Generalized Einstein structures on Lie groups

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Plan of the talk

1. Courant algebroids: exact, heterotic and odd exact
2. Generalized Einstein equation
3. Left-invariant setting
4. Classification results in 3 dimensions

Courant algebroids

Context

Generalized geometry as study of geometric structures on generalized tangent bundles $\mathbb{T}M = TM \oplus T^*M$ or more general Courant algebroids $E \to M$.

Definition

A Courant algebroid (CA) is a vector bundle $E \to M$ with

1. scalar product $\langle \cdot, \cdot \rangle$,
2. bilinear bracket $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ and
3. homomorphism $\pi : E \to TM$ (anchor)

such that

A1) $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$,

A2) $\pi(u)\langle v, w \rangle = \langle [u, v], w \rangle + \langle v, [u, w] \rangle$,

A3) $\langle [u, v] + [v, u], w \rangle = \pi(w)\langle u, v \rangle$, $\forall u, v, w \in \Gamma(E)$.

Exact Courant algebroids

Definition

A Courant algebroid is called exact if the complex

$$0 \to T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \to 0$$

is exact.

Theorem (Severa)

Every exact CA over M is isomorphic to $\mathbb{T}M$ endowed with the canonical scp, anchor and following bracket

$$[X+\xi, Y+\eta]_H := \mathcal{L}_X(Y+\eta) - \iota_Y d\xi + \iota_X \iota_Y H,$$

where $H \in \Omega^3_{\mathrm{cl}}(M)$. The class $[H] \in H^3_{dR}(M)$ classifies the exact CA up to isomorphism.

Heterotic Courant algebroids

Fact (García Fernández, Baraglia-Hekmati)

Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ be a quadratic Lie algebra. Any G principal bundle $P \to M$ with connection θ (*Lie* $G = \mathfrak{g}$) such that

$$dH + \langle F^\theta \wedge F^\theta\rangle_{\mathfrak{g}} = 0, \quad (F^\theta = \text{ curvature})$$

for some $H \in \Omega^3(M)$ defines a Courant algebroid as follows.

1. The underlying vector bundle is $E = TM \oplus T^*M \oplus \mathrm{ad}\, P$ endowed with the obvious scp and anchor.
2. The bracket is given by

$$\begin{aligned}
&[X + \xi + r, Y + \eta + s] := \\
&[X + \xi, Y + \eta]_H + 2\langle s, \iota_X F\rangle_{\mathfrak{g}} - 2\langle r, \iota_Y F\rangle_{\mathfrak{g}} + 2\langle \nabla r, s\rangle_{\mathfrak{g}} \\
&+\nabla_X s - \nabla_Y r - [r, s]_{\mathfrak{g}} - F(X, Y),
\end{aligned}$$

where ∇ is the induced connection on $\mathrm{ad}\, P$, F is its curvature and $r, s \in \Gamma(\mathrm{ad}\, P)$. These CAs are called heterotic.

Odd exact Courant algebroids

Specializing to $\dim G = 1$, $\mathfrak{g} = \mathbb{R}$ with canonical scp and redefining $F \to -F \in \Omega^2_{\mathrm{cl}}(M)$ we arrive at

$$[X+\xi+\lambda, Y+\eta+\mu]_{H,F} := [X+\xi, Y+\eta]_H - 2\mu\iota_X F + 2\lambda\iota_Y F + 2\mu d\lambda + X(\mu) - Y(\lambda) + F(X,Y),$$

where $\lambda, \mu \in C^\infty(M)$ and $dH + F \wedge F = 0$.

Definition (Rubio)

$E = TM \oplus T^*M \oplus \mathbb{R} \to M$ with the obvious scp, anchor and with the bracket $[\cdot,\cdot]_{H,F}$ on $\Gamma(E)$ is called a Courant algebroid of type B_n ($n = \dim M$). An odd exact Courant algebroid is a CA which is isomorphic to a CA of type B_n.

