Conformal transformations of Cahen-Wallach spaces

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SPECTRUM AND SYMMETRY FOR GROUP ACTIONS IN DIFFERENTIAL GEOMETRY II
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Semi-Riemannian symmetric spaces

- A semi-Riemannian manifold (M, g) is a *symmetric space* if at each $p \in M$ there is an isometry ϕ_p that fixes p and with $d\phi_p|_p = -\mathbf{1}$.
- Symmetric spaces are locally symmetric, $\nabla R = 0$, and homogeneous, G/H, where $G = \langle \phi_p \phi_q \mid p, q \in M \rangle$ is the transvection group, and the isotropy group H in G is equal to the holonomy group of (M, g).
- A symmetric space (M, g) is *decomposable* if locally g is a product metric, and *indecomposable* otherwise. A decomposable symmetric space is universally covered by a global product of symmetric spaces.
- Riemannian indecomposable symmetric spaces (and more generally, semi-Riemannian symmetric spaces with irreducible holonomy H) were classified by E. Cartan and M. Berger, Lorentzian ones by M. Cahen & N. Wallach.

Theorem (Cahen & Wallach, '70)

Let (M, g) be an indecomposable Lorentzian symmetric space.

Algebraic version: The transvection Lie algebra of (M, g) is either (semi)simple or solvable.

Geometric version: (M, g) either has constant sectional curvature or is universally covered by a Cahen-Wallach space.

Constant sectional curvature: Minkowski, de Sitter and anti-de Sitter space

Cahen-Wallach space: $M = \mathbb{R}^{n+2}$ with Lorentzian metric

$$g_Q = 2\,\mathrm{d}v\,\mathrm{d}t + \mathrm{d}\mathbf{x}^\top \mathrm{d}\mathbf{x} + \mathbf{x}^\top Q\,\mathbf{x}\,\mathrm{d}t^2 = 2\,\mathrm{d}v\,\mathrm{d}t + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + Q_{ij}x^ix^j\,\mathrm{d}t^2,$$

where $(t, x^1, \ldots, x^n, v) = (t, \mathbf{x}, v)$ are coordinates on $\mathbb{R}^{n+2}$ and $Q = (Q_{ij})$ is a symmetric $(n \times n)$-matrix with $\det Q \neq 0$. We denote $(\mathbb{R}^{n+2}, g_Q)$ by $\mathbb{R}^{1,n+1}_Q$.

- ∂_v is a parallel null vector field
- curvature $R(\partial_t, \partial_i, \partial_j, \partial_t) = Q_{ij}$, otherwise zero,
  $\mathrm{Hol} = \mathbb{R}^n \subset \mathrm{Stab}(\partial_v) \subset \mathbf{SO}(1, n+1)$.

Clifford-Klein problem for indecomposable Lorentzian symmetric spaces

Which groups Γ can act on G/H such that $\Gamma\backslash G/H$ is a (compact) manifold?

Which indecomposable Lorentzian symmetric spaces admit compact quotients by isometries?

- Only finite groups can ac properly discontinuous on de Sitter space $\mathbb{S}^{1,n-1}$ $\rightsquigarrow$ no compact quotients [Calabi & Markus '62].
- Universal AdS $\widetilde{\mathbb{H}}^{1,n+1}$ has compact isometric quotients only if n is odd [Kulkarni, '81].
- Compact isometric quotients of Cahen-Wallach spaces: Kath & Olbrich '19.

Together with completeness results this gives a classification of indecomposable compact locally symmetric Lorentzian manifolds:

- Compact Lorentzian manifolds with constant sectional curvature are complete [Carriere '89, Klingler '96] and hence an isometric quotient of Minkowski or odd-dimensional universal AdS.
- Compact local Cahen-Wallach spaces are complete [Schliebner & L '16] and hence an isometric quotient of $\mathbb{R}^{1,n+1}_Q$.

What are the compact quotients of Cahen-Wallach spaces by a group of conformal transformations, yielding a compact conformal manifold?

Essential conformal transformations and the Lichnerowicz conjecture

$H \subseteq \mathrm{Conf}(M, g)$ is inessential if $\exists\, \hat{g} \in [g]$ such that $H \subseteq \mathrm{Isom}(M, \hat{g})$, and essential otherwise. A single $\phi \in \mathrm{Conf}(M, g)$ is (in)essential if $H = \langle \phi \rangle$ is.

Lichnerowicz conjecture '64

If (M, g) is a Riemannian manifold with an essential conformal transformation, then it is conformally diffeomorphic to Euclidean $\mathbb{R}^n$ or the round $\mathbb{S}^n$.

Proof by Ferrand and Obata '71 (compact) and Ferrand '96 (non-compact)

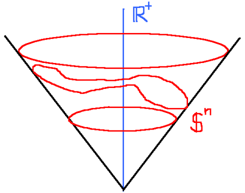
- Key ingredient [Alekseevsky 72]: The conformal group G of a Riemannian manifold (M, g) is inessential $\iff$ G acts properly.
  Proof in the case of M compact: here G proper $\iff$ G compact. Then: $G \subset \mathrm{Isom} \implies G$ compact. Conversely, if G is compact, then

$$\hat{g} = \int_G \phi^* g \, \mathrm{d}\phi \ \in \ [g] \quad \text{and } G\text{-invariant.}$$

- Ferrand analysed the dynamics of a non-proper conformal group to show that $M = \mathbb{S}^n$ or $M = \mathbb{R}^n$.

The flat model of conformal geometry

- Let C be the light cone without origin in $\mathbb{R}^{p+1,q+1}$. Its projectivisation, $C^{p,q} = \mathbb{P}C$ is finitely covered by $\mathbb{S}^p \times \mathbb{S}^q$ and hence compact, and carries a conformal structure induced from the flat metric on $\mathbb{R}^{p+1,q+1}$.



- Its conformal group $\mathbf{O}(p+1, q+1)/_{\{\pm 1\}}$ is not compact.
- For Riemannian conformal structures: $\mathrm{Conf}(\mathbb{S}^n) = \mathbf{O}(1, n+1)/_{\{\pm 1\}}$
- $\mathrm{Conf}(\mathbb{R}^n, g_{Euclid}) = \mathbf{CO}(n) \ltimes \mathbb{R}^n$ and homotheties $\mathbf{x} \mapsto \lambda \mathbf{x}$ are essential conformal transformations, however on $\mathbb{R}^n \setminus \{0\}$ they are isometries for $\hat{g} = \frac{1}{||x||^2} g_{Euclid}$, and hence inessential.
  I.e. when removing points, we loose essential conformal transformations by gaining metrics in the conformal class.
- This illustrates why completeness is not needed as assumption in the non-compact case.

The Lichnerowicz conjecture in other signatures

The original Lichnerowicz conjecture is false in other signatures:

- plenty of non-compact Lorentzian manifolds with essential conformal transformations that are not conformally flat [e.g. Podoksenov '89], in particular Lorentzian pp-waves, including Cahen-Wallach spaces.
- Examples of compact Lorentzian manifolds that are not diffeomorphic to the "Einstein universe" $C^{1,n-1} = \mathbb{S}^1 \times \mathbb{S}^{n-1}/_{\pm 1}$ by Frances '05.
- Both results extend to other signatures beyond Lorentzian.

All known compact examples are conformally flat (Weyl tensor = $W = 0$).

Generalised Lichnerowicz conjecture:

A compact pseudo-Riemannian manifold with essential conformal transformations is conformally flat.

The Generalised Lichnerowicz conjecture

- Frances '15: counterexamples in all signatures except in Lorentzian.
  Constructed as quotients of *locally symmetric metrics* on $\mathbb{R}^n \setminus \{0\}$ with two parallel null vector fields by a cyclic group $\langle\gamma\rangle$ of homotheties.
  The essential conformal transformations on the quotient lift to homotheties on $\mathbb{R}^n \setminus \{0\}$ that commute with γ.

Lorentzian Lichnerowicz conjecture (LLC)

A compact Lorentzian manifold with essential conformal transformations is conformally flat.

- Melnick & Pecastaing 2019: LLC holds for real analytic Lorentzian mfd's with finite $\pi_1(M)$.
- Melnick & Frances 2021: LLC holds for 3-dim'l real analytic Lorentzian mfd's
- Pecastaing: Let $H \subset \mathrm{Conf}(M, g)$ be essential.
  - [2018] LLC holds if H is semi-simple.
  - [2023] If H is solvable, then the nilradical of H is essential.
    LLC holds if (M, g) is real analytic and $\mathfrak{nil}(\mathfrak{conf}(M, g))$ is not abelian.

