The prescribed cross curvature problem on $\mathbb{S}^3$

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Riemannian Geometry

Let M be a smooth manifold.

The study of the geometry of M is achieved by equipping M with a Riemannian metric g, which is a smooth assignment of an inner product to each tangent space. We can write g in local co-ordinates with $g_{ij} = g(\partial_{x_i}, \partial_{x_j})$.

The Ricci curvature is a qausi-linear second order partial differential operator $\mathrm{Ric} : \{\text{Riemannian metrics}\} \to \{\text{symmetric } (0,2)\text{-tensor fields}\}$. Ricci curvautre is diffeomorphism-invariant, and is therefore not elliptic.

Riemannian Geometry

Many important problems in Riemannian geometry are studied through the study of solutions to equations involving Ricci curvature:

- The *Einstein equation* $\mathrm{Ric}(g) = \lambda g$ for some constant λ. Solutions are the 'ideal' geometries;
- The *prescribed Ricci curvature equation* $\mathrm{Ric}(g) = T$ asks what symmetric tensors T are actually available as Ricci curvature of some metric g.
- The *Ricci flow* $\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g)$ evolves a Riemannian metric in the direction of its curvature to hopefully 'improve' its geometry.

Ricci flow is an excellent tool for positive curvature geometries, but is less useful for negative curvature geometries.

Cross curvature

- Let (M, g) be a smooth Riemannian manifold with its scalar curvature $S(g) = \mathrm{tr}_g \mathrm{Ric}(g)$ and its Einstein tensor field $E(g) = \mathrm{Ric}(g) - \frac{S(g)g}{2}$.
- Let $\mathcal{E}(g)$ be the linear operator corresponding to $E(g)$, i.e., $\mathcal{E}(g) = E(g)^\sharp$.
- The *cross curvature* of g is the symmetric $(0, 2)$-tensor field

$$X(g) = \det(\mathcal{E}(g))g(\mathcal{E}^{-1}\cdot, \cdot),$$

which exists whenever $\mathcal{E}(g)$ is invertible on all of M.
- Studied almost exclusively for **three-dimensional manifolds**.

Cross curvature flow

- Chow-Hamilton, ’04:
  - Introduction of the cross curvature flow $\frac{\partial g}{\partial t} = X(g)$ on three-manifolds;
  - Second contracted Bianchi identities for $X(g)$;
  - The IVP is well-posed for initial metrics of negative (or positive) sectional curvature (Nash-Moser approach);
  - Monotonicity formulas;
  - Negative sectional curvature is preserved.
- Buckland, ’05: a more geometric approach to well-posedness of IVP (DeTurck trick).
- Andrews-Chen-Fang-McCoy, ’15: Convergence to hyperbolic metrics in case the initial metric satisfies an integrability condition.

The prescribed cross curvature equation

Conjecture (Hamilton, '08):

Given any positive symmetric $(0,2)$-tensor field T on $\mathbb{S}^3$, there exists a unique Riemannian metric g satisfying

$$X(g) = T.$$

- Gkigkitzis '08: Conjecture is true if g and T are both assumed to be left-invariant on $SU(2)$.
- Hartley: local existence for positive T.

The prescribed cross curvature equation

Regularity:

Theorem (B-Pulemotov, '23)

If $g \in C^3$ is a metric solving $X(g) = T$ in a neighbourhood of the point p, and $T(p) > 0$, then $g \in C^{k,\alpha}$ whenever $T \in C^{k,\alpha}$, $(k, \alpha) \in \mathbb{N} \times (0, 1)$.

Compare with:

Theorem (DeTurck-Kazdan, '81)

If $g \in C^2$ is a metric solving $\mathrm{Ric}(g) = T$ in a neighbourhood of the point p, and $T(p)$ is non-degenerate, then $g \in C^{k,\alpha}$ whenever $T \in C^{k,\alpha}$, $(k, \alpha) \in \mathbb{N} \times (0, 1)$.

The additional assumptions that $g \in C^3$ and $T(p) > 0$ are related to the fact that $X(g)$ is fully non-linear (with principal symbol dependent on the Einstein tensor), whereas $\mathrm{Ric}(g)$ is quasi-linear.

The prescribed cross curvature equation

Close to the round sphere:

Theorem (B-Pulemotov, '23)

Let g_0 be the round metric on $\mathbb{S}^3$ with Einstein constant 2. If $k > \frac{9}{2}$ and T is close to g_0 in H^k, then there is a g so that $X(g) = T$. If T is smooth, then g is smooth.

Proof is by decomposing $H^k(S^2T^*M) = H^k(\ker\delta) \oplus \delta^* H^{k+1}(T^*M)$ because:

- $X'(g_0)$ is elliptic and injective on $H^k(\ker\delta)$;
- $X'(g_0)$ is not elliptic on $\delta^* H^{k+1}(T^*M)$, but this is precisely the tangent space of the action of the H^{k+1} diffeomorphism group.

The prescribed cross curvature equation

With symmetries:

Theorem (B-Pulemotov, '23)

Suppose $T > 0$ is $O(2) \times O(2)$-invariant and 'diagonal' on $\mathbb{S}^3$. Then there is a g on $\mathbb{S}^3$ satisfying $X(g) = T$.

The group action is given by the embedding $\mathbb{S}^3 \subset \mathbb{R}^2 \times \mathbb{R}^2$, with $O(2)$ acting on each $\mathbb{R}^2$ factor. The principal orbits are $\mathbb{S}^1 \times \mathbb{S}^1$, and the singular orbits are two copies of $\mathbb{S}^1$.