Generalized metrics

Definition

A generalized (pseudo-Riemannian) metric on a Courant algebroid $E \to M$ is a subbundle $E_- \subset E$ such that $\langle \cdot, \cdot \rangle_{E_- \times E_-}$ is nondegenerate and

$$\pi|_{E_-} : E_- \to TM$$

is an isomorphism. It is called Riemannian if E_- is negative definite.

Induced structures:

$$\begin{aligned}
\mathcal{G}^{\mathrm{end}} &\in \Gamma(\mathrm{End}\, E), \quad \mathcal{G}^{\mathrm{end}}|_{E_\pm} = \pm \mathrm{Id}, \quad E_+ := (E_-)^\perp, \\
\mathcal{G} &:= \langle \mathcal{G}^{\mathrm{end}} \cdot, \cdot \rangle \in \Gamma(\mathrm{Sym}^2 E^*), \\
g &:= -s^* \mathcal{G} \quad \text{(pseudo-Riemannian metric)},
\end{aligned}$$

where $s : TM \to E_-$ denotes the inverse of $\pi|_{E_-}$.

Generalized connections

Definition

A generalized connection on $E \to M$ is a linear map

$$D : \Gamma(E) \to \Gamma(E^* \otimes E), \quad v \mapsto Dv = (u \mapsto D_u v),$$

such that

1. $D_u(fv) = u(f)v + fD_u v$,
2. $u\langle v, w\rangle = \langle D_u v, w\rangle + \langle v, D_u w\rangle$, for all $u, v, w \in \Gamma(E), f \in C^\infty(M)$, where $u(f) := \pi(u)(f)$.

Torsion (Gualtieri)

$T := T^D \in \Gamma(\wedge^2 E^* \otimes E)$ is defined by

$$T(u, v) := D_u v - D_v u - [u, v] + (Du)^* v.$$

Generalized Levi-Civita connections

Definition

Let $\mathcal{G}$ be a generalized metric on E. A torsion-free generalized connection D is called a Levi-Civita generalized connection for D if $D\mathcal{G} = 0$.

Curvature

The map $\Gamma(E) \times \Gamma(E) \times \Gamma(E) \to \Gamma(E)$,

$$(u, v, w) \mapsto D_u D_v w - D_v D_u w - D_{[u,v]} w$$

restricts to tensors $R_+^D : E_+ \otimes E_- \otimes E_+ \to E_+$ and $R_-^D : E_- \otimes E_+ \otimes E_- \to E_-$.

Theorem (García Fernández)

The generalized Ricci tensors $(v, w) \mapsto Ric_\pm(v, w) := \mathrm{tr}(u \mapsto R_\pm^D(u, v)w)$ depend only on $\mathcal{G}$ and $\delta_D : \Gamma(E) \to C^\infty(M)$, $u \mapsto \mathrm{tr}(Du)$.

Generalized Einstein equation

Definition

- Let $\mathcal{G}$ be a generalized metric on E,

$$\delta : \Gamma(E) \to C^{\infty}(M)$$

  a divergence operator (i.e. $\delta(fv) = v(f) + f\delta(v)$ for all $u, v \in \Gamma(E), f \in C^{\infty}(M)$) and D a LC generalized connection with divergence δ and Ricci tensors $Ric^{\delta}_{\pm}$.
- Then

$$Ric^{\delta} := Ric^{\delta}_{-} \oplus Ric^{\delta}_{+} \in \Gamma(E^{*}_{+} \otimes E^{*}_{-} \oplus E^{*}_{-} \otimes E^{*}_{+})$$

  is called the total Ricci tensor of the pair $(\mathcal{G}, \delta)$.
- $\mathcal{G}$ is called a generalized Einstein metric with divergence δ if

$$Ric^{\delta} = 0.$$

Exact Courant algebroids over Lie groups

General context

We consider the generalized tangent bundle $\mathbb{T}G$ of a Lie group G of dimension n with the Dorfman bracket $[\cdot,\cdot]_H$ defined by a left-invariant $H \in \Omega^3_{\mathrm{cl}}(G)$. The corresponding CA is denoted by $(\mathbb{T}G)_H$.

Proposition

Let $\mathcal{G}$ be a left-inv. generalized metric on $(\mathbb{T}G)_H$. Then $\exists$ left-inv. 3-form $H' \in [H]$ and an isomorphism of Courant algebroids $(\mathbb{T}G)_H \cong (\mathbb{T}G)_{H'}$ mapping $\mathcal{G}$ to a generalized metric of the form

$$\mathcal{G}_g(X+\xi, Y+\eta) = \frac{1}{2}(g(X,Y) + g^{-1}(\xi,\eta)),$$

$X+\xi, Y+\eta \in \Gamma(\mathbb{T}G)$, where g is a left-inv. pseudo-Riemannian metric.

Space of left-invariant Levi-Civita generalized connections

Proposition

Let $\mathcal{G}$ be a left-inv. generalized metric on $(\mathbb{T}G)_H$ and denote by $E = E_+ \oplus E_-$ the corresponding decomposition of

$$E = E(\mathfrak{g}) := \mathfrak{g} \oplus \mathfrak{g}^*.$$

- Then the space of left-inv. Levi-Civita generalized connections of $\mathcal{G}$ is an affine space modeled on

$$\mathfrak{so}(E)^{\langle 1 \rangle}_{\mathcal{G}} := \ker(\partial : E^* \otimes \mathfrak{so}(E)_{\mathcal{G}} \to \wedge^3 E^*) = \Sigma_+ \oplus \Sigma_-,$$

$$\Sigma_\pm := \ker(\partial : E^*_\pm \otimes \mathfrak{so}(E_\pm) \to \wedge^3 E^*_\pm),$$

- where

$$(\partial\alpha)(u, v, w) = \sum_{\text{cycl}} \langle \alpha(u)v, w \rangle, \quad u, v, w \in E.$$

Space of left-inv. LC generalized connections continued

Proposition

A particular divergence-free element is

$$D^0 = \frac{1}{3}\mathcal{B}|_{\wedge^3 E_+} \oplus \frac{1}{3}\mathcal{B}|_{\wedge^3 E_-} \oplus \mathcal{B}|_{E_+ \otimes \wedge^2 E_-} \oplus \mathcal{B}|_{E_- \otimes \wedge^2 E_+},$$

where $\mathcal{B}(u, v, w) := \langle [u, v]_H, w \rangle$ and D^0 is considered as $\in E^* \otimes \wedge^2 E^* \cong E^* \otimes \mathfrak{so}(E)$.

Proof (sketch)

$$T^{D^0} = \partial D_0 - \mathcal{B} = \mathcal{B} - \mathcal{B} = 0.$$

□

LC generalized connections with prescribed divergence

Left-invariant divergence operators on $\mathbb{T}G$ are identified with elements $\delta \in (\mathbb{T}_e G) \cong E^*$.

Proposition

Let $\mathcal{G}$ be a generalized metric on $(\mathbb{T}G)_H$, $\dim G \geq 2$, and $\delta \in E^*$. Then there exists a left-inv. LC generalized connection D with $\delta_D = \delta$.

Proof (idea)

We show that $\delta_{D^0} = 0$ and that the map

$$\lambda : \mathfrak{so}(E)^{\langle 1 \rangle}_{\mathcal{G}} \to E^*, \quad \lambda(S)(v) := \mathrm{tr}(Sv),$$

is surjective. □

Adapted bases

- Let $\mathcal{G}$ be a left-inv. generalized metric on $(\mathbb{T}G)_H$. Up to isomorphism, $\mathcal{G} = \mathcal{G}_g$.
- Let $(v_a) = (v_1, \ldots, v_n)$ be a g-ONB of $\mathfrak{g}$. Then

  $$e_a := v_a + gv_a \in \mathfrak{g} \oplus \mathfrak{g}^* = E$$

  defines a $\mathcal{G}$-ONB $(e_a)_{a=1,\ldots,n}$ of E_+ and

  $$e_{n+a} := v_a - gv_a$$

  defines a $\mathcal{G}$-ONB $(e_i)_{i=n+1,\ldots,2n}$ of E_-.
- In total we obtain a $\mathcal{G}$-ONB

  $$(e_A)_{A=1,\ldots,2n}$$

  of E adapted to the decomposition $E = E_+ \oplus E_-$.

Components of Dorfman bracket

Note that

$$\eta_{AB} := \langle e_A, e_B \rangle = \varepsilon_A \delta_{AB},$$

where $\varepsilon_a = g(v_a, v_a) \in \{\pm 1\}$ for $a = 1, \ldots, n$ and
$\varepsilon_i = -g(v_{i-n}, v_{i-n}) = -\varepsilon_{i-n}$ for $i = n+1, \ldots, 2n$.

Dorfman coefficients

We denote by $\mathcal{B}_{ABC} := \langle [e_A, e_B]_H, e_C \rangle$ the components of
$\mathcal{B} := \langle [\cdot, \cdot]_H, \cdot \rangle \in \wedge^3 E^*$ in the basis (e_A).

Remark

The bracket $[\cdot, \cdot]_H : \wedge^2 E \to E$ is a Lie bracket on $E = \mathfrak{g} \oplus \mathfrak{g}^*$.
Moreover, $(E, [\cdot, \cdot]_H, \langle \cdot, \cdot \rangle)$ is a quadratic Lie algebra.

Components of canonical LC generalized connection

Generalized LC coefficients

The coefficients

$$\omega_{ABC} := \langle D^0_{e_A} e_B, e_C \rangle$$

of the canonical divergence-free LC generalized connection D^0 are related to the Dorfman coefficients by:

$$\omega_{abc} = \frac{1}{3}\mathcal{B}_{abc}, \qquad \omega_{ijk} = \frac{1}{3}\mathcal{B}_{ijk},$$
$$\omega_{ibc} = \mathcal{B}_{ibc}, \qquad \omega_{ajk} = \mathcal{B}_{ajk},$$

where $a, b, c \in \{1, \ldots, n\}$, $i, j, k \in \{n+1, \ldots, 2n\}$.

Ricci curvature of left-invariant generalized metrics

Theorem

Let g a left-inv. pseudo-Riemannian metric on a Lie group G and $\delta \in E^*$. Then the Ricci curvature of any LC generalized connection compatible with $(\mathcal{G} = \mathcal{G}_g, \delta)$ is given by

$$
\begin{aligned}
Ric^{\delta}_{+}(u_-, u_+) &= -\mathrm{tr}(\Gamma_{u_-} \circ \Gamma_{u_+}) + \delta(\mathrm{pr}_{E_+}[u_-, u_+]_H), \\
Ric^{\delta}_{-}(u_+, u_-) &= -\mathrm{tr}(\Gamma_{u_-} \circ \Gamma_{u_+}) + \delta(\mathrm{pr}_{E_-}[u_+, u_-]_H),
\end{aligned}
$$

where $u_\pm \in E_\pm$ and

$$\Gamma_{u_\pm} : E_\mp \to E_\pm, \quad \Gamma_{u_\pm} = \mathrm{pr}_{E_\pm} \circ [u_\pm, \cdot]_H|_{E_\mp}.$$

Ricci curvature of left-invariant generalized metrics continued

Corollary

The components of the Ricci tensor of the pair $(\mathcal{G}, \delta)$ in an adapted basis (e_A) take the form

$$Ric^{\delta}_{ia} = \mathcal{B}^{j}_{bi}\mathcal{B}^{b}_{aj} + \mathcal{B}^{c}_{ia}\delta_c, \quad Ric^{\delta}_{ai} = \mathcal{B}^{j}_{bi}\mathcal{B}^{b}_{aj} + \mathcal{B}^{j}_{ai}\delta_j,$$

where $a, b, c \in \{1, \ldots, n\}$, $i, j, k \in \{n+1, \ldots, 2n\}$ and $\mathcal{B}^{C}_{AB} (= \mathcal{B}_{ABD}\eta^{DC})$ are the components of the Dorfman bracket.