Conformal quotients

Let $(\widetilde{M}, g)$ be a semi-Riemannian manifold and let $\Gamma \subseteq \mathrm{Conf}(\widetilde{M}, g)$ act properly discontinuously so that M/Γ is a manifold.

- M inherits a conformal structure $\mathbf{c}$ that is locally represented by g.
- If $\widetilde{M}$ is simply connected, then any conformal transformation of M lifts to $\widetilde{M}$. Hence, if $\phi \in \mathrm{Conf}(M, \mathbf{c})$ is inessential, then so is $\widetilde{\phi}$.
- The converse is not true:
  If ϕ is essential, $\widetilde{\phi}$ may be inessential for a metric that does not descend to the quotient.
  If $\psi \in \mathrm{Conf}(\widetilde{M}, g)$ is essential, it may not descend to the quotient to give an essential conformal transformation.
  In case that it does, $\psi = \widetilde{\phi}$, then ϕ is essential.

The conformal group of spaces with parallel Weyl tensor

- Frances' examples in signature $(2+k, 2+\ell)$ were constructed from indecomposable symmetric spaces.
- Compact isometric quotients will not give anything conformally interesting:

Theorem (Cahen & Kerbrat '82)

Let (M, g) be a connected semi-Riemannian manifold of dimension $m \geq 4$ with $\nabla W = 0$ and $W \neq 0$. Then every (local) conformal diffeomorphism ϕ is a homothety, i.e. $\Phi^* g = \mathrm{e}^{2s} g$, with $s \in \mathbb{R}$.

- In particular, if $\nabla W = 0$ and $\Gamma \subset \mathrm{Isom}(M, g)$ cocompact and prop disc, then $W = 0$ or $\mathrm{Conf}(M/\Gamma, g) = \mathrm{Isom}(M/\Gamma, g)$.
- Cahen-Wallach spaces are locally symmetric, so unless $W = 0$, their conformal group is equal to their homothety group

$$G := \mathrm{Conf}(\mathbb{R}_Q^{1,n+1}) = \mathrm{Isom}(\mathbb{R}_Q^{1,n+1}) \rtimes \mathbb{R}.$$

- $\mathbb{R}_Q^{1,n+1}$ has $W = 0 \iff Q = \lambda \mathbf{1}$.
- Try compact conformal quotients.

The isometry group of a Cahen-Wallach space $\mathbb{R}^{1,n+1}_Q$

$$g_Q \;=\; \underbrace{2\,\mathrm{d}v\,\mathrm{d}t + \mathrm{d}\mathbf{x}^\top \mathrm{d}\mathbf{x}}_{\text{Minkowski metric}} + \mathbf{x}^\top Q\,\mathbf{x}\,\mathrm{d}t^2, \qquad (t,\mathbf{x},v) \in \mathbb{R}^{n+2}, \det(Q) \neq 0.$$

- The Euclidean group $\mathbf{Euc}(1) = \mathbb{R} \rtimes \mathbb{Z}_2$ and the centraliser C_Q of Q in $\mathbf{O}(n)$

$$\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \longmapsto \begin{pmatrix} E_{c,\epsilon}(t) := \epsilon\, t + c \\ A\boldsymbol{x} \\ \epsilon v \end{pmatrix}, \qquad \begin{array}{l} E_{c,\epsilon} \in \mathbb{R} \rtimes \mathbb{Z}_2, \\ A \in C_Q = \{A \in \mathbf{O}(n) \mid [A,Q] = 0\}. \end{array}$$

- Heisenberg group $\mathbf{Hei}_n$ (of dim $2n+1$): set $V_Q := \left\{ \beta : \mathbb{R} \to \mathbb{R}^n \mid \ddot{\beta} = Q\beta \right\}$,

$$\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \longmapsto \begin{pmatrix} t \\ \boldsymbol{x} + \beta(t) \\ v + b - \dot{\beta}(t)^\top \left(\boldsymbol{x} + \frac{\beta(t)}{2} \right) \end{pmatrix}, \quad \beta \in V_Q, \quad b \in \mathbb{R},$$

$\mathbf{Hei}_n = V_Q \times_\omega \mathbb{R}$ with $\omega(\beta,\gamma) := \frac{1}{2}\left(\beta(0)^\top \dot{\gamma}(0) - \dot{\beta}(0)^\top \gamma(0)\right)$.

The conformal group of a Cahen-Wallach space

Isometries:

$$\mathrm{Isom}(\mathbb{R}^{1,n+1}_Q) = \mathbf{Hei}_n \rtimes_\alpha (\mathbf{Euc}(1) \times C_Q), \qquad \alpha(c,\epsilon,A) : (b,\beta) \mapsto \left(\epsilon b, A\beta \circ E^{-1}_{c,\epsilon}\right)$$

Transvections $= \mathbf{Osc} = \mathbf{Hei}_n \rtimes \mathbb{R}$ with isotropy

$$L_Q := \left\{ \beta \in V_Q \mid \beta(0) = 0 \right\} \subset \mathbf{Hei}_n, \qquad \text{Lagrangian subspace in } V_Q,$$

so $\mathbb{R}^{1,n+1}_Q = \mathbf{Osc}/L_Q$ and $\mathrm{Hol} = L_Q = \mathbb{R}^n$.

Corollary

Let $\mathbb{R}^{1,n+1}_Q$ be a Cahen-Wallach space with $Q \neq \lambda\mathbf{1}$ and $n \geq 2$. Then

$$\mathrm{Conf}(\mathbb{R}^{1,n+1}_Q) = \mathrm{Isom}(\mathbb{R}^{1,n+1}_Q) \rtimes \mathbb{R} = \mathbf{Hei}_n \rtimes (\mathbf{Euc}(1) \times C_Q \times \mathbb{R}) \;=:\; G,$$

where $\mathbb{R}$ is generated by the linear homotheties

$$\begin{pmatrix} t \\ \mathbf{x} \\ v \end{pmatrix} \overset{h_s}{\longmapsto} \begin{pmatrix} t \\ \mathrm{e}^s \mathbf{x} \\ \mathrm{e}^{2s} v \end{pmatrix}, \qquad s \in \mathbb{R}.$$

Which are essential and which subgroups act properly discontinuous and cocompactly?

Essential conformal transformation of Cahen-Wallach spaces

Lemma

A homothety of a semi-Riemannian manifold with fixed point is essential.

If $\phi^* g = \mathrm{e}^{2s} g$ and ϕ is an isometry of $\mathrm{e}^{2f} g$, then at fixed point p:

$$(\mathrm{e}^{2f} g)|_p = (\phi)^*(\mathrm{e}^{2f} g)|_p = \mathrm{e}^{2f\circ\phi(p)}((\phi)^* g)|_p = \mathrm{e}^{2s}\mathrm{e}^{2f(p)} g|_p = \mathrm{e}^{2s}(\mathrm{e}^{2f} g)|_p. \qquad ⚡$$

Theorem (Teisseire & L '22)

Let ϕ be a non-isometric homothety of a Cahen-Wallach space. Then

(1) *ϕ is essential* $\iff$ (2) *ϕ has a fixed point* $\iff$

(3) *the* **Euc**(1)*-component of ϕ has a fixed point, i.e., $\epsilon = -1$ or $c = 0$.*

In particular, when the Cahen-Wallach space is not conformally flat, every essential conformal transformation is a homothety with a fixed point.

(2) $\iff$ (3):

$$\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \overset{\phi}{\longmapsto} \begin{pmatrix} E_\phi(t) := \epsilon\, t + c \\ \mathrm{e}^s A\boldsymbol{x} + \beta(t) \\ \epsilon\left(\mathrm{e}^{2s} v + b - \langle \dot\beta(t), A\boldsymbol{x} + \frac{\beta(t)}{2}\rangle\right) \end{pmatrix},$$

Essential $\Longrightarrow$ fixed point: ϕ without FP $\overset{wlog}{\Longrightarrow} t \circ \phi = t + c$ with $c \neq 0$
$\Longrightarrow \langle \phi \rangle$ has a locally finite fundamental domain $D = \mathbb{R}^{n+1} \times (0, c)$.
Let $h : \mathbb{R} \to \mathbb{R}_{>0}$ be a bump function of t with $h|_{[0,c]} \equiv 1$ and support in $(-\frac{c}{4}, \frac{5c}{4})$.
Then define functions $\{h_k\}_{k \in \mathbb{Z}}$ on $\mathbb{R}^{n+2}$ by

$$h_0(t, \mathbf{x}, v) = h(t), \qquad h_k := h_0 \circ \phi^{-k}, \qquad \text{i.e.} \quad h_k = h_{k+1} \circ \phi$$

