On Twisted Elliptic Genera

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Abstract We study the twisted elliptic genera associated to the BPS strings in the twisted circle compactification of 6d $(1,0)$ SCFTs. Such objects can arise when the 6d gauge algebra allows outer automorphism. We study several aspects of twisted elliptic genera including twisted elliptic blowup equations, $\Gamma_1(N)$ modular bootstrap and spectral flow symmetry. This is a short summary of the author's talk at MATRIX Program "2D Supersymmetric Theories and Related Topic", and based on a joint work with Kimyeong Lee and Xin Wang.

1 Introduction

For the last decade, there has been huge progress on 6d $(1,0)$ superconformal field theories (SCFTs) in both classification [1] and computation. These 6d SCFTs are geometrically engineered by F-theory compactified on local elliptic Calabi-Yau (CY) threefolds, and contain BPS strings with worldsheet theories as some highly nontrivial 2d $(0,4)$ gauge theories. Upon circle compactification and deformations, they produce 5d Kaluza-Klein (KK) theories which are conjectured to flow to all 5d SCFTs by decoupling matters. A simple example is the 6d E-string theory, geometrically engineered by local half-K3 Calabi-Yau threefold. The worldsheet theories for k E-strings are certain 2d $(0,4)$ $O(k)$ gauge theories [2]. Upon circle compactification, E-string theory gives the well-known 5d $SU(2)$ theory with 8 fundamentals. On the partition function level, the above correspondence can be summarized as the following relation chain

$$\mathbb{E}^{\text{2d (0,4) SCFT}} = Z^{\text{6d (1,0) SCFT}}_{\mathbb{R}^4\times T^2} = Z^{\text{5d KK}}_{\mathbb{R}^4\times S^1} = Z^{\text{ref. top.}}_{\text{local elliptic CY3}}. \tag{1}$$

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This states that the partition function of a 6d $(1,0)$ SCFT on 6d Omega background should be equal to the elliptic genera of the 2d worldsheet theories, and the Nekrasov partition function of the 5d KK theory, and the refined topological string partition function on the underlying local elliptic Calabi-Yau threefold. The relation chain (1) has been tested extensively for many theories including E-string theory.

When a 6d (1,0) SCFT has a discrete global symmetry, one can do twisted circle compactification to a 5d KK theory [3]. There are two possibilities:

1. gauge algebra allows outer automorphism: folding vector multiplet.
2. quiver structure has discrete symmetry: folding tensor multiplet.

Here we focus on the first kind. In such situations, fractional KK charges naturally appear upon circle compactification and the local elliptic Calabi-Yau threefolds are generalized to local genus-one fibered ones. For partition functions, we expect a twisted version of relation chain (1):

$$\mathbb{E}^{\text{2d (0,4) SCFT}}_{\text{twisted}} = Z^{\text{6d (1,0) SCFT}}_{\mathbb{R}^4\times S^1\times S^1,\text{twisted}} = Z^{\text{5d KK}}_{\mathbb{R}^4\times S^1} = Z^{\text{ref. top.}}_{\text{local genus-one fibered CY3}}. \tag{2}$$

We are interested in the twisted elliptic genera arising here.

Now we show some simple examples of twisted 6d SCFTs. A 6d $(1,0)$ SCFT with rank-one tensor branch is specified by self-intersection number $\mathfrak{n}$, gauge algebra G, flavor symmetry F and matter represention R. The $(\mathfrak{n},G,F,R)$ is highly constrained by the Calabi-Yau condition. The gauge algebras that allow outer automorphism are classified by twisted affine Lie algebras which are $E_6^{(2)}$, $D_{r+4}^{(2)}$, $A_r^{(2)}$ and $D_4^{(3)}$ [4, Chapter 8]. Removing the affine node, we denote the truncated algebra as $\mathring{G}$ which is the 5d low energy gauge algebra. The twisted matter content $\mathring{R}$ are in the representation of $\mathring{G}$ and can have fractional KK charges. The flavor symmetry normally get reduced which we denote as $\mathring{F}$. We summarize some typical twisted 6d $(1,0)$ rank one theories with $\mathfrak{n}\geq 3$ in Table 1. The main goal of this work is to compute the twisted elliptic genera of all these theories and study their properties.

Table 1 Some typical twisted 6d $(1,0)$ rank-one theories with $\mathfrak{n}\geq 3$. The subscripts in $\mathring{R}$ denote the KK charge, and $\mathbf{V},\mathbf{F},\mathbf{S}$ denote vector, fundamental and spinor representation respectively. The constants c are used in the modular bootstrap in Sect 3. There are four pure gauge theories: $\mathfrak{n}=6$ with $E_6^{(2)}$, $\mathfrak{n}=4$ with $D_4^{(2)},D_4^{(3)}$ and $\mathfrak{n}=3$ with $A_2^{(2)}$.

