Isothermic submanifolds of symmetric R-spaces

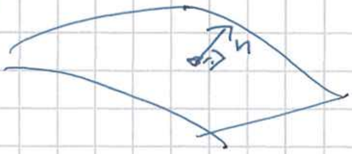
1/

/w Donaldson (Neil), Pedit & Pinkall   Crelle $\underline{660}$ (2011) 191-243

Plan:
- classical (~1900's) thy of isothermic surfaces in $\mathbb{R}^3$ (or $\mathbb{R}^3 \cup \{\infty\} = S^3$)
- replace S^3 by symmetric R-space G/P ← real semisimple; ← parabolic with abelian nilrad
- (if time) curves in $\mathbb{RP}^1$ — integrable dynamics ⇝ KdV, mKdV } Miura

1. Classical isothermic surfaces

$x : \Sigma^2 \hookrightarrow \mathbb{R}^3$ $n : \Sigma \to S^2$ Gauss map



Invariants: $I = dx \cdot dx$, $II = -dn \cdot dx$ (quadratic forms)

Bonnet: Determine x up to rigid motion.

Def$^{\underline{n}}$ (Bour 1862) x isothermic if $\exists$ (locally) conf. curv. line (CCL) coords

ie u,v s.t.
$$I = e^{2\varphi}(du^2 + dv^2) \quad \text{conf.}$$
$$II = e^{2\varphi}(\kappa_1 du^2 + \kappa_2 dv^2) \quad \text{curv. line.}$$

principal curvatures

Examples

- cones, cylinders & surfaces of revolution ← param profile curve by hyperbolic arclength ⇒ no regularity !
- quadrics [confocal quadrics] ← a bit mysterious
- CMC - const mean curv. $H \left(:= \frac{1}{2}(\kappa_1 + \kappa_2)\right)$

Symmetries

* Christoffel (1867): x isothermic iff $\exists\, \check{x} : \Sigma \to \mathbb{R}^3$ (locally) s.t.
  - $I, \check{I}$ conf.
  - $dx(T_p\Sigma) = d\check{x}(T_p\Sigma)\ \forall p \in \Sigma$ (parallel tangent planes) (1)
  - $\det(dx^{-1} \circ d\check{x}) < 0$

(& then $\check{x}$ also isothermic by symmetry)

Pf u, v CCL $\Leftrightarrow d\left(\frac{x_v}{\|x_u\|^2} du - \frac{x_v}{\|x_u\|^2} dv \right) = 0$
$=: d\check{x}$

$\square$

Alternate views:

A. $(1) \Leftrightarrow d\check{x} = dx \circ Q \quad Q \in \mathrm{Sym}_0(T\Sigma) \quad (1)'$

Now $dx \circ Q$ clsd $\Leftrightarrow \begin{cases} 1/\ d^\nabla Q = 0 \Leftrightarrow I(Q-,-)^{2,0} \in H^0(K) \\ 2/\ [A, Q] = 0 \end{cases}$

A: shape op.

Punchline:

X isothermic $\Leftrightarrow \exists\, q \in H^0(K^2)$ (locally) s.t. $[q, \mathrm{II}] = 0$
(then when $q = dz^2$, $z = u + iv$ for u, v CCL)

Example H const $\Leftrightarrow \mathrm{II}^{2,0}$ (Hopf?) holomorphic $\therefore$ isothermic.

B. $(1)' \Leftrightarrow \begin{cases} dx \mathbin{\dot\wedge} d\check{x} = 0 & \text{in } \Omega^2_2(\mathbb{R}) \\ dx \mathbin{\hat\wedge} d\check{x} = 0 & \text{in } \Omega^2_2(\Lambda^2\mathbb{R}^3) \end{cases} \quad (1)''$

($\dot\wedge$: dot prod.)

$\Updownarrow$

$dx \wedge d\check{x} = 0 \quad \text{in } \Omega^2_2(C\ell(\mathbb{R}^3)) \quad (1)'''$

Examples

- X minimal $(H = 0) \Leftrightarrow n = \check{X}$ (converse is Weierstrass-Enneper)

- $H \neq 0$ const then $\check{X} = X + \frac{1}{H} n$ also has same H
  or $X + \frac{1}{2H} n$ has const K.

* Conf. invariance: $T: \mathbb{R}^3 \cup \{\infty\} \circlearrowleft$ conf. diffeo (ex: invert in 2-sphere; these generate)
or X isothermic $\Rightarrow T \circ X$ isothermic

Remark All our examples break this symmetry but:
- H const in 3-dim^{l} space-form isothermic ($\mathrm{II}^{2,0}$ holo comes from Codazzi eq^n)

* ~~Darboux transfs~~

∴ contemplate $f : \Sigma \to S^3$ (with stereoproj $x : \Sigma \to \mathbb{R}^3$ if nec)

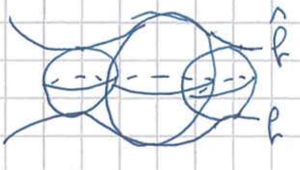
* **Darboux transfs** ∴ (Darboux 1899) ~~isoth~~ f isothermic iff

$\exists$ (loc.) $\hat{f} : \Sigma \to S^3$ & $S : \Sigma \to \{$2-spheres in $S^3\}$ s.t.

1/ $I_f, I_{\hat{f}}$ conf.

2/ $f, \hat{f}$ have same curv. lines

3/ $S(p)$ tangent to $f, \hat{f}$ at p ∴



+ then (by symmetry) $\hat{f}$ isothermic.

$\hat{f}$ sol^{n} of $S' \times S'$ integ. sys of linear diff eq^{n}s with quadratic constraint

depends on 4 parameters: $\{$ initial pos $\in S^3$; $m \in \mathbb{R}^\times$ spectral param.

(say $\hat{f}$ is m-Darboux transf of f)

[Aside: when are these $f, \hat{f}, S$ with 1/ & 3/ ?

Darboux claims: either $f, \hat{f}$ congruent or Darboux ~~transfs~~ pair

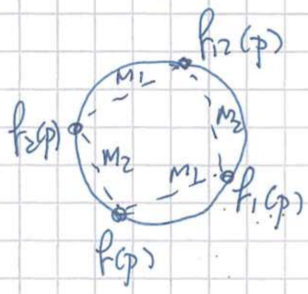
BUT he missed a case: $f, \hat{f}$ dual pair of Willmore surfaces with S conf. Gauss map (which Blaschke found)].

Bianchi (1905) Permutability ∴ given f & f_1, f_2 $\quad f_i$ m_i-Darboux of f

then (can choose initial cond^{n} so that) $\exists f_{12}$ isothermic with

f_{12} m_2-Darboux of f_1 AND m_1-Darboux transf of f_2

Demoulin (1910) ∴ f, f_1, f_2, f_{12} pointwise concircular with $\overline{\text{const.}}$ cross-ratio m_1/m_2



* T-transform (Bianchi 1905) "associated family"
Calapso 1903 up to conf. deformation

Given f isothermic $\exists$ 1-param ~~family~~ f_t, $t \in \mathbb{R}$, isothermic $f = f_0$
with common conf. str & II_0 [Non-isothermic determined by conf. str. & II_0]

[Aside: what determines $f : \Sigma^m \to S^n$ up to conf. diffeo? Obv. invariants: – conf. str on Σ
– $\cancel{\Xi}_0 (N\Sigma, \nabla^\perp)$
– $\mathrm{II}_0 \in S^2_0 T^*\Sigma \otimes N\Sigma$

Thm $m \geq 3$ these suffice.
$m = 2$ also need Möbius structure (à la Calderbank) = $SL(2,\mathbb{C})$ Cartan geometry induced by ~~extra~~ conf. Gauss map]

Kamberov-P-P.

[isoth. surfaces also appear in Euclidean Bonnet pair story]

Example f CMC H in 3-dim^l space-form curv K

f_t is Lawson family of such with $H_t^2 + K_t$ const.

All these transfs "commute" eg T-transf of Darboux is Darboux of T-transf
(shifted spectral param...)

$\therefore$ Suggests:

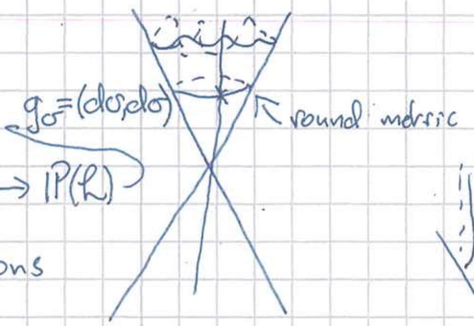
Thm (Cieśliński-Goldstein-Sym, 1995) Isothermic surfaces are an integrable system.

Let us see why! (ans: write down zero-curvature rep^n).

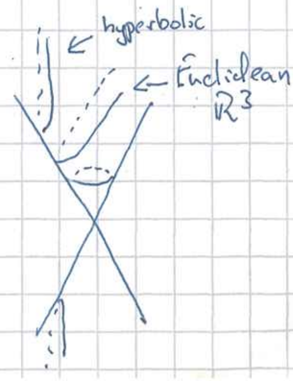
2. S^3 as symmetric R-space

Conf. geometry of S^3: conf. diffeo grp is $O_+(4,1)$ —

$\mathcal{L} \subset \mathbb{R}^{4,1}$ ← 5-dim^l $(,)$ sig $++++-$
↑ lightcone $\qquad \mathbb{P}(\mathcal{L}) \cong S^3$



Metrics given by sections σ of $\mathcal{L}^\times \to \mathbb{P}(\mathcal{L})$
Space-form metrics = conic sections



Fix $\ell \in \mathbb{P}(\mathcal{L})$, set $P = \mathrm{Stab}(\ell)$ with Lie alg $\mathfrak{p} \leq \mathfrak{o}(4,1)$.

FACT: P is parabolic subgrp i.e.

$\mathfrak{p}^{\mathbb{C}}$ contains max solvable subalg of $\mathfrak{o}(5,\mathbb{C})$

$\Updownarrow$

$\mathfrak{p}^{\perp}$ ← Killing polar

$\mathfrak{p}^{\perp}$ is a nilpt subalg of $\mathfrak{o}(4,1)$ (it is nilrad of $\mathfrak{p}$)

In our case $\mathfrak{p}^{\perp}$ is <u>abelian</u> (it is inf. translations of $\mathbb{R}^3 = \mathbb{P}(\mathcal{L}) \setminus \{[e]\}$)

With $G = O_+(4,1)$ $\mathfrak{g} = \mathfrak{o}(4,1)$, differentiate $g \mapsto g\ell$

$G \to S^3$

to get $T_e S^3 \cong \mathfrak{g}/\mathfrak{p}$

$\therefore$ $T_e^* S^3 \cong \mathfrak{p}^{\perp} \leq \mathfrak{g}$ via Killing form | this is Moment map $T^*S^3 \to \mathfrak{g}^* \cong \mathfrak{g}$

Watch carefully: $q \in \Gamma S^2 T^*\Sigma \leq \Omega^1_{\Sigma}(T^*\Sigma)$

$T^*\Sigma \hookrightarrow f^{-1} T^*S^3$ as $\mathrm{ann}(N\Sigma)$

$\cong$

$\mathrm{stab}(f)^{\perp} \leq \Sigma \times \mathfrak{g} =: \underline{\mathfrak{g}}$

$\therefore$ $q \rightsquigarrow \eta \in \Omega^1_{\Sigma} \otimes \underline{\mathfrak{g}}$

<u>Thm</u> $q \in H(K^2)$ ~~hol~~ with $[q, \mathrm{II}_0] = 0$ (i.e. f isothermic)

$\Updownarrow$

$d\eta = 0$

More: $\mathrm{stab}(f)^{\perp}$ bundle of abelian subalgs $\therefore$ $[\eta \wedge \eta] = 0$

$\therefore$ $\boxed{f \text{ isothermic iff } d + t\eta \text{ flat connection on } \underline{\mathfrak{g}}\ \forall t \in \mathbb{R}}$

Whole classical thy follows:

- For $t \in \mathbb{R}$ $\exists$ (loc) $\Phi_t : \Sigma \to G$ s.t. $\Phi_t \cdot (d + t\eta) = d$

then $f_t := \Phi_t f$ isothermic (with $\eta_t = \mathrm{Ad}\,\Phi_t\,\eta$)

— this is T-transf of Bianchi/Calapso

- View $\hat{f}: \Sigma \to S^3$ ~~as~~ as null line subbundle of $\mathbb{R}^{4,1}$

$\hat{f}$ is m-Darboux transf $\Leftrightarrow$ • $f \cap \hat{f}$ = zero section
• $\hat{f}$ $d + m\eta$-$\parallel$

[This is Darboux linear system: quadratic constraint is $\hat{f}$ is null]

3. ~~Isoth~~ Symmetric R-spaces

Main structure: ~~stuff~~ $f: \Sigma \to G/P$

- G real noncpct semisimple
- P parabolic with abelian unipotent radical

Think: block upper Δ's

These are the symmetric R-spaces ← "racine"

Characterisations:
- real flag manifolds $G/P = K/K \cap P$ K max cpct ~~s.t.~~ which are Riemannian symmetric K-spaces
- [Nagano] Riem. symmetric spaces with Lie grp ~~strictly~~ of diffeo strictly larger than isometry grp
- Hermitian symmetric spaces & their real forms.

Examples
- Grassmannians $G_k(\mathbb{K}^n)$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ $G = SL(n, \mathbb{K})$
- $Lag(\mathbb{R}^{2n})$ $G = Sp(2n, \mathbb{R})$
- $J^{\pm}(\mathbb{R}^{2n})$ $G = SO(2n, \mathbb{C})$
- $SO(n)$ as $SO(n,n)$-space (ex: what is the action?)
- $\mathbb{P}^2(\mathbb{O})$ ← $G = E_6$ & some other E_6, E_7 spaces (read it off the Satake diagram)
- quadrics $\mathbb{P}(\mathcal{L}^{p,q})$ (products of spheres)

Uniform approach:

$N = \mathrm{Ad}\, G\, \mathfrak{p}$ conj. class of $\mathfrak{p}$ since $\mathfrak{p}$ self normalising

$\mathfrak{f}: \Sigma^k \to N \quad \leftrightsquigarrow \quad \mathfrak{f} \leq \underline{\mathfrak{g}}$ bundle of parabolics.

Def: $\mathfrak{f}$ isothermic if $\exists\, \eta \neq 0 \in \Omega^1_\Sigma(\mathfrak{g})$

- η takes values in $\mathfrak{f}^\perp$
- $d\eta = 0$

- Duality: $\mathfrak{p}, \mathfrak{q}$ complementary (opposite) if

$$\mathfrak{g} = \mathfrak{p}^\perp \oplus (\mathfrak{p} \cap \mathfrak{q}) \oplus \mathfrak{q}^\perp$$

↑ graded Lie alg.

NOT opt/nop oper duality.

FACT $\mathfrak{p} \in N$ then any $\mathfrak{q}$ compl. has $\mathfrak{q} \in N^*$ — dual symmetric R-space.

same conj class

Ex $\cancel{G_k(\mathbb{K}^n)}$ $G_k(\mathbb{K}^n)^* = G_k(\mathbb{K}^n)$

$J^\pm(\mathbb{R}^{2n})^* = J^\pm$ or $J^\mp$ if $n = 2m$ or not

$S^{n*} = S^n$

* Darboux transf: $\hat{\mathfrak{f}}: \Sigma^k \to N^*$ s.t.

- $\hat{\mathfrak{f}}(p)$ compl $\mathfrak{f}(p)$ $\forall p \in \Sigma$
- $\hat{\mathfrak{f}} \leq \underline{\mathfrak{g}}$ $d + m\eta$ - ||.

Stereoproj: Fix $\mathfrak{p}_\infty \in N^*$

$\Omega_{\mathfrak{p}_\infty} \subset N$

$= \{\mathfrak{p} \in N \mid \mathfrak{p} \text{ compl to } \mathfrak{p}_\infty\}$ "Big cell"

For $\mathfrak{p}_0 \in \Omega_{\mathfrak{p}_\infty}$

$$\mathfrak{p}_\infty^\perp \xrightarrow{\wedge} \Omega_{\mathfrak{p}_\infty} \quad \text{diffeo}$$
$$x \longmapsto \exp x\, \mathfrak{p}_0$$

inverse stereo proj
— affine chart on Grassmannian
— Cayley param of $SO(n)$
etc.

BRUNNEN

Ch. transf.: $x : \Sigma \to p_\infty^\perp \rightsquigarrow f : \Sigma \to N$ isothermic if

$\exists\ \check{x} : \Sigma \to p_0^\perp$ s.t. $[dx \wedge d\check{x}] = 0$

Circles: $N = N^*$ p_0, p_1, p_∞ mutually comp.

$p_1 = \exp x\, p_0 \quad x \in p_\infty^\perp$

$t \mapsto p_t = \exp t p_0$ Then $C := \{p_t \mid t \in \mathbb{R}\} \cup \{p_\infty\} \cong \mathbb{R}P^1$

"circle"

Ex $l_1, l_2, l_2 \in G_2(\mathbb{R}^4)$ 3-skew lines

$\rightsquigarrow$ quadric $\rightsquigarrow$ family of lines incl. l_1, l_2, l_3.

+ Demoulin's Thm holds ∇.