Mirror symmetry from families of Calabi-Yau manifolds

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Abstract We report interesting examples of Calabi-Yau threefolds where birational geometry and geometry of Fourier-Mukai partners of a Calabi-Yau manifold arise naturally from studying its mirror family of Calabi-Yau manifolds. We define mirror symmetry in terms of families of Calabi-Yau manifolds in general. We will find for our examples that all expected geometry of Calabi-Yau manifolds arises from special boundary points, called LCSL points, in the parameter space of mirror Calabi-Yau manifolds.

1 Introduction

Mirror symmetry of Calabi-Yau manifolds has been a central topic in theoretical physics and mathematics since its discovery in the beginning of the 1990s. In particular, its surprising application to enumerative geometry started by Candelas et al [7] draw significant attention from both theoretical physicists and mathematicians. Intensive study over the last three decades has led two main mathematical formulations of mirror symmetry. One is based on the geometry of Calabi-Yau manifolds, called geometric mirror symmetry due to Strominger-Yau-Zaslow [27] and also Gross-Siebert [12, 13], and the other focuses on equivalences of two different categories; the derived category of coherent sheaves and the derived Fukaya category in symplectic geometry for a Calabi-Yau manifold. The latter approach is called homological mirror symmetry [24] and describes mirror symmetry as a derived equivalence interchanging the two different categories defined for Calabi-

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Yau manifolds X and its mirror $\check{X}$. While both approaches to mirror symmetry are extensively developed, to fully understand the symmetry, exploring it in many examples of Calabi-Yau manifolds is particularly important since it is still very hard to describe $\check{X}$ explicitly from X by these two approaches.

In this article, based on a series of the present authors' works [18, 19, 20, 21, 22], we will report on mirror symmetry of two interesting classes of Calabi-Yau manifolds; one is Calabi-Yau manifolds related to determinantal quintics in four dimensional projective space, and the other is Calabi-Yau manifolds which are fibered by abelian surfaces. For these Calabi-Yau manifolds, by studying mirror families, it has been observed that mirror symmetry nicely incorporates the problems of birational geometry and also Fourier-Mukai partners of Calabi-Yau manifolds.

2 Families of Calabi-Yau manifolds and mirror symmetry

2.1 Calabi-Yau manifolds

Complex manifolds which admit Ricci-flat Kähler metric are called Calabi-Yau manifolds in general. Abelian varieties or complex tori are simple examples of such Calabi-Yau manifolds. Here in this note, however, we adopt the following definition of Calabi-Yau manifolds in a narrow sense, where abelian varieties and complex tori are excluded.

Definition 1. We call a non-singular projective variety X of dimension d Calabi-Yau manifold if it satisfies $c_1(X)=0$ and $H^i(X,\mathscr{O}_X)=0\,(1\leq i\leq d-1)$.

Given a Calabi-Yau manifold X, we can describe the deformation of its complex structures by the Kodaira-Spencer theory. In particular, due to the theorem by Bogomolov-Tian-Todorov, all infinitesimal deformations are unobstructed and given by the cohomology $H^1(X,TX)$. We denote by $\mathscr{M}$ the deformation space, and write by X_p the Calabi-Yau manifold represented by $p\in\mathscr{M}$. According to the deformation theory, we have a linear map from the tangent space

$$KS_p : T_p\mathscr{M}\to H^1(X_p,TX),$$

which is called Kodaira-Spencer map.

Definition 2. Let X be a Calabi-Yau manifold and $\mathfrak{X},\mathscr{M}$ be complex manifolds. If there exists a surjective morphism $\pi:\mathfrak{X}\to\mathscr{M}$ satisfying the following conditions: (0) $\pi^{-1}(p_0)\simeq X$, (1) differential π_* is submersive, (2) the fiber $\pi^{-1}(p)$ is compact for all $p\in\mathscr{M}$, (3) the Kodaira-Spencer maps are isomorphic for all $p\in\mathscr{M}$, then we call $\mathfrak{X}$ a deformation family of X.

It is often required that the morphism $\pi:\mathfrak{X}\to\mathscr{M}$ is proper, however, we do not require this property following [23, Thm. 2.8].

2.2 A-structure and B-structure

For a given Calabi-Yau manifold, we introduce two different but similar structures. In what follows, we assume Calabi-Yau manifolds are of dimension three.

2.2.1 A-structure

Since a Calabi-Yau manifold X is projective by definition, we can take a positive line bundle L with its first Chern class $c_1(L) \in H^{1,1}(X,\mathbb{Z})$ being primitive. We take a Kähler form κ satisfying $[\kappa] = [c_1(L)]$. The Kähler form κ is a $(1,1)$ form, and determines the nilpotent linear map $L_\kappa : H^{p,q}(X) \to H^{p+1,q+1}(M)$ in the Hard Lefschetz theorem. For Calabi-Yau manifolds, since it acts trivially on $H^3(X) = \oplus_{p+q=3} H^{p,q}(X)$, we can restrict it on $H^{even}(X) = \oplus_{p+q:\text{even}} H^{p,q}(X)$ and obtain

$$L_\kappa : H^{even}(X,\mathbb{Q}) \to H^{even}(X,\mathbb{Q}), \tag{1}$$

where we use the fact that κ is a real $(1,1)$ form.

For coherent sheaves $\mathscr{E},\mathscr{F}$ on X, we denote the Riemann-Roch (RR) pairing by

$$\chi(\mathscr{E},\mathscr{F}) = \sum_{i=0}^{3} (-1)^i \dim Ext^i(\mathscr{E},\mathscr{F}).$$

This RR pairing is a pairing on the Grothendieck group $K(X)$ of coherent sheaves on X. The numerical K group $K_{num}(X) := K(X)/\equiv$ is defined by the quotient by the radical of RR pairing. $K_{num}(X)$ defines a free abelian group. For Calabi-Yau manifolds, due to the Serre duality, RR pairing introduces a skew symmetric form on $K_{num}(X)$, which we denote by $(K_{num}(X),\chi(-,-))$. Moreover, by Lefshetz (1,1) theorem and Hard-Lefschetz theorem, we can see that the Chern character homomorphism $ch : K_{num}(X) \to H^{even}(X,\mathbb{Q})$ gives rise to an isomorphism $ch(K_{num}(X)) \otimes \mathbb{Q} \simeq H^{even}(X,\mathbb{Q})$ for Calabi-Yau threefolds. Namely, by the homomorphism $ch : