New developments on Einstein manifolds with symmetry

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Einstein manifolds

(M^n, g) complete Riemannian manifold is Einstein if $\mathrm{Ric}_g = \lambda\, g, \quad \lambda \in \mathbb{R}$.

2nd order, (weakly) elliptic PDE: $-\frac{1}{2}\Delta_g g_{ij} + Q_{ij}(g, \partial g) = \lambda g_{ij}$ in harmonic coordinates

In this talk: $\lambda < 0$ (may assume $\lambda = -1$).

Examples

- (Cartan 1929) Irreducible symmetric spaces $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^n$, $\mathsf{SL}_n(\mathbb{R})/\mathsf{SO}(n)$, $\mathsf{SL}_n(\mathbb{C})/\mathsf{SU}(n)$, etc. They all admit compact quotients (Borel 1963).
- (Aubin-Yau 1976) Kähler-Einstein manifolds: (X, ω) compact Kähler, $c_1(X) < 0 \implies \exists!$ $\bar{\omega}$ with $[\bar{\omega}] = [\omega]$ such that $\mathrm{Ric}_{\bar{\omega}} = -\bar{\omega}$.
- (Anderson 2006), (Bamler 2012), (Fine, Premoselli 2020) Examples from constructions related to hyperbolic manifolds.

Fact These are all known compact examples.

Einstein manifolds with symmetry

(M^n, g) $\quad \mathrm{Ric}_g = -g$

Cocompact symmetry $\exists$ connected Lie group G acting on (M^n, g)

- Isometrically; $\quad \tau : \mathrm{G} \to \mathrm{Isom}(M, g)$
- Properly; $\quad \tau(\mathrm{G})$ closed
- **Cocompactly**. $\quad M/\mathrm{G}$ compact

Remark M and G are non-compact (Bochner '48) M cpt $+$ $\mathrm{Ric} < 0 \implies \dim \mathrm{Isom}(M, g) = 0$

Simplest case Homogeneous manifolds: $M/\mathrm{G} = pt$

- Symmetric spaces;
- Einstein solvmanifolds (S, g), S solvable, $\pi_1 \mathrm{S} = 1$, g Einstein and left-invariant.
  (Alekseevskii '75) (Cortes '94) (Heber '98) (Lauret '10) (Nikolayevsky '11) (Jablonski '15)

Conjecture (Alekseevskii 1975) (M^n, g) homogeneous, $\mathrm{Ric}_g = -g \implies M \simeq_{\mathrm{Diff}} \mathbb{R}^n$.

The Alekseevskii and splitting conjectures

Conjecture (Alekseevskii 1975) (M^n, g) homogeneous, $\mathrm{Ric}_g = -g \implies M \simeq_{\mathrm{Diff}} \mathbb{R}^n$.

Theorem A (Böhm, L. 2021) Conjecture holds. Moreover, $(M^n, g) \simeq_{\text{isom}}$ Einstein solvm.

Key ingredient: structure result for Einstein manifolds with **non-transitive** cocompact symmetry. Based on this, we propose:

Splitting conjecture (Böhm, L. 2023) (M^n, g), $\mathrm{Ric}_g = -g$ + cocompact symmetry $\implies M$ splits isometrically as a product of a compact Einstein manifold and an Einstein solvmanifold.

Main result for this talk:

Theorem B (Böhm, L. 2023) Splitting conjecture holds if the isometry group is unimodular.

G is unimodular if $\det \mathrm{Ad}_x = 1$, $\forall x \in \mathrm{G}$. E.g. compact, semisimple, reductive, nilpotent, etc.

Main result

Theorem B (Böhm, L. 2023) $(M^n, g), \mathrm{Ric}_g = -g$, with cocompact L-symmetry, L unimodular
$\implies \quad M^n \simeq_{\mathrm{isom}} B \times \mathsf{L}/\mathsf{K}, \quad B$ compact Einstein, $\quad \mathsf{L}/\mathsf{K}$ symmetric space.

Remarks

- Orbit space M/L may have arbitrary dimension and may be singular.
- Compactness of M/L is needed: e.g. horospheres in $\mathbb{R}H^n$.
- We believe unimodularity is not necessary (cf. Splitting conjecture).
- Not true for $\mathrm{Ric}_g = +g$: $M = S^n$, $\mathsf{L} = \mathsf{SO}(n)$.

What about $\mathrm{Ric}_g = 0$?

Theorem C $(M^n, g), \mathrm{Ric}_g = 0$, cocompact symmetry $\implies \tilde{M}^n \simeq_{\mathrm{isom}} \bar{M} \times \mathbb{R}^k$, $\bar{M}$ compact.

Proof Essentially the Cheeger-Gromoll splitting theorem ($\mathrm{Ric} \geqslant 0$ + line $\implies$ $\mathbb{R}$-splitting).

Main result: proof strategy

Theorem C (Böhm, L. 2023) (M^n, g), $\mathrm{Ric}_g = -g$, with cocompact L-symmetry, L unimodular
$\implies \quad M^n \simeq_{\mathrm{isom}} B \times \mathrm{L}/\mathrm{K}, \quad B$ compact Einstein, L/K symmetric space.

Slice Theorem $\implies \exists M_{\mathrm{reg}} \subset M$ open, dense, L-invariant, such that $M_{\mathrm{reg}}/\mathrm{L} =: B$ smooth.
Riemannian submersion $\pi : M_{\mathrm{reg}} \to M_{\mathrm{reg}}/\mathrm{L} \rightsquigarrow$ study $\mathrm{Ric}_g = -g$ using O'Neill's formulas.

We will show that:

1. L is semisimple. Only uses $\mathrm{Ric} < 0$
2. L acts polarly. $\iff A = 0$ for the Riemannian submersion $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/\mathrm{L}$
3. L-orbits are symmetric spaces. But a priori not necessarily Einstein!
4. L-orbits are totally geodesic. $\iff T = 0$

Recall: $A = 0 + T = 0 \implies$ local isometric splitting.

Negative Ricci with symmetry: the β^+-volume density

Setup (M,g) $\mathrm{Ric}_g < 0$ + L-symmetry + L unimodular + M/L compact and smooth.

Assume: $M/\mathsf{L} = B$ smooth, L = Heisenberg group. $\{e_1, e_2, e_3\}$ basis for $\mathfrak{l}$, $[e_1, e_2] = e_3$.
g induces $\hat{g} : B \to \mathrm{GL}(\mathfrak{l})/\mathrm{O}(\mathfrak{l})$ inner products on $\mathfrak{l}$
Assume $\hat{g}$ diagonal, i.e. $\hat{g}(b) = h_{11}(b)\, e^1 \otimes e^1 + h_{22}(b)\, e^2 \otimes e^2 + h_{33}(b)\, e^3 \otimes e^3$

GIT weights for $\mathfrak{l}$: $\beta^+ = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$ cf. $\mathbb{C}H^2$

β^+-weighted volume density of the orbits: $v_{\beta^+} : B \to \mathbb{R}$, $v_{\beta^+}(b) := \left(h_{11}(b) h_{22}(b) h_{33}^2(b)\right)^{1/3}$

$X_i = e_i^*$ Killing fields on (M,g), $\mathrm{Ric}_{ii} := \mathrm{Ric}_g(X_i, X_i)$, $\mathrm{Ric}^{\mathsf{L}}_{ii} := \mathrm{Ric}^{\mathsf{L}}_{\hat{g}(b)}(X_i, X_i)$

$$\Delta \log v_{\beta^+} - \langle \mathrm{H}, \nabla \log \mathrm{v}_{\beta^+} \rangle = \left(\tfrac{1}{3} \mathrm{Ric}^{\mathsf{L}}_{11} + \tfrac{1}{3} \mathrm{Ric}^{\mathsf{L}}_{22} + \tfrac{2}{3} \mathrm{Ric}^{\mathsf{L}}_{33} \right) + \left(\tfrac{1}{3} \|A_{X_1}\|^2 + \tfrac{1}{3} \|A_{X_2}\|^2 + \tfrac{2}{3} \|A_{X_3}\|^2 \right)$$
$$- \left(\tfrac{1}{3} \mathrm{Ric}_{11} + \tfrac{1}{3} \mathrm{Ric}_{22} + \tfrac{2}{3} \mathrm{Ric}_{33} \right) \geqslant - \left(\tfrac{1}{3} \mathrm{Ric}_{11} + \tfrac{1}{3} \mathrm{Ric}_{22} + \tfrac{2}{3} \mathrm{Ric}_{33} \right)$$

$\mathrm{Ric} < 0$ yields a contradiction, unless all GIT weights β^+ vanish. $\implies$ L semisimple.

The frame bundle

Setup (M, g) $\mathrm{Ric}_g < 0$ + L-symmetry + L unimodular + M/L compact.

In general, M/L need not be smooth. We need to argue on the **frame bundle**

$$FM = \{(p, u) : p \in M, u : \mathbb{R}^n \to T_pM \text{ linear isometry}\}$$

$FM \to M$ is an $\mathsf{O}(n)$-principal bundle. Its sections are **frames**.