Then $\left\{\operatorname{supp}(h_k)\right\}_{k \in \mathbb{Z}}$ is locally finite, the function $\sum_{k \in \mathbb{Z}} h_k$ is well defined and has no zeros. Hence, we get a partition of unity on $\mathbb{R}^{n+2}$

$$f_k := \frac{h_k}{\sum_{k \in \mathbb{Z}} h_k} \quad \text{with} \quad f_k = f_{k+1} \circ \phi.$$

If $\phi^* g = \mathrm{e}^{2s} g$, we set

$$f := -s \sum_{k \in \mathbb{Z}} k f_k,$$

which yields that ϕ is an isometry for $\mathrm{e}^{2f} g$ as

$$f \circ \phi = -s \sum_{k \in \mathbb{Z}} k f_{k-1} = f - s \sum_{k \in \mathbb{Z}} f_k = f - s.$$

Therefore, ϕ is not essential. $\lightning$

Compact quotients of Cahen-Wallach spaces of imaginary type

A Cahen-Wallach space $\mathbb{R}^{1,n+1}_Q$ is of *imaginary type* if Q is negative definite and of *real type* if Q is positive definite, otherwise of *mixed type* [Kath & Olbrich '19].

$\rightsquigarrow$ Different behaviour of the solutions of $\ddot{\beta} = Q\beta$,

$$\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \stackrel{\phi}{\longmapsto} \begin{pmatrix} \epsilon t + c \\ e^s A\boldsymbol{x} + \beta(t) \\ \dots \end{pmatrix}, \qquad \begin{array}{rcll} \beta^i(t) & = & \beta^i_0 \cos(\mu t) + \frac{\beta^i_1}{\mu} \sin(\mu t), & \text{for ev } -\mu^2 \text{ of } Q \\ \beta^i(t) & = & \beta^i_0 \cosh(\lambda t) + \frac{\beta^i_1}{\lambda} \sinh(\lambda t), & \text{for ev } \lambda^2 \text{ of } Q \end{array}$$

Theorem (Teisseire & L '22)

Let $\mathbb{R}^{1,n+1}_Q$ be a Cahen-Wallach space of imaginary type and with $Q \neq \lambda\mathbf{1}$, and let $\Gamma \subset G$ acting properly discontinuously and with compact quotient $M = \mathbb{R}^{n+2}/\Gamma$.

Then Γ is contained in the isometries.

In particular, M is locally isometric to $\mathbb{R}^{1,n+1}_Q$ and $\mathrm{Conf}(M) = \mathrm{Isom}(M)$.

Step 1 of the proof

If $\Gamma \subset G$ acts cocompactly and contains a homothety

$$\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \overset{\gamma}{\longmapsto} \begin{pmatrix} t + a \\ \mathrm{e}^{s} A \boldsymbol{x} \\ \mathrm{e}^{2s} v \end{pmatrix} \quad \in \quad \mathbf{Euc}(1) \times C_Q \times \mathbb{R} \simeq \mathbf{G}/_{\mathbf{Hei}_n},$$

then Γ cannot act properly discontinuously.

- Γ acts cocompactly $\implies$ Γ not cyclic, i.e. $\exists\, \phi \in \Gamma \setminus \langle \gamma \rangle$.
- Γ prop disc $\implies$ ϕ has $c \neq 0$.
- The sequence in the Γ-orbit of 0 accumulates,

$$y_k = \gamma^{-k} \phi \gamma^{k}(0) = \begin{pmatrix} c \\ \mathrm{e}^{-ks} (A^{\top})^{k} \beta(ka) \\ \mathrm{e}^{-2ks} \left(b - \left\langle \dot{\beta}(ka), \tfrac{1}{2} \beta(ka) \right\rangle \right) \end{pmatrix} \xrightarrow{k \to \infty} \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix},$$

  because β is bounded as a solution of $\ddot{\beta} = Q\beta$ with Q negative definite.
- This convergence contradicts proper discontintuity of Γ.

Note: We prove this under the more general assumption that γ satisfies that $0 < s^2 - \lambda^2 a^2 = (s - \lambda c)(s + \lambda c)$ for all *positive* eigenvalues λ^2 of Q.

Step 2 of the proof

Every $\hat{\gamma} \in G$ with $\epsilon = 1$ is conjugate to a homothety in $\mathbb{R} \times C_{\mathbf{O}(n)}(S) \times \mathbb{R}$.

