Higgs bundles and SYZ geometry

Franz Pedit, UMass Amherst

(Joint with Sebastian Heller and Charles Ouyang)

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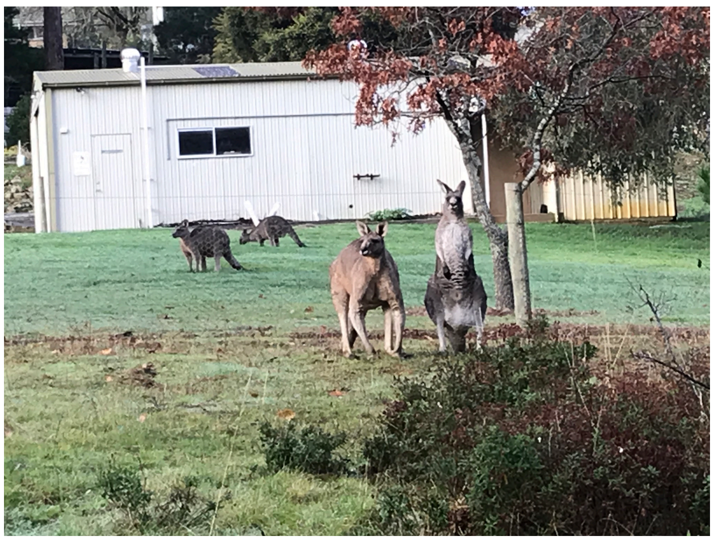

MATRIX workshop
Spectrum and Symmetry for Group Actions in Geometry

July 24–August 4, 2023

What we will do

- MathPhysics motivation
- Monge–Ampère metrics
- Affine spheres
- Parabolic Higgs bundles, self-duality equations, and surface group representations
- A little Diophantine geometry
- Calabi-Yau 3-folds fibered by special Lagrangian 3-tori

MathPhysics

Space-time

$$\mathbb{R}^4 \times X\,, \qquad X = \text{Calabi-Yau 3-fold: Ricci flat Kähler, } K_X \cong \underline{\mathbb{C}}$$

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Brane (endpoints of open strings) duality

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- $S \subset X$ special Lagrangian:

$$\omega_{|S} = 0 = \operatorname{Re}\Omega_{|S}$$

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$$6 = \dim_{\mathbb{R}} \check{X} = 2 \dim H^1(S, \mathbb{R}) \rightsquigarrow b_1(S) = 3 \rightsquigarrow S \cong T^3$$

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3. $B_0 = B \cup \Gamma$ with $\Gamma \subset B_0$ a trivalent graph (singularity locus) over which the fibrations degenerate.

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- $X_{|B} = TB/\Lambda$ with $\Lambda \subset TB$ full rank ∇-parallel lattice bundle.

- Calabi–Yau structure on $X_{|B}$ is completely determined by a Monge–Ampère metric $\nabla d\phi$ on B for a smooth

$$\phi: B \to \mathbb{R}, \qquad \det_B \nabla d\phi = 1\,.$$

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on $(B, \nabla, \det_B)$ for $\dim_{\mathbb{R}} B \geq 3$ are hard to come by.

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Blaschke (1923):

- Solutions ϕ of Monge–Ampère equations are precisely the graph functions

$$z = \phi(b)$$

of parabolic affine spheres over B.

- 3-dimensional parabolic affine spheres can be obtained from 2-dimensional elliptic or hyperbolic affine spheres via coning (Calabi 1972, Baues–Cortéz 2003).

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Levi-Civita of Blaschke metric

$$\nabla = \nabla^g + g^{-1} \circ C\,, \qquad C = -\tfrac{1}{2}\nabla g \in \Gamma(TM^{*\odot 3}) \text{ cubic Pick form}$$

C is g-tracefree.

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Inclusion $T\tilde{M} \hookrightarrow \mathbb{R}^{n+1}$ is a closed 1-form which integrates to an affine hypersurface

$$f \colon \tilde{M} \to \mathbb{R}^{n+1} , \qquad \gamma^* f = \rho_\gamma \circ f$$

equivariant with respect to d_V-monodromy representation

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Note: parabolic affine spheres, $H = 0$, carry flat special affine structure ∇, affine normal ξ is constant and have graph parametrization

$$f(p) = p + \xi\phi(p) \,, \quad \phi \text{ solves Monge-Ampère equation}$$

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$$B^{n+1} = M \times (0,1) \xrightarrow{p} M$$

Then

$$F : TB \to p^* V \,, \quad F(v, \mu \partial_r) = (rv, \mu H)$$

is bundle isomorphism; $\nabla_B := F^* d_V$ and $\det_B := F^* \det_V$ satisfy:

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3. $\phi : B \to \mathbb{R}\,, \qquad \phi(r) = -H \int_0^r (1 - H\rho^{n+1})^{\frac{1}{n+1}}\, d\rho\,, \quad H = \pm 1$
   is convex, $\det_B \nabla_B d\phi = 1$ and thus $\nabla_B d\phi$ is a Monge–Ampère metric on B.

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Tzitzéica equation $\Leftrightarrow d_V$ flat $\Leftrightarrow F^D + [\Phi, \Phi^{\dagger}] = 0\,, \quad \bar{\partial}^D \Phi = 0$

$H = 1$: $\mathbf{SU}_3$ self-duality eqns; $H = -1$: $\mathbf{SU}_{2,1}$ self-duality eqns.

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Equivalent to constructing Monge-Ampère metrics

$$\nabla d\phi : TB \times TB \to \mathbb{R}, \qquad \det_B \nabla d\phi = 1$$

on flat special affine real 3-manifold $B = B_0 \setminus \Gamma$ with integral monodromy: flat torsion free ∇, parallel $\det_B \in \Omega^3(B, \mathbb{R})$, $\rho_\nabla : \pi_1(B) \to \mathbf{SL}_3(\mathbb{Z})$.

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Solving Hitchin's self-duality equations on $K^{-1} \oplus \underline{\mathbb{C}} \oplus K \to M$,

$$F^D + [\Phi, \Phi^\dagger] = 0\,, \quad \bar{\partial}^D \Phi = 0\,, \qquad \Phi = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ Q & 0 & 0 \end{pmatrix}$$

provides, via coning, examples of such $B = M \times (0, 1)$ if

$$d_V = D + \Phi + \Phi^\dagger$$

has integral monodromy.

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Y-vertex is obtained by coning a thrice punctured 2-sphere $M = S^2 \setminus \{p_1, p_2, p_3\}$. So $B_0 = S^2 \times [0, 1)$ is open 3-ball and $B = M \times (0, 1)$ is obtained by removing a Y-vertex from an open 3-ball.

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We need to solve the self-duality equations

$$F^D + [\Phi, \Phi^\dagger] = 0\,, \qquad \bar{\partial}^D \Phi = 0$$

equivalently, the Tzitzéica equation, for $g = e^{2u} g_0$ and $Q \in \Gamma(K^3)$

$$2 \triangle_{g_0} u + 2|Q|^2_{g_0} e^{-4u} - He^{2u} + 1 = 0\,, \quad \bar{\partial}\, Q = 0$$

over thrice punctured sphere, where g_0 is hyperbolic metric.

Loftin-Yau-Zaslow (JDG 2005)

There exists solution $u\colon S^2 \setminus \{p_1, p_2, p_3\} \to \mathbb{R}$ of

$$2\,\triangle_{g_0} u + 2|Q|^2_{g_0} e^{-4u} - He^{2u} + 1 = 0\,, \quad \bar{\partial}\, Q = 0$$

for sufficiently small meromorphic cubic differential Q with quadratic poles at the punctures $p_k \in S^2$ in the elliptic affine sphere case, $H = -1$. The metric $g = e^{2u} g_0$ is asymptotic to a radially symmetric metric at the punctures.

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No information on the monodromy $\rightsquigarrow$ C-Y 3-fold $X = TB$ fibered by sLag 3-planes, and not 3-tori.

LYZ use “wrong” affine sphere equations:
elliptic affine spheres are $\mathbf{SU}_{2,1}$ self-duality equations, for which the non-Abelian Hodge correspondence does not hold.
But it does hold for the compact $\mathbf{SU}_3$ case corresponding to hyperbolic affine spheres.

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$M = \bar{M} \setminus \{p_1, \ldots, p_n\}$ punctured compact Riemann surface and $\mathfrak{D} = \sum p_k$ the singularity divisor. There are bijections between the following moduli spaces (Hitchin 1987, Simpson 1990):

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1. Degree zero holomorphic vector bundles $W \to \bar{M}$ of rank r and Higgs fields $\Phi \in H^0(K\mathbf{sl}(W)O(\mathfrak{D}))$ with nilpotent residues $\operatorname{Res}_{p_k} \Phi$—the Dolbeault space $\mathcal{M}_D$.

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2. "Tame" solutions to the self-duality equations

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on M—the self-duality space $\mathcal{M}_{sd}$.

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Nilpotency structure of $\operatorname{Res}_{p_k}\Phi$ is the same as unipotency structure of monodromy $\rho_k \in \mathbf{SL}_r(\mathbb{C})$ around p_k.

Non-Abelian Hodge over the thrice-punctured sphere

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Set $\bar{M} = S^2$, $M = S^2 \setminus \{p_1, p_2, p_3\}$ the thrice-punctured sphere, singularity divisor $\mathfrak{D} = p_1 + p_2 + p_3$.

Want to apply non-Abelian Hodge to our Higgs bundle

$$V \otimes \mathbb{C} = K^{-1} \oplus \underline{\mathbb{C}} \oplus K \to M\,, \qquad \Phi = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ Q & 0 & 0 \end{pmatrix}$$

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Need to extend (the holomorphically trivial) bundle $V \otimes \mathbb{C}$ to $\bar{M} = S^2$ with $Q \in H^0(K^3_{S^2} \mathcal{O}(2\mathfrak{D}))$. This results in the Higgs bundle

$$W = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(1)\,, \qquad \Phi_Q = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ Q & 0 & 0 \end{pmatrix} \in H^0(K_{S^2} \mathbf{sl}(W) \mathcal{O}(\mathfrak{D}))$$

$$\mathrm{Res}_{p_k} \Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbf{sl}(W_{p_k})\,.$$

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$K^3_{S^2} \mathcal{O}(2\mathfrak{D}) = \mathcal{O}(-2)^3 \mathcal{O}(6) = \mathcal{O}$, so have $\mathbb{C}$-family of such Higgs bundles (W, Φ_Q).

Non-Abelian Hodge and C-Y metrics

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Theorem (S. Heller, Ouyang, - , 2022):

1. The $\mathbb{C}$-family (W, Φ_Q) parametrizes (real analytically) the $\mathbf{SL}_3(\mathbb{R})$ Hitchin component $\mathcal{C} \subset \mathcal{M}_B$ for the thrice-punctured sphere:

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2. The corresponding solutions to the self-duality equations give a $\mathbb{C}$-family of solutions to the Tzitzéica equation for hyperbolic affine spheres on the thrice-punctured sphere asymptotic to the hyperbolic cusp metric at the punctures $p_k \in S^2$.

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3. Coning provides a $\mathbb{C}$-family of non-isometric Monge-Ampère metrics on B, the open 3-ball deleted a Y-vertex, and thus a $\mathbb{C}$-family of non-isometric C-Y metrics on the sLag fibration $X = TB \to B$.

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The $\mathbb{C}$-family of C-Y metrics on $X_{|B} = TB$ descends to a sLag 3-torus fibrations $TB/\Lambda \to B \iff$

$\Gamma \subset TB$ is ∇-parallel lattice bundle $\iff$

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Need to understand the monodromy of solutions to the Tzitzéica equation for hyperbolic affine spheres, or equivalently, for which Higgs bundles (W, Φ_Q) the—via non-Abelian Hodge—corresponding representation ρ_Q is integral.

A Diophantine problem cont.

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Theorem (S. Heller, Ouyang, - , 2022):

1. The character map

$$\mathcal{X}\colon \mathcal{M}_B \to \mathbb{C}^3, \qquad \mathcal{X}_\rho = \begin{pmatrix} \mathrm{tr}(\rho_1\rho_2^{-1}) \\ \mathrm{tr}(\rho_1^{-1}\rho_2) \\ \mathrm{tr}(\rho_1\rho_2\rho_1^{-1}\rho_2^{-1}) \end{pmatrix}$$

is a biholomorphism onto the cubic affine variety

$$\mathcal{F} = \{414 - 108x + x^3 - 108y + 21xy + y^3 - (51 - 9x - 9x + xy)z + z^2 = 0\}$$

A Diophantine problem cont.

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2. $\mathcal{F}(\mathbb{R}) = \mathcal{F} \cap \mathbb{R}^3$ corresponds via $\mathcal{X}$ to the real representations in $\mathcal{M}_B$ and has two connected components: the Hitchin component and the component of the trivial representation.
3. $\mathcal{F}(\mathbb{Z}) = \mathcal{F} \cap \mathbb{Z}^3$ has infinitely many points in each component which correspond via $\mathcal{X}$ to integral representations $\rho\colon \pi_1(M) \to \mathbf{SL}_3(\mathbb{Z})$.

Some examples

(s,t)		x	y	z	ρ_1	ρ_2
$(1,3)$		84	84	256	$\begin{pmatrix}1 & 9 & 0\\ 0 & 1 & 9\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & -7\\ 0 & -1 & 8\\ 1 & 0 & 4\end{pmatrix}$
$(3,20)$		35	99	643	$\begin{pmatrix}1 & 11 & 32\\ 0 & 97 & 288\\ 0 & -32 & -95\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 3\\ 0 & 21 & 64\\ 1 & -6 & -18\end{pmatrix}$
$(\frac{7}{5},\frac{18}{5})$		93	129	327	$\begin{pmatrix}1 & 18 & -5\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & -1\\ 2 & -1 & 2\\ 7 & -1 & -4\end{pmatrix}$

At this point, we do not know whether all integer points $\mathcal{F}(\mathbb{Z}) \subset \mathcal{F}$ in the character variety give rise to integral representations, nor can we characterize all integer points.

Theorem (S. Heller, Ouyang, - , 2022): There exists infinitely many non-isometric Calabi-Yau metrics on sLag 3-torus fibrations

$$\pi \colon X = TB/\Lambda \to B$$

where B is an open 3-ball deleted a Y-vertex.