Key property. May lift g to a metric on FM such that $\mathsf{L} \circlearrowright M$ lifts to a **free** $\mathsf{L} \circlearrowright FM$.

$$\begin{array}{ccc} FM & \longrightarrow & FM/\mathsf{L} \\ \big\downarrow {\scriptstyle /\mathsf{O}(n)} & & \big\downarrow {\scriptstyle /\mathsf{O}(n)} \\ M & \longrightarrow & M/\mathsf{L} \end{array}$$

Argue as in non-singular case, but for L-action on FM, everything $\mathsf{O}(n)$-equivariant.

Main result: proof strategy

Theorem C (Böhm, L. 2023) $(M^n, g), \mathrm{Ric}_g = -g$, with cocompact L-symmetry, L unimodular
$\implies \quad M^n \simeq_{\text{isom}} B \times \mathrm{L}/\mathrm{K}, \quad B$ compact Einstein, $\quad \mathrm{L}/\mathrm{K}$ symmetric space.

1. ~~L is semisimple. Only uses $\mathrm{Ric} < 0$~~ ✓
2. L acts polarly. $\iff$ $A = 0$ for the Riemannian submersion $M_{\text{reg}} \to M_{\text{reg}}/\mathrm{L}$
3. L-orbits are symmetric spaces. But a priori not necessarily Einstein
4. L-orbits are totally geodesic. $\iff$ $T = 0$

The action is polar

Setup (M,g) L-invariant, $\mathrm{Ric}_g = -g$ + L semisimple + M/L compact.

Riemannian submersion $\pi : M_{\mathrm{reg}} \to M_{\mathrm{reg}}/\mathsf{L}$, $TM = \mathcal{H}^{\mathsf{L}} \oplus^{\perp} \mathcal{V}^{\mathsf{L}}$, $\mathcal{V}^{\mathsf{L}} := \ker d\pi$.

$$\langle A_X Y, U\rangle = -\langle \nabla_X U, Y\rangle, \qquad X, Y \in \Gamma(\mathcal{H}^{\mathsf{L}}), \qquad \text{Integrability tensor}$$

$$\langle T_U V, X\rangle = \langle \nabla_U X, V\rangle, \qquad U, V \in \Gamma(\mathcal{V}^{\mathsf{L}}) \qquad \text{Second fundamental form}$$

Action is **polar** $\iff$ there exists a section $\sigma : \Sigma \to M$ intersecting all orbits orthogonally.
$\iff$ $A = 0$ (Heintze, Liu, Olmos '06)

$$(P)_{\mathsf{L}} \qquad \langle \nabla_X U^*, Y\rangle = 0, \qquad \forall X, Y \in \Gamma(\mathcal{H}^{\mathsf{L}}), \quad \forall U \in \mathfrak{l}.$$

($U^* \in \Gamma(\mathcal{V}^{\mathsf{L}})$ vertical Killing field associated to $U \in \mathfrak{l}$)

Recall the **Iwasawa dec.** $\mathsf{L} = \mathsf{KAN}$, K max. compact, $\mathsf{G} := \mathsf{AN}$ solvable, N nilradical of G.

Lemma If $(P)_{\mathsf{N}}$ holds for all 'Iwasawa' N then $(P)_{\mathsf{L}}$ holds.

But $(P)_{\mathsf{N}}$ holds by the structure results used to prove **Theorem A**, applied to the G-action.

Main result: proof strategy

Theorem C (Böhm, L. 2023) (M^n, g), $\mathrm{Ric}_g = -g$, with cocompact L-symmetry, L unimodular $\Longrightarrow$ $M^n \simeq_{\text{isom}} B \times \mathsf{L}/\mathsf{K}$, B compact Einstein, L/K symmetric space.

1. ~~L is semisimple. Only uses $\mathrm{Ric} < 0$~~ ✓
2. ~~L acts polarly. $\iff$ $A = 0$ for the Riemannian submersion $M_{\text{reg}} \to M_{\text{reg}}/\mathsf{L}$~~ ✓
3. L-orbits are symmetric spaces. But a priori not necessarily Einstein
4. L-orbits are totally geodesic. $\iff$ $T = 0$

The orbits are symmetric spaces

Setup (M^n, g) L-invariant, $\mathrm{Ric}_g = -g$ + L semisimple + M/L compact + $A = 0$.

$\Sigma \subset M$ section for the L-action may be non-compact, $p \in \Sigma$, $\mathrm{H} := \mathrm{L}_p$ $\mathrm{L} \cdot p \simeq \mathrm{L}/\mathrm{H}$

(Heber '98) $A = 0 \implies \exists$ Iwasawa $\mathrm{L} = \mathrm{KAN}$ such that $\mathrm{A} \cdot p \perp \mathrm{N} \cdot p$.