$\mathfrak{n}$	G	$\mathring{G}$	$\mathring{R}$	$\mathring{F}$	c
6	$E_6^{(2)}$	F_4	$-$	$-$	5/4
4	$D_4^{(3)}$	G_2	$-$	$-$	5/9
4	$D_{r+4}^{(2)}$	B_{r+3}	$2r(\mathbf{V}_0\oplus\mathbf{1}_{1/2})$	$\mathfrak{sp}(2r)$	3/4
4	$E_6^{(2)}$	F_4	$\mathbf{F}_0\oplus\mathbf{F}_{1/2}\oplus\mathbf{1}_0\oplus\mathbf{1}_{1/2}$	$\mathfrak{sp}(1)$	1
3	$A_2^{(2)}$	C_1	$-$	$-$	5/16
3	$D_4^{(2)}$	B_3	$\mathbf{V}_0\oplus\mathbf{1}_{1/2}\oplus\mathbf{S}_0\oplus\mathbf{S}_{1/2}$	$\mathfrak{sp}(1)\times\mathfrak{sp}(1)$	1/2
3	$D_4^{(3)}$	G_2	$\mathbf{F}_0\oplus\mathbf{F}_{1/3}\oplus\mathbf{F}_{2/3}\oplus\mathbf{1}_0\oplus\mathbf{1}_{1/3}\oplus\mathbf{1}_{2/3}$	$\mathfrak{sp}(1)$	1/2

2 Twisted Elliptic Blowup Equations

Blowup equations as the functional equations of Nekrasov partition functions [5] were proposed by Nakajima and Yoshioka [6] for 4d $\mathcal{N}=2$ $SU(N)$ gauge theories to prove the Nekrasov conjecture. They originated by comparing the Nekrasov partition functions on the Omega background $\mathbb{C}^2_{\varepsilon_1,\varepsilon_2}$ and on its one-point blown-up. By blowing up and down the exceptional divisor, it can be expected that the two partition functions should be closely connected. Blowup equations are later generalized to various 4d/5d/6d gauge theories and become powerful tools to solve partition functions and refined BPS invariants. For the elliptic genera associated to 6d $(1,0)$ SCFTs, such functional equations were established in [7, 8, 9, 10] and called elliptic blowup equations. Now we want to further generalize them to twisted elliptic blowup equations and solve the twisted elliptic genera for all rank-one twisted theories. Some cases have been recently studied in [11]. For the total partition function Z of refined topological strings on a non-compact Calabi-Yau threefold X, the general form of blowup equations was proposed in [12]:

$$\begin{aligned}
&\Lambda(\varepsilon_1,\varepsilon_2,m_i)Z(\varepsilon_1,\varepsilon_2,t_i+\pi\mathrm{i}B_i)\\
=&\sum_{\mathbf{k}\in\mathbb{Z}^{b_4}}(-1)^{|\mathbf{k}|}Z(\varepsilon_1,\varepsilon_2-\varepsilon_1,t_i+(C_{ij}k_j+B_i/2)\varepsilon_1+\pi\mathrm{i}B_i)\\
&\qquad\times Z(\varepsilon_1-\varepsilon_2,\varepsilon_2,t_i+(C_{ij}k_j+B_i/2)\varepsilon_2+\pi\mathrm{i}B_i)\,,
\end{aligned}\tag{3}$$

Here b_4 counts the number of compact divisors of X. The t_i is the Kähler parameter and C_{ij} is the intersection numbers of divisors and curves in X. The B_i is the flux for the corresponding Kähler parameter. The $\Lambda(\varepsilon_1,\varepsilon_2,m_i)$ is a function that only depends on the mass parameters m_i of the theory. The $\Lambda(\varepsilon_1,\varepsilon_2,m_i)\equiv 0$ case is called vanishing, otherwise unity. By geometric engineering, the partition function can be written by 6d gauge quantities as

$$Z(\varepsilon_1,\varepsilon_2,t_i)=e^{\mathscr{F}^{\mathrm{cls}}}Z_0(\varepsilon_1,\varepsilon_2,m_{\mathring{G}},m_{\mathring{F}})\left(1+\sum_{d=1}^{\infty}Q^d\mathbb{E}_d(\varepsilon_1,\varepsilon_2,m_{\mathring{G}},m_{\mathring{F}})\right),\tag{4}$$

Here $\mathscr{F}^{\mathrm{cls}}$ is the classical free energy, Z_0 is the one-loop partition function, $\mathbb{E}_d$ is the d-string twisted elliptic genus and Q counts the number of strings. As the classical and one-loop part have known formulas, by substituting (4) into (3), we can derive the functional equations of $\mathbb{E}_d$ which we call twisted elliptic blowup equations.

We find that for all twisted theories in Table 1, there exist unity blowup equations such that the twisted d-string elliptic genera $\mathbb{E}_d(\tau,m_{\mathring{G}},m_{\mathring{F}},\varepsilon_{1,2})$ can be explicitly solved. We are particular interested in the reduced twisted one-string elliptic genus defined by

$$\mathbb{E}_1^{\mathrm{red}}(\tau,\varepsilon_+)=\frac{\theta_1(\tau,\varepsilon_1)\theta_1(\tau,\varepsilon_2)}{\eta(\tau)^2}\mathbb{E}_1(\tau,\varepsilon_1,\varepsilon_2).\tag{5}$$

The simplicity of $\mathbb{E}_1^{\mathrm{red}}$ is that it only depends on $\varepsilon_+=(\varepsilon_1+\varepsilon_2)/2$ rather than both $\varepsilon_1,\varepsilon_2$. We will discuss its modular bootstrap and spectral flow in the later sections.