Dorfman coefficients in terms of structure constants and 3-form

Proposition

The Dorfman coefficients $\mathcal{B}_{ABC}$ have the following expression in terms of the structure constants $\kappa_{abc} = g([v_a, v_b]_{\mathfrak{g}}, v_c)$ of the Lie algebra $\mathfrak{g}$ and the coefficients of the 3-form H in the ONB (v_a):

$$\begin{aligned}
\mathcal{B}_{ajk} &= \frac{1}{2}(H_{aj'k'} - \kappa_{aj'k'} + \kappa_{j'k'a} - \kappa_{k'aj'}), \\
\mathcal{B}_{ibc} &= \frac{1}{2}(H_{i'bc} + \kappa_{i'bc} - \kappa_{bci'} + \kappa_{ci'b}), \\
\mathcal{B}_{abc} &= \frac{1}{2}(H_{abc} + (\partial\kappa)_{abc}), \\
\mathcal{B}_{ijk} &= \frac{1}{2}(H_{i'j'k'} + (\partial\kappa)_{i'j'k'}),
\end{aligned}$$

where $i' := i - n$ for all $i \in \{n+1, \ldots, 2n\}$ and $(\partial\kappa)_{abc} = \kappa_{abc} + \mathrm{cycl.}$

Scaling and twisted scaling

Proposition

Let $\mu > 0$ and $\varepsilon = \pm 1$. Then the generalized metric $\mathcal{G}_g$ on $(\mathbb{T}G)_H$ is Einstein with divergence δ if and only if $\mathcal{G}_{g'}$ on $(\mathbb{T}G)_{H'}$ is Einstein with divergence δ', where

$$g' = \varepsilon\mu^{-2}g, \quad H' = \varepsilon\mu^{-2}H, \quad \delta' = \mu\delta.$$

Proposition (Lüschen)

Let $\mu > 0$ and $\varepsilon = \pm 1$. Then $\mathcal{G}_g$ on $(\mathbb{T}G)_H$ is Einstein with divergence δ if and only if $\mathcal{G}_{g'}$ on $(\mathbb{T}G)_{H'}$ is Einstein with divergence δ', where

$$g' = \varepsilon\mu^{-2}g, \quad H' = -\varepsilon\mu^{-2}H, \quad \delta' = \mu\bar{\delta},$$

where $\bar{\delta} \in E^*$ is defined by $\bar{\delta}_a = \delta_{a+n}$ for $a \in \{1, \ldots, n\}$ and $\bar{\delta}_i = \delta_{i-n}$ for $i \in \{n+1, \ldots, 2n\}$.

Example: left-inv. generalized Einstein metrics with $H = 0$ and $\delta = 0$

Proposition

If $H = 0$ and $\delta = 0$, then

$$Ric^{\delta}_{+}(v - gv, u + gu) = Ric^{\delta}_{-}(u + gu, v - gv) = Ric^{g}(u, v) + (\nabla_u \tau)(v),$$

for all $u, v \in \mathfrak{g}$, where $\tau(v) := \mathrm{tr}\,\mathrm{ad}_v$ and ∇ = LC connection.