The prescribed cross curvature equation

Proof 1/2 (Setting up the ODE problem)

- The principal part of the cohomogeneity one action is $(0,1)\times\mathbb{S}^1\times\mathbb{S}^1$; we will look at

$$T = dr^2 + y_1(r)^2 d\theta_1^2 + y_2(r)^2 d\theta_2^2,\ \ g = h(r)^2 dr^2 + f_1(r)^2 d\theta_1^2 + f_2(r)^2 d\theta_2^2.$$

- Using $l_i = -\frac{f_i'}{h}, \phi_1 = \frac{y_1}{y_2}, \phi_2 = \frac{y_2}{y_1}, \sigma = y_1 y_2$, the equations become

$$\left(\frac{l_1'}{\phi_1}\right)' = \frac{l_1 l_2^2}{\sigma}, \qquad \left(\frac{l_2'}{\phi_2}\right)' = \frac{l_1^2 l_2}{\sigma}, \qquad r \in (0,1).$$

- The required boundary conditions come from insisting that both g and T are actually smooth at the singular orbits:
  - At $r=0$, l_1 is even, $l_1(0)=-1$, l_2 is odd, y_1 is odd, $y_1'(0)=1$, y_2 is even and $y_2(0)>0$.
  - At $r=1$, l_2 is even, $l_2(1)=1$, l_1 is odd, y_2 is odd, $y_2'(1)=-1$, y_1 is even and $y_1(0)>0$.

The prescribed cross curvature equation

Proof 2/2 (Solving the ODEs):

- Study the IVP near each singular orbit; with the arbitrary prescription of two real parameters at each end, the local IVP is well-posed.
- Show that, for any solution of

$$\begin{aligned} \left(\frac{l_1'}{\phi_1}\right)' &= \frac{l_1(pl_2 + (1-p)\sin\frac{\pi r}{2})^2}{\sigma}, \\ \left(\frac{l_2'}{\phi_2}\right)' & \quad \frac{l_2(pl_1 - (1-p)\cos\frac{\pi r}{2})^2}{\sigma_p}, \qquad r \in (0,1), \end{aligned}$$

  the four parameters are bounded independently of $p \in [0,1]$.
- When $p = 0$, the equations are linear, decoupled and non-degenerate; topological degree theory implies existence for $p = 1$.

The prescribed cross curvature equation

With more symmetries:

Theorem (B-Pulemotov, '23)

Suppose $T > 0$ is $O(3)$-invariant on $\mathbb{S}^3$. Then there is an $O(3)$-invariant g on $\mathbb{S}^3$ satisfying $X(g) = T$.

Compare:

Theorem (Hamilton, '84)

Suppose $T > 0$ is $O(3) \times \mathbb{Z}_2$-invariant on $\mathbb{S}^3$. Then there is a g and a $c > 0$ on $\mathbb{S}^3$ satisfying $\mathrm{Ric}(g) = cT$ if T itself has positive Ricci curvature.

Existence within class of $O(3) \times \mathbb{Z}_2$-invariant metrics may fail without the assumption that T has positive Ricci curvature.

The prescribed cross curvature equation

Proof 1/2 (Setting up the ODE problem)

- The principal part of the cohomogeneity one action is $(-1,1)\times\mathbb{S}^2$; let Q be the standard metric on $\mathbb{S}^2$, look at $T = dr^2 + y(r)^2 Q$ and $g = h(r)^2 dr^2 + f(r)^2 Q$.
- Using $l = -\frac{f'}{h}, \sigma = y^2$, the equations become

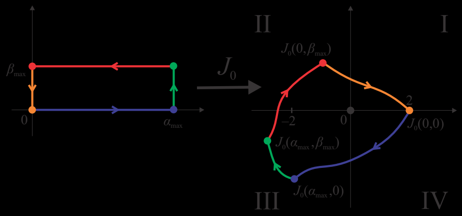
$$l'' = \frac{l^3 - l}{\sigma}.$$

- Boundary conditions come from insisting that both g and T are smooth at the singular orbits: At $r \pm 1$, l is even, $l = \pm 1$, σ is vanishing and even with $\sigma'' = 2$.
- We also need to insist that $l'(r) > 0$ for each $r \in (-1,1)$, for the metric to be positive-definite.

The prescribed cross curvature equation

Proof 2/2 (Solving the ODE)

- At each singular orbit, the local IVP is well-posed with the prescription of one real parameter.
- Show that, for any continuous deformation of σ_p, $p \in [0, 1]$, solutions to $l'' = \frac{l^3 - l}{\sigma_p}$ have the two parameters α, β bounded independently of p.
- There is a special choice of σ_0 so that the problem has a non-zero degree, and the solution has $l' > 0$ on the interior.
- Topological degree theory implies the existence of a continuum of solutions hitting every possible p. The condition $l' > 0$ survives through the continuum.



The prescribed cross curvature equation

Theorem (B-Pulemotov, '23) There is an $O(3)$-invariant $T > 0$ on $\mathbb{S}^3$ so that $X(g) = T$ has at least three distinct solutions. Two of these solutions are isometric, but not isometric to the third.

Therefore, the uniqueness component of Hamilton's conjecture is false.

The prescribed cross curvature equation

Proof

- We are trying to find a choice of positive σ so that $l'' = \frac{l^3 - l}{\sigma}$ has at least three solutions, with l crossing from -1 to 1.
- If we insist that σ is even about $r = 0$, we can always find a solution l which is odd about $r = 0$.
- This solution encounters a pitchfork bifurcation by making σ quite small on the interior.

Some concluding remarks

- These results suggest that $X(g) = T$ might always be globally solvable on $\mathbb{S}^3$ for $T > 0$ (though not uniquely).
- This is in constrast to the equation $Ric(g) = cT$ for $T > 0$ on $\mathbb{S}^3$; there are cohomogeneity one $T > 0$ for which there are no *cohomogeneity one* solutions g.
- Is there a useful variational interpretation?