3 Modular Bootstrap for $\Gamma_1(N)$

For untwisted 6d $(1,0)$ SCFTs, the modular ansatz for reduced one-string elliptic genus was proposed in [13] which involves Jacobi form on $SL(2,\mathbb{Z})$. For twisted cases, it is natural to expect from modularity that the modular ansatz should be established on the principal congruence subgroup $\Gamma_1(N)$. Here N is the twist coefficient given by:

$$N=\begin{cases}2,\ A^{(2)}_{2r-1},D^{(2)}_r,E^{(2)}_6,\\ 3,\ D^{(3)}_4,\\ 4,\ A^{(2)}_{2r}.\end{cases} \tag{6}$$

From the aspect of geometry, the modular ansatz for N-section Calabi-Yau threefolds has been proposed in [14], see also [15]. Combining together, we propose the following modular ansatz for the reduced twisted one-string elliptic genus:

$$\mathbb{E}^{\rm red}_1(\tau,\varepsilon_+)=\frac{\mathcal{N}(\tau,\varepsilon_+)}{\eta(\tau)^{12(\mathfrak{n}-2)-4+24\delta_{\mathfrak{n},1}}\Delta_{2N}(\frac{\tau}{N})^s\phi_{-2,1}(\tau,2\varepsilon_+)^{h^\vee_{\mathring{G}}-1}}, \tag{7}$$

where

$$s=\frac{N}{N-1}\Big(c-\frac{\mathfrak{n}-2}{2}-\delta_{\mathfrak{n},1}\Big). \tag{8}$$

Here c is the constant given in Table 1. The numerator $\mathcal{N}(\tau,\varepsilon_+)$ is a Jacobi form of weight $6(\mathfrak{n}-2)+2Ns+12\delta_{\mathfrak{n},1}-2h^\vee_{\mathring{G}}$ and index $4(h^\vee_{\mathring{G}}-1)+\mathfrak{n}-h^\vee_G$ and

$$\mathcal{N}(N\tau,\varepsilon_+)\in M_*(N)[\phi_{-2,1}(N\tau,\varepsilon_+),\phi_{0,1}(N\tau,\varepsilon_+)]. \tag{9}$$

Here $M_*(N)$ is the ring of holomorphic modular forms of $\Gamma_1(N)$. Besides, the Δ_{2N} are some cusp forms on $\Gamma_0(N)$:

$$\Delta_4(\tau)=\frac{\eta(2\tau)^{16}}{\eta(\tau)^8},\quad \Delta_6(\tau)=\frac{\eta(3\tau)^{18}}{\eta(\tau)^6},\quad \Delta_8(\tau)=\frac{\eta(2\tau)^8\eta(4\tau)^{16}}{\eta(\tau)^8},$$

and $\phi_{-2,1}(\tau,z),\phi_{0,1}(\tau,z)$ are Eichler-Zagier's generators for weak Jacobi forms. We successfully fix the numerator for all theories in Table 1, check them against the elliptic genera solved from blowup equations and find perfect agreements.

4 Spectral Flow Symmetry

Spectral flow is a characteristic feature of 2d $\mathcal{N}=2$ SCFTs. The spectral flow for reduced one-string elliptic genera associated to 6d $(1,0)$ SCFTs was studied in [16, 13]. Such flow relates the R-R elliptic genera we have discussed and the NS-R elliptic genera. For the reduced twisted one-string elliptic genera, we find the spectral flow from the R-R sector to NS-R sector is induced by the following transformation:

$$\mathbb{E}^{\mathring{R}_{\mathrm{KK}}}_{\mathrm{NS-R}}\Big(q,v\Big)=\pm\Big(\frac{q^{1/4}}{v}\Big)^{\mathfrak{n}-h^\vee_G}\mathbb{E}^{\mathring{R}_{\mathrm{KK}}}_{\mathrm{R-R}}\Big(q,\frac{q^{1/2}}{v}\Big). \tag{10}$$

Here $q=e^{2\pi i\tau},v=e^{2\pi i\varepsilon_+}$. On the other hand, the spectral flow shifts the KK charges of hypermultiplets by half. For 6d $(1,0)$ pure gauge $G^{(n)}$ theories, the above two transformations imply the following spectral flow symmetry for the reduced twisted one-string elliptic genera:

$$\mathbb{E}^{G^{(n)}}_1\Big(q,\frac{q^{1/2}}{v}\Big)=\Big(-\frac{q^{1/2}}{v^2}\Big)^{\mathfrak{n}-3}\,\mathbb{E}^{G^{(n)}}_1(q,v). \tag{11}$$

We have explicitly checked this symmetry for $A_2^{(2)},D_4^{(2)},D_4^{(3)},E_6^{(2)}$ theories.

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