Scalar curvature along Ebin geodesics

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The Einstein-Hilbert action

Let M be a smooth, connected and closed manifold, and $\mathcal{M}$ the set of Riemannian metrics on M. The *Einstein-Hilbert action* is the function $\mathcal{S} : \mathcal{M} \to \mathbb{R}$ with

$$\mathcal{S}(g) = \int_M S(g) dV_g,$$

where $S(g)$ is the scalar curvature of g, and dV_g is the volume form of g.

Theorem

Critical points of the Einstein-Hilbert functional subject to the constraint $\int_M dV_g = 1$ are precisely Einstein metrics.

Topology and Geometry on $\mathcal{M}$

$\mathcal{M}$ is an infinite-dimensional Fréchet manifold. For any $g \in \mathcal{M}$ and $h, k \in T_g\mathcal{M}$, the L^2 or *Ebin* Riemannian structure E on $\mathcal{M}$ is found by specifying

$$E_g(h,k) = \int_M g(h,k)dV_g.$$

Theorem (Freed-Groisser, '89)

An Ebin geodesic starting at g with initial velocity h is given by

$$\gamma(t)(X,Y) = \left(q(t)^2 + r^2t^2\right)^{\frac{n}{2}} g\left(\exp\left(\frac{4}{\sqrt{ntr(H_0^2)}}\arctan(\frac{r(t)}{q(t)})H_0\right)X, Y\right),$$

where H_0 is the trace-free part of $g^{-1}h$, $q(t) = 1 + \frac{t}{4}tr(H_0)$, $r = \frac{1}{4}\sqrt{ntr(H_0^2)}$.

Volume forms and diffeomorphisms

Fix a volume form μ on M, and consider $\mathcal{M}_\mu \subset \mathcal{M}$, the set of Riemannian metrics g with volume form μ.
$\mathcal{M}_\mu$ contains all Riemannian structures:

Theorem (Moser, '65)

For any two Riemannian metrics g_1, g_2 on M, there is a diffeomorphism $\phi : M \to M$ and a scale $c > 0$ so that $c\phi^*(g_1)$ has the same volume form as g_2.

The intrinsic geodesics of $\mathcal{M}_\mu \subset \mathcal{M}$ have the form $\gamma_t(\cdot,\cdot) = g_0(\exp(tH)\cdot,\cdot)$, where $tr(H) = 0$.

Main idea: study the asymptotics of the E-H functional using these geodesics.

Reducing the complexity of the Einstein-Hilbert functional

To understand the E-H functional

$$\mathcal{S}(g) = \int_M S(g) dV_g,$$

Moser's result implies that it suffices to pick a reference Riemannian metric g_0 with volume form μ_0, and study

$$\int_M S(\gamma_H(t))\mu_0,$$

where $\gamma_H(t)(\cdot,\cdot) = g_0(\exp(tH)\cdot,\cdot)$, with H self-adjoint (w.r.t. g_0) and traceless.

So we can focus on the scalar curvature of $\gamma_H(t)$, rather than the scalar curvature of $\gamma_H(t)$ multiplied by its volume element.

Scalar curvature along homogeneous Ebin geodesics

If G is a compact Lie group of diffeomorphisms on M, then we define $\mathcal{M}_G$ to be the G-invariant Riemannian metrics on M. Critical points of the E-H functional restricted to unit-volume metrics on $\mathcal{M}_G$ are G-invariant Einstein metrics.

If G acts transitively on M, then

- $M = G/H$ is a homogeneous space, where H is the isotropy subgroup of a fixed point,
- $\mathcal{M}_G$ is finite-dimensional, and
- the scalar curvature of a metric $g \in \mathcal{M}_G$ is constant.

Scalar curvature along homogeneous Ebin geodesics

Let Q be a unit-volume bi-invariant metric on G. Choose $\mathfrak{m}$ to be the Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then homogeneous Ebin geodesics starting at Q on M are given by

$$\gamma(t) = \sum_{i=1}^{l} e^{tv_i} Q|_{\mathfrak{m}_i},$$

where $\{v_i\}_{i=1}^{l} \subset \mathbb{R}$, and $\mathfrak{m} = \bigoplus_{i=1}^{l} \mathfrak{m}_i$ is a choice of $Ad(H)$-irreducible and Q-orthogonal decomposition. The volume 1 constraint is $\sum_{i=1}^{l} d_i v_i = 0$, where $d_i = \dim(\mathfrak{m}_i)$. We can also assume that $\sum_{i=1}^{l} d_i v_i^2 = 1$.