- $\Longrightarrow$ generalized Einstein equation reduces to Ricci soliton equation

$$Ric^{g} + \nabla\tau = 0.$$

Classification of left-inv. generalized Einstein structures in dimension 3

Cases

$$\begin{cases} G \text{ unimodular} \begin{cases} \delta = 0 \\ \delta \neq 0 \end{cases} \\ G \text{ non-unimodular} \begin{cases} \delta = 0 \\ \delta \neq 0 \end{cases} \end{cases}$$

Milnor operator

Choosing an orientation, the Lie bracket on $(\mathfrak{g}, g)$ is encoded in $L \in \mathrm{End}(\mathfrak{g})$ defined by

$$[u, v]_{\mathfrak{g}} = L(u \times v), \quad u, v \in \mathfrak{g}.$$

- L is symmetric if and only if $\mathfrak{g}$ is unimodular.

Normal forms of L

Proposition

Let $\mathfrak{g}$ be unimodular and oriented. Then $\exists$ ONB (v_a) such that $g(v_1, v_1) = g(v_2, v_2)$ and L is represented by one of the following:

$$L_1(\alpha,\beta,\gamma) = \mathrm{diag}(\alpha,\beta,\gamma), \quad L_2(\alpha,\beta,\gamma) = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \alpha & -\beta \\ 0 & \beta & \alpha \end{pmatrix},$$

$$L_3(\alpha,\beta) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \frac{1}{2}+\alpha & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2}+\alpha \end{pmatrix}, \quad L_4(\alpha,\beta) = -L_3(-\alpha,-\beta),$$

$$L_5(\alpha) = \begin{pmatrix} \alpha & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & \alpha & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & \alpha \end{pmatrix}.$$

Classification in the unimodular case for $\delta = 0$

Theorem

Any divergence-free gen. Einstein structure on a 3-dim. unimodular Lie algebra is isomorphic to one of the following:

1. $\mathfrak{g}$ abelian $H = 0$, g flat any signature.
2. $\mathfrak{g}$ simple, $H \neq 0$, g of nonzero constant curvature; definite iff $\mathfrak{g} = \mathfrak{so}(3)$, indefinite iff $\mathfrak{g} = \mathfrak{so}(2,1)$.
3. $\mathfrak{g}$ metabelian, $H = 0$, g-flat; definite on $[\mathfrak{g}, \mathfrak{g}]$ iff $\mathfrak{g} = \mathfrak{e}(2)$ and indefinite on $[\mathfrak{g}, \mathfrak{g}]$ iff $\mathfrak{g} = \mathfrak{e}(1,1)$.
4. $\mathfrak{g} = \mathfrak{heis}$, $H = 0$, g flat indefinite.

Corollary

Divergence-free gen. Einstein structures do exist on any unimodular Lie group and have underlying pseudo-Riemannian metrics of constant curvature.

Classification in the unimodular case for $\delta \neq 0$

Theorem

Any unimodular Lie algebra admits a gen. Einstein metric with $\delta \neq 0$ (as well as one with $\delta = 0$). These structures appear in families.

Remark

Not all normal forms for L are consistent with the generalized Einstein equation. The possibilities are:

1. diagonal type, more precisely, scalar or of of rank 2 with double eigenvalue, $\mathfrak{g} \not\cong \mathfrak{heis}$,
2. $L_3(\alpha, 0)$ and $L_4(\alpha, 0)$ with $\mathfrak{g} = \mathfrak{h}eis$ (if $\alpha = 0$) or $\mathfrak{g} = \mathfrak{e}(1,1)$ if $\alpha \neq 0$.
3. $L_5(0)$ and $\mathfrak{g} = \mathfrak{e}(1,1)$.

- The normal forms $L_3(\alpha, 0)$ and $L_4(\alpha, 0)$ for $\alpha \neq 0$ and $L_5(0)$ do only occur if $\delta \neq 0$.

Summary of further results

Non-unimodular case

Classification of Einstein structures on non-unimodular 3-dimensional Lie algebras is given in same article with David Krusche. It includes Ricci solitons of nonconstant curvature among other families of examples.

Odd exact case

Classification of generalized Einstein structures on odd exact Courant algebroids over 3-dimensional Lie groups is obtained in joint work in progress with Liana David.