $\Gamma = D_4$,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow D_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1. \tag{65}$$

The quantum symmetry B is determined by its image on the generators $\{a, b\}$, and we list all possibilites in table 1 (including cases in which we turn on discrete torsion in the $[X/\Gamma]$ orbifold, in addition to the quantum symmetry).

Table 1 A list of all possible quantum symmetries, anomalies resolved, and corresponding orbifolds $[X/\Gamma]_B$ for the case $\Gamma = D_4$. The first two columsn describe the quantum symmetry; the column $d_2(B)$ gives the image of the quantum symmetry, and hence the anomaly that can be resolved; and the last two columns give the physical theory equivalent to $[X/\Gamma]_B$, for either choice of discrete torsion in the $\Gamma = D_4$ orbifold.

$B(a)$	$B(b)$	$d_2(B)$	$[X/\Gamma]_B$ w/o d.t.	$[X/\Gamma]_B$ with d.t.
1	1	–	$[X/G] \coprod [X/G]_{\mathrm{d.t.}}$	$[X/\langle b \rangle]$
-1	1	–	$[X/\langle b \rangle]$	$[X/G] \coprod [X/G]_{\mathrm{d.t.}}$
1	-1	$\langle b \rangle$	$[X/\langle b \rangle]$	$[X/\langle ab \rangle]$
-1	-1	$\langle b \rangle$	$[X/\langle ab \rangle]$	$[X/\langle a \rangle]$

The first row of table 1 describes the case of no quantum symmetry. Nothing is resolved, and the physical theories are (copies of) the G orbifold. The last two rows are more interesting. These describe cases in which an anomaly in the subgroup $\langle b \rangle \subset G$ can be resolved. The resulting physical theories, listed in the last two columns (corresponding to either choice of discrete torsion in the $\Gamma = D_4$ orbifold) are orbifolds by subgroups not containing $\langle b \rangle$, and so by assumption are anomaly-free. In particular, we see that the Wang-Wen-Witten prescription works, explicitly.

For our next resolution of the anomalous $[X/G]$ orbifold, for the same G as before, we take $\Gamma = \mathbb{H}$, the eight-element group of unit quaternions,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1. \tag{66}$$

We list all possibilites for the quantum symmetry in table 2.

Table 2 A list of all possible quantum symmetries, anomalies resolved, and corresponding orbifolds $[X/\Gamma]_B$ for the case $\Gamma = \mathbb{H}$. The first two columsn describe the quantum symmetry; the column $d_2(B)$ gives the image of the quantum symmetry, and hence the anomaly that can be resolved; and the last column gives the physical theory equivalent to $[X/\Gamma]_B$. (No discrete torsion is possible for this choice of Γ.)

$B(a)$	$B(b)$	$d_2(B)$	$[X/\Gamma]_B$
1	1	–	$[X/G] \coprod [X/G]_{\text{d.t.}}$
-1	1	$\langle a \rangle, \langle ab \rangle$	$[X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$[X/\langle a \rangle]$
-1	-1	$\langle a \rangle, \langle b \rangle$	$[X/\langle ab \rangle]$

The first row of table 2 describes the case of no quantum symmetry. In this case, all of the nontrivial quantum symmetries can be used to resolve an anomaly, and in each case, the resulting physical theory $[X/\Gamma]_B$ is equivalent to an orbifold which does not intersect an anomalous subgroup. Again, we see that the Wang-Wen-Witten prescription works.

For our next resolution of the anomalous $[X/G]$ orbifold, for the same G as before, we take $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1. \tag{67}$$

We list all possibilites for the quantum symmetry in table 3.

Table 3 A list of all possible quantum symmetries, anomalies resolved, and corresponding orbifolds $[X/\Gamma]_B$ for the case $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$. The first two columsn describe the quantum symmetry; the column $d_2(B)$ gives the image of the quantum symmetry, and hence the anomaly that can be resolved; and the last two columns give the physical theory equivalent to $[X/\Gamma]_B$, for either choice of discrete torsion in the Γ orbifold.

$B(a)$	$B(b)$	$d_2(B)$	$[X/\Gamma]_B$ w/o d.t.	$[X/\Gamma]_B$ with d.t.
1	1	–	$[X/G] \coprod [X/G]$	$[X/G]_{\text{d.t.}} \coprod [X/G]_{\text{d.t.}}$
-1	1	$\langle ab \rangle$	$[X/\langle b \rangle]$	$[X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$[X/\langle a \rangle]$	$[X/\langle a \rangle]$
-1	-1	$\langle b \rangle$	$[X/\langle ab \rangle]$	$[X/\langle ab \rangle]$

The details here are different, but we see the same overall pattern. As before, the first row corresponds to the case of no quantum symmetry. In each of the next three rows, we see that it is possible to resolve an anomaly, and if one chooses B such that $d_2(B)$ describes the anomaly, then the resulting physical theory $[X/\Gamma]_B$ is equivalent to an orbifold by a subgroup which does not contain the anomalous subgroup. As before, Wang-Wen-Witten works.

So far we have picked 'minimal' resolutions. For our next resolution of the anomalous $[X/G]$ orbifold, for the same G as before, we take $\Gamma = \mathbb{Z}_2 \times \mathbb{H}$,

$$1 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{H} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1, \tag{68}$$

so that in effect there is an extra $\mathbb{Z}_2$. We list all possibilities for the quantum symmetry in table 4.

Table 4 A list of all possible quantum symmetries, anomalies resolved, and corresponding orbifolds $[X/\Gamma]_B$ for the case $\Gamma = \mathbb{Z}_2\mathbb{H}$. The first two columsn describe the quantum symmetry; the column $d_2(B)$ gives the image of the quantum symmetry, and hence the anomaly that can be resolved; and the last column gives the physical theory equivalent to $[X/\Gamma]_B$.

$B(a)$	$B(b)$	$d_2(B)$	$[X/\Gamma]_B$
1	1	–	$\coprod_2 \left([X/G] \coprod [X/G]_{\text{d.t.}}\right)$
-1	1	$\langle a \rangle, \langle ab \rangle$	$\coprod_2 [X/\langle b \rangle]$
1	-1	$\langle b \rangle, \langle ab \rangle$	$\coprod_2 [X/\langle a \rangle]$
-1	-1	$\langle a \rangle, \langle b \rangle$	$\coprod_2 [X/\langle ab \rangle]$

The details are different, but the pattern is the same: by picking B such that the anomaly is described by $d_2(B)$, the resulting physical theory is non-anomalous, as predicted by Wang-Wen-Witten.

8 Conclusions

In this article we have reviewed work on decomposition, the observation that sometimes one local quantum field theory is equivalent to a disjoint union of other local quantum field theories, known as universes. This arises when a d-dimensional quantum field theory has a $(d-1)$-form symmetry. After reviewing examples of decomposition and its properties (such as multiverse interference effects), we discussed the application to the anomaly-resolution procedure of Wang-Wen-Witten [62].

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