Scalar curvature along homogeneous Ebin geodesics

The scalar curvature is given by

$$S(\gamma(t)) = \frac{1}{2}\sum_{i=1}^{l} d_i b_i e^{-v_i t} - \frac{1}{4}\sum_{i,j,k=1}^{l} [ijk] e^{t(v_i - v_j - v_k)},$$

where

- $[ijk] = \sum_{\mu,\nu,\rho} Q([X_\mu, Y_\nu], Z_\rho)^2$, $\{X_\mu\}, \{Y_\nu\}, \{Z_\rho\}$ are Q-orthonormal bases for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$ respectively,
- $b_i = -\frac{B_{\mathfrak{m}_i}}{Q|_{\mathfrak{m}_i}}$, B is the Killing form of $\mathfrak{g}$.

Asymptotic behaviour is determined by a competition between the two non-negative terms $\frac{1}{2}\sum_{i=1}^{l} d_i b_i e^{-v_i t}$ and $\frac{1}{4}\sum_{i,j,k=1}^{l}[ijk]e^{t(v_i - v_j - v_k)}$.

Scalar curvature along homogeneous Ebin geodesics

Recall

$$S(\gamma(t)) = \frac{1}{2}\sum_{i=1}^{l} d_i b_i e^{-v_i t} - \frac{1}{4}\sum_{i,j,k=1}^{l} [ijk] e^{t(v_i - v_j - v_k)}.$$

Since $\sum_{i=1}^{l} d_i v_i = 0$ and $\sum_{i=1}^{l} d_i v_i^2 = 1$, at least one v_i is negative, so there are terms with potentially exponential growth.
If v_k is the smallest, and $v_j < v_i$, then $[ijk]e^{t(v_i - v_j - v_k)}$ will dominate all of the $d_i b_i e^{-v_i t}$ terms **unless** $[ijk] = 0$.

Therefore, $S(\gamma(t))$ converges to $-\infty$ except for special choices of v and decomposition of $\mathfrak{m}$.

Scalar curvature along homogeneous Ebin geodesics

The vanishing of some of the $[ijk]$ terms is related to the existence of intermediate subgroups between H and G. Böhm-Wang-Ziller associate a graph to G/H which contains information about intermediate subgroups.

Theorem (Böhm-Wang-Ziller, '04)

Let G/H be a compact homogeneous space. If the graph of G/H has at least two non-toral components, then G/H admits a homogeneous Einstein metric.

This is proven by the Mountain-Pass Theorem, once they have enough compactness.

Theorem (Böhm-Wang-Ziller, '04)

Let G/H be a compact homogeneous space. If the graph of G/H has at least two non-toral components, then G/H admits a homogeneous Einstein metric. Let G/H be a homogeneous space. Then for each $\epsilon > 0$, the scalar curvature functional satisfies the Palais-Smale compactness condition on the set of unit-volume, G-invariant Riemannian metrics that have scalar curvature at least ϵ.

Scalar curvature along Ebin geodesics

Motivating question: can these techniques be used to construct Einstein metrics **without** homogeneous symmetry?
To do this, we would need to

- obtain a satisfactory compactness theory, and
- understand scalar curvature asymptotics.

The first looks quite challenging:

Theorem (Böhm, '98)

For each $n \in [5, 9]$, there exists a sequence of unit-volume Einstein metrics on $\mathbb{S}^n$ which converge to a non-smooth Einstein metric with positive scalar curvature (thus breaking Palais-Smale compactness).

Scalar curvature along Ebin geodesics

What about curvature asymptotics? Recall that Ebin geodesics starting at a Riemannian metric g_0 look like

$$\gamma_H(t)(\cdot,\cdot) = g_0(\exp(tH)\cdot,\cdot),$$

where H is traceless, and self-adjoint with respect to g_0.

Theorem (Böhm-B-Clarke)

Suppose $\dim(M) \geq 5$. There exists an open and dense set of H (in the Whitney C^∞ topology) for which

$$\lim_{t\to\infty} S(\gamma_H(t)) = -\infty$$

uniformly on M.

Scalar curvature along Ebin geodesics

First, we need a formula for the scalar curvature. Let us first assume that there is a local g_0-orthonormal frame $\{e_1, \cdots, e_n\}$ diagonalising H with strictly increasing eigenvalues $(\lambda_1, \cdots, \lambda_n)$.

Lemma

We have

$$R(g_t) = \sum_{i \neq j \neq k} e^{(\lambda_k - \lambda_i - \lambda_j)t} \cdot F_{ij}^k + \sum_{i=1}^{n} e^{-\lambda_i t} \cdot (t^2 F_i^{(2)} + t F_i^{(1)} + F_i^{(0)}),$$

where $F_{ij}^k = -\frac{1}{4} g_0([e_i, e_j], e_k)^2$, and $F_i^{(j)}$ is independent of t.

With enough of the F_{1j}^k terms (with $1 < j < k$), we can perturb (using theory of under-determined PDES) so that at least one is non-zero, and scalar curvature goes to $-\infty$.

Scalar curvature along Ebin geodesics

But what if we have eigenvalue bifurcations? For any traceless H and any point $p \in M$, let $(\lambda_1, \cdots, \lambda_L)$ be the eigenvalues of H in increasing order, occuring with multiplicities $(m_1, \cdots, m_L)$. Then there is a local g_0-orthonormal frame $e = \bigcup_{i=1}^{L} \{e_{i_a}\}_{a=1}^{m_i}$ in which H takes the block diagonal form

$$H = \begin{pmatrix} \lambda_1 I_{m_1} + S_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{m_2} + S_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_L I_{m_L} + S_L \end{pmatrix},$$

where S_i is a traceless and symmetric $m_i \times m_i$ matrix. We have $\lambda_1 < 0$, unless all λ_is vanish.

Scalar curvature along Ebin geodesics

To compute the scalar curvature, we use the $\gamma_H(t)$-orthonormal frame $\bigcup_{i=1}^{L}\{e^t_{i_a}\}_{a=1}^{m_i}$, where for $i = 1, ..., L$ and $\tilde{a} = 1, ..., m_i$

$$e^t_{i_{\tilde{a}}} = \frac{1}{\sqrt{\alpha_i(t)}} \cdot \sum_{a=1}^{m_i} \exp\left(-\tfrac{S_i t}{2}\right)_{a\tilde{a}} \cdot e_{i_a}, \qquad \alpha_i(t) = \exp(\lambda_i t).$$

Then we define the Christoffel symbols

$$(\Gamma_t)^{k_c}_{i_a, j_b} = \gamma_H(t)([e^t_{i_a}, e^t_{j_b}], e^t_{k_c}) + \gamma_H(t)([e^t_{k_c}, e^t_{j_b}], e^t_{i_a}) + \gamma_H(t)([e^t_{k_c}, e^t_{i_a}], e^t_{j_b}).$$

The scalar curvature $S(\gamma_H(t))$ is given by

$$S(\gamma_H(t)) = 2\sum_{i_a, j_b} e^t_{i_a}(\Gamma_t)^{i_a}_{j_b, j_b} - \sum_{i_a, j_b, k_c}\left((\Gamma_t)^{k_c}_{i_a, i_a}(\Gamma_t)^{k_c}_{j_b, j_b} + (\Gamma_t)^{j_b}_{i_a, k_c}(\Gamma_t)^{j_b}_{k_c, i_a}\right).$$

Scalar curvature along Ebin geodesics

The term $\sum_{i_a,j_b,k_c} \left((\Gamma_t)^{j_b}_{i_a,k_c}(\Gamma_t)^{j_b}_{k_c,i_a}\right)$ contains a sum of squares of $\gamma_H(t)([e^t_{i_a}, e^t_{j_b}], e^t_{k_c})$. The ones with $1 = i \leq j < k$ will have the most powerful exponential growth, provided

$$g_0([e_{i_a}, e_{j_b}], e_{k_c}) \neq 0$$

for some $a \leq m_i$, $b \leq m_j$, $c \leq m_k$.

If $n = \dim(M)$ and we have at least $n+1$ of these terms, we can perturb the g_0-orthonormal frame so that at least one of them is non-zero.

Scalar curvature along Ebin geodesics

How can we ensure there are enough of these terms?
If for example, M is two dimensional, and H has a local (x, y) co-ordinate expression

$$H = \begin{pmatrix} x & y \\ y & -x \end{pmatrix},$$

then the existence of a point with $H = 0$ cannot be perturbed away. However, if M is at least three-dimensional, then we can generically avoid $H = 0$.

Scalar curvature along Ebin geodesics

What about other problematic multiplicities? $\mathrm{Sym}_0(n)$ admits a stratification according to these eigenvalue multiplicities.

Theorem

This stratification satisfies Whitney's transversality conditions.

Therefore, we can generically assume that all of the intersections of H with non-trivial multiplicities are transversal. This implies that not all eigenvalue multiplicities can occur. Starting in dimension five, we get enough terms.

Some concluding remarks

- We have shown that, generically, scalar curvature converges uniformly to $-\infty$ along volume-form-preserving Ebin geodesics, starting in dimension five.
- The work of Böhm-Wang-Ziller shows that there can exist Ebin-geodesics where scalar curvature tends to $+\infty$.
- In dimension two, Gauss-Bonnet is an obstruction.
- In dimension three:
  - There are choices of H admitting points with $\lim_{t \to \infty} S(\gamma_H(t)) = \infty$ that cannot be perturbed away;
  - E-H functional generically goes to $-\infty$;
  - Any orientable three-manifold is parallelisable.
- In dimension four, E-H functional generically goes to $-\infty$, but it is hard to say much more.
- New Einstein metrics? Do Palais-Smale sequences converge to Einstein metrics with singularities?