- We can conjugate $\hat{\gamma}$ by $(b, \beta) \in \mathbf{Hei}_n$ into $\mathbf{Euc}(1) \times C_Q \times \mathbb{R} \iff$

$$e^{\hat{s}} \hat{A}\, \beta(t - \hat{c}) - \beta(t) = \hat{\beta}(t), \quad \text{for } \beta \in V_Q. \qquad (*)$$

- Since A commutes with Q we solve this on each eigenspace separately.
- For negative eigenvalue $-\mu^2$, $\beta \in V_Q$ is given as

$$\beta(t) = \beta_0 \cos(\mu t) + \tfrac{\beta_1}{\mu} \sin(\mu t).$$

$$(*) \iff \underbrace{\begin{pmatrix} \mu(e^s \cos(\mu c) A - \mathbf{1}) & e^s \sin(\mu c) A \\ -\mu e^s \sin(\mu c) A & e^s \cos(\mu c) A - \mathbf{1} \end{pmatrix}}_{=:M} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \mu \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix}$$

$$\det(M) = \mu^n \det\left(e^{2s} A^2 - 2e^s \cos(\mu c) A + \mathbf{1}\right) = \prod_{i=1}^{n} \left(e^{2s} z_i^2 - 2e^s \cos(\mu c) z_i + 1\right),$$

where $z_i \in \mathbb{S}^1$ are the eigenvalues of A. But since $s \neq 0$, $\det(M)$ has no roots on the unit circle, so we can invert M and solve $(*)$.

Note: Again our proof works also for positive eigenvalues of Q under the assumption that $\hat{s}^2 \neq \lambda^2 \hat{c}^2$.

Cocompact groups in the centraliser of an essential homothety

If ϕ is an essential homothety of $\mathbb{R}^{1,n+1}_Q$ and if there was a properly discontinuous, cocompact Γ such that $\phi \in N_G(\Gamma)$, then ϕ would descend to the quotient, remain essential, and would provide a counterexample to the LLC.

Examples by Frances in signature $(p+2, q+2)$ are constructed in this way: two commuting homotheties ϕ and γ that are essential, $\Gamma = \langle\gamma\rangle$, so that $\Gamma \in C_G(\phi)$. Then ϕ descends to the quotient as an essential conformal transformation.

Theorem (Teisseire & L '22)

A group of conformal transformations of a conformally curved Cahen-Wallach space centralising an essential conformal transformation cannot act properly discontinuously and cocompactly.

This leaves the possibility of such Γs normalising an essential homothety and of course the possibility of an essential conformal transformation on the quotient that does not come from one on the cover.

Idea of the proof

Easy case: $\phi = diag(1, \mathrm{e}^s \mathbf{1}, \mathrm{e}^{2s})$ a *linear homothety*. Then

$$C_G(\phi) = \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R} \ni \gamma : \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \mapsto \begin{pmatrix} \epsilon\, t + c \\ e^s A \boldsymbol{x} \\ \epsilon\, e^{2s} v \end{pmatrix}.$$

- Since Γ acts freely, $\epsilon = 1$ and $c \neq 0$ unless $\gamma = \mathbf{1}$. Hence, the projection

  $\rho : \Gamma \to \mathbb{R}^2, \qquad \gamma = (c, A, s) \mapsto (c, s)$ is an injective homomorphism

  since $Ker(\rho) = \Gamma \cap C_{\mathbf{O}(n)}(S) \neq \{1\}$ contradicts proper discontionuity.
- $\Gamma \simeq \rho(\Gamma) \simeq \mathbb{Z}^2 \subset \mathbb{R}^2$, but this contradicts cocompactness and prop disc: If $\Gamma = \langle \gamma_1, \gamma_2 \rangle \simeq \mathbb{Z}^2$ with corresponding c_1 and c_2, and p_n and q_n such that $\frac{p_n}{q_n} \to \frac{c_1}{c_2}$, then $\gamma_1^{p_n} \gamma_2^{-q_n}$ has $c_n \to_n 0$ and hence $\gamma^{p_n} \phi^{-q_n}(0) \to 0$. $\lightning$

General case:

$$\begin{array}{ccccccc} G \supset C_G(\phi) & \overset{!!!}{\hookrightarrow} & G/_{\mathbf{Hei}_n} = \mathbf{Euc}(1) \times C_{\mathbf{O}(n)}(S) \times \mathbb{R} & & \\ \cup & & \cup & & \\ \Gamma & \simeq & \hat{\Gamma} & \overset{!!!}{\hookrightarrow} & \mathbb{R}^2 \end{array}$$

Again $\hat{\Gamma} \simeq \mathbb{Z}^2$ and a contradiction.