$v_{\mathrm{N}} := \sqrt{\det \hat{g}_{ij}}$ volume density of N-orbits, computed along Σ.

$$\Delta \log v_{\mathrm{N}} - \langle \mathrm{H}^{\mathrm{L}}, \nabla \log \mathrm{v}_{\mathrm{N}} \rangle = \dim \mathfrak{n} + \sum_{\mathrm{i} \in \mathfrak{n}} \mathrm{Ric}^{\mathrm{L}}(\mathrm{U}_{\mathrm{i}}^*, \mathrm{U}_{\mathrm{i}}^*) \geqslant 0$$

Equality $\iff$ $\mathrm{Q} := \mathrm{MAN} \leqslant \mathrm{L}$ **minimal parabolic** acts transitively on L-orbits. $\iff \mathrm{L} = \mathrm{HQ}$

By previous work, we also know $\nabla \log v_{\mathrm{N}} = -\mathrm{H}^{\mathrm{N}} \in \mathcal{V}^{\mathrm{L}} \implies \log v_{\mathrm{N}}$ constant along Σ !
$\implies \mathrm{Q}^0 \cdot p = \mathrm{L} \cdot p$ for all p.

If L split semisimple (e.g. $\mathrm{L} = \mathrm{SL}_m(\mathbb{R})$) then $\mathrm{Q}^0 = \mathrm{AN} \implies \mathrm{L} \cdot p \simeq \mathrm{L}/\mathrm{K}$ symmetric.
Also, $\Sigma \simeq M/\mathrm{L}$ is compact.

Main result: proof strategy

Theorem C (Böhm, L. 2023) $(M^n, g), \mathrm{Ric}_g = -g$, with cocompact L-symmetry, L unimodular
$\Longrightarrow \quad M^n \simeq_{\text{isom}} B \times \mathsf{L}/\mathsf{K}, \quad B$ compact Einstein, L/K symmetric space.

1. ~~L is semisimple. Only uses $\mathrm{Ric} \leq 0$~~ ✓
2. ~~L acts polarly. $\iff A = 0$ for the Riemannian submersion $M_{\text{reg}} \to M_{\text{reg}}/\mathsf{L}$~~ ✓
3. ~~L-orbits are symmetric spaces. But a priori not necessarily Einstein~~ ✓
4. L-orbits are totally geodesic. $\iff T = 0$

The orbits are totally geodesic

Setup (M^n, g) L-invariant, $\mathrm{Ric}_g = -g$ + L-orbits are symmetric + $\Sigma \subset M$ compact section.

$\mathfrak{l} = \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_r$ simple factors, $\mathfrak{l}_i = \mathfrak{k}_i \oplus \mathfrak{a}_i \oplus \mathfrak{n}_i$
$\mathsf{L} \cdot p \simeq \mathsf{L}/\mathsf{K} \implies \exists \lambda_i < 0$ such that

$$\mathrm{Ric}^{\mathsf{L}}(U_i^*, U_i^*) = \lambda_i \, \| U_i^* \|^2, \qquad U_i \in \mathfrak{l}_i.$$

For $f_{U_i} := \frac{1}{2} \| U_i^* \| : \Sigma \to \mathbb{R}$ this implies

$$\Delta f_{U_i} - \langle \mathrm{H}^{\mathsf{L}}, \nabla \mathrm{f}_{\mathrm{U_i}} \rangle = (1 + \lambda_\mathrm{i}) \, \|\mathrm{U_i}\|^2 + 2 \|\mathrm{T}_{\mathrm{U_i}}\|^2$$

Maximum principle $\implies \lambda_i \leqslant -1$ for all $i = 1, \ldots, r$.
On the other hand, tracing on the $\mathfrak{n}_i$ directions we obtain

$$\sum_i (\dim \mathfrak{n}_i) \, \lambda_i = \mathrm{tr}_{\mathsf{N}} \, \mathrm{Ric}^{\mathsf{L}} = - \dim \mathfrak{n}$$

$\implies \lambda_i = -1$ for all $i = 1, \ldots, r$!!! Maximum principle $\implies T_{U_i} \equiv 0 \; \forall i \implies T \equiv 0.$ □

Future directions

Splitting conjecture (M^n, g), $\mathrm{Ric}_g = -g$ + cocompact symmetry $\implies$ M splits isometrically as a product of a compact Einstein manifold and an Einstein solvmanifold.

Main result today proves it in the case of unimodular symmetry. For the general case, still open even in cohomogeneity one.

Dynamical Alekseevskii conjecture $\pi_1 M = 1$, $M \not\simeq \mathbb{R}^n$ $\implies$ any homogeneous Ricci flow on M has a finite time singularity.

Most results discussed today are about **non-existence** of Einstein metrics.

Question Can one produce new inhomogeneous examples of Einstein metrics with non-compact symmetry groups?

See Adam Thompson's recent paper: "Inhomogeneous deformations of Einstein solvmanifolds" arxiv:2305.05923

Thank you!