# A problem about virtual polytopes

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**Abstract** This is a short summary on the problem of finding canonical representatives for elements in the Grothendieck group of polytopes.

## 1 Background

Let $\mathscr{K}$ be the set of polytopes in $\mathbb{R}^d$. Recall that $\mathscr{K}$ is equipped with the following two operations. For $P, Q \in \mathscr{K}$, $\lambda \geq 0$

$$\begin{aligned} &\text{Minkowski addition:} \quad P+Q=\{x+y \mid x \in P, y \in Q\} \\ &\text{scalar multiplication:} \quad \lambda P=\{\lambda x \mid x \in P\} \end{aligned}$$

The Minkowski addition turns $\mathscr{K}$ into a semigroup, with the identity element $\{0\}$. The semigroup $(\mathscr{K},+)$ satisfies the following *cancellation property*. For $P, Q, R \in \mathscr{K}$

$$P+R=Q+R \Leftrightarrow P=Q$$

This allows us to construct the *Grothendieck group* of $\mathscr{K}$. Concretely, let $\mathscr{P}=\mathscr{K} \times \mathscr{K} / \sim$, where

$$(P_1, Q_1) \sim (P_2, Q_2) \text{ if and only if } P_1+Q_2=P_2+Q_1$$

The additive structure on $\mathscr{P}$ is given by $(P_1, Q_1)+(P_2, Q_2)=(P_1+Q_1, P_2+Q_2)$. There is a the canonical embedding

$$\mathscr{K} \hookrightarrow \mathscr{P}, P \mapsto (P, \{0\})$$

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In other words, there is a canonical way to make sense of "Minkowski subtraction" in $\mathscr{K}$, and we can also write the equivalence class represented by $(P,Q)$ as $P-Q$.

This construction seems to be known already in the early 20th century, first officially appearing in [7]. Elements in $\mathscr{P}$ are called *virtual polytopes* by Pukhlikov and Khovanskii [6]. It appears in McMullen's work on the polytope algebra [3]. Recently it also attracts attention from topologists in the context of Thurston norm [1]. For the history of virtual polytopes, see [5].

If one thinks of polytopes as they are, this construction seems a little bit surprising. However, it is very natural from a convex geometric perspective, since a convex body $P$ is determined by its support function $h_P$ and $h_{P+Q} = h_P + h_Q$. Clearly, one can subtract two functions. The difference of two functions is a function, but the difference of two convex bodies is an equivalence class. The natural question is, can we find a canonical representative for that equivalence class?

## 2 Question statement

The word "canonical" can mean different things. One natural way to construct canonical representatives is to define some minimality condition, as is shown in the following example.

*Example 1.* The nonzero rational numbers $\mathbb{Q}^\times$ are the Grothendieck group of the nonzero integers, with respect to its multiplicative structure. Each class in $\mathbb{Q}^\times$ has a canonical representative (up to multiplication by a unit) whose numerator and denominator both have the smallest absolute value. Since the integers have unique factorization, $\frac{a}{b}$ is a canonical representative when $a$ and $b$ are coprime. A more interesting example is the ring of quadratic integers $\mathbb{Z}[\sqrt{-5}]$. Unlike $\mathbb{Z}$, $\mathbb{Z}[\sqrt{-5}]$ doesn't have unique factorization. Let $R$ be its Grothendieck group of the nonzero elements. An element in $R$ may have two representatives $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ such that $a_1$ and $b_1$ are coprime, $a_2$ and $b_2$ are coprime, but $a_1$ and $a_2$ are not multiples of each other. For instance,

$$2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5}) \Rightarrow \frac{2}{1-\sqrt{-5}} = \frac{1+\sqrt{-5}}{3}$$

However, since $\mathbb{Z}[-5] \subset \mathbb{C}$, each element in $R$ has a canonical representative up to a unit whose numerator and denominator both have the smallest modulus.

Question of such flavor first appeared in the context of quasidifferential calculus [4]. Lately, it was considered by Tran and the first author in the context of finding minimal representation of piecewise linear functions [9]. In summary, there are several ways to define what "minimal" means. They can be put into the following framework. Let $(S,\prec)$ be an partially ordered set and $f : \mathscr{K} \to S$ be a map. For two representatives of the same class $P_1 - Q_1 = P_2 - Q_2$, we say

$$P_1 - Q_1 \prec P_2 - Q_2 \text{ if } f(P_1) \prec f(P_2) \text{ and } f(Q_1) \prec f(Q_2)$$

- Take $S$ to be $\mathscr{K}$, $f$ to be the identity map, and $\prec$ to be inclusion, one gets the minimality condition defined by Pallaschke and Urbanski [4]. We call this *strong minimality*. It was shown that strongly minimal representatives are unique for $d = 2$, but not for higher dimensions. [2, 8]
- Take $S$ to be the real numbers, $f$ to be the $d$-dimensional volume, and $\prec$ to be the usual $\leq$, we get a minimality condition that minimizes the volume. We call this *volume minimality*. Volume-minimal representatives are unique when $d = 2$ (this is a consequence of the uniqueness of strongly minimal representatives, although direct elementary proofs can be found). The answer is unknown for higher dimensions.
  This minimality condition resembles Example 1 most. It is probably the most intriguing one since the volume function extends to virtual polytopes, and there is a strong connection with algebraic geometry where the (mixed) volume of convex bodies correspond to intersections of divisors. To our knowledge, this notion of minimality hasn't appeared in any literature.
- Take $S$ to be $\mathbb{N}^{d+1}$, $f$ to be the map extracting the $f$-vectors, and $\prec$ to be the product order induced by $\leq$ on $\mathbb{N}^{d+1}$, one gets a minimality condition that minimizes the combinatorial complexity, so we may call this *combinatorial minimality*. A version of this where $f$ extracts only the vertices is considered in [9]. Note that this only makes sense for polytopes, while the above two minimality conditions apply for all compact convex bodies.

*Question 1.* In any of the above minimality criteria, what does the space of all minimal representatives look like?

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