Let $\mathbb{R}^{1,3}_{-\mathbf{1}}$ be a conformally flat Cahen-Wallach space of imaginary type of dim 4.
Let Γ be generated by the following isometries,

$$\gamma\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} := \begin{pmatrix} t+\frac{\pi}{2} \\ \boldsymbol{x} \\ v \end{pmatrix}, \qquad \eta\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} := \begin{pmatrix} t \\ \boldsymbol{x} \\ v+1 \end{pmatrix},$$

where $\mathbf{x} = (x^1, x^2)$ and

$$\zeta\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} := \begin{pmatrix} t \\ \boldsymbol{x}+\beta(t) \\ v-\langle\beta(t), \boldsymbol{x}\rangle \end{pmatrix}, \quad \text{with} \quad \beta(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

Then $\mathbb{R}^4/\Gamma = \mathbb{T}^4$ is compact.

Example: isometric quotient when $Q > 0$

For $r \in \mathbb{N}_{\geq 3}$, consider the polynomial $f(x) = x^2 - rx + 1$ with roots $\rho \neq \rho^{-1}$. Let $\mathbb{R}^{1,3}_Q$ be the conformally flat Cahen-Wallach space of real type defined by $Q = (\ln|\rho|)^2\mathbf{1}$. Let Γ be the group generated by

$$\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \overset{\alpha}{\mapsto} \begin{pmatrix} t \\ \boldsymbol{x} \\ v+1 \end{pmatrix}, \quad \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \overset{\gamma}{\mapsto} \begin{pmatrix} t+1 \\ \boldsymbol{x} \\ v \end{pmatrix}, \quad \beta(t) = \begin{pmatrix} \rho^t \\ \\ \rho^{-t} \end{pmatrix} \quad \text{and} \quad \hat{\beta}(t) = \beta(t+1).$$

Then β and $\hat{\beta}$ are linearly independent in V_Q with

$$\hat{\beta}(t+1) - r\hat{\beta}(t) + \beta(t) = \begin{pmatrix} \rho^t f(\rho) \\ \\ \rho^{-t} f(\rho^{-1}) \end{pmatrix} = 0,$$

and hence $\Gamma \subset \mathrm{Isom}(\mathbb{R}^{1,3}_Q)$ is a lattice, i.e. acts cocompactly on $\mathbb{R}^{1,3}_Q$.
Replacing γ by a homothety

$$\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \overset{\hat{\gamma}}{\mapsto} \begin{pmatrix} t+1 \\ \rho\boldsymbol{x} \\ \rho^2 v \end{pmatrix},$$

yields a non-discrete group with $\hat{\gamma}^{-k}\alpha\hat{\gamma}^k = \alpha_{\rho^{1-2k}}$ converging to the identity. ☹

Example: compact conformal quotient of an open subset of $\mathbb{R}^{1,n+1}_Q$

Consider $U := \mathbb{R}^{1,n+1}_Q \setminus \{(t,0,0) \mid t \in \mathbb{R}\}$ and Γ generated by

$$\begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \overset{\gamma}{\mapsto} \begin{pmatrix} t+1 \\ \boldsymbol{x} \\ v \end{pmatrix}, \qquad \begin{pmatrix} t \\ \boldsymbol{x} \\ v \end{pmatrix} \overset{\eta}{\mapsto} \begin{pmatrix} t \\ 2\boldsymbol{x} \\ 4v \end{pmatrix},$$

Γ acts properly discontinuously and cocompactly on U, $M = U/\Gamma = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^n$,
Γ is normalised by $H = \mathbb{R} \times C_Q \times \mathbb{R}$.
Unfortunately, by removing the fixed points, we gain metrics, so all $\phi \in H$ are inessential, because they are isometries of

$$\hat{g} = \frac{1}{\sqrt{\|\boldsymbol{x}\|^4 + (v)^2}} g_Q.$$

We do not know if there are essential conformal transformations on $M = U/\Gamma$ which do not lift or such that the lift is inessential ...

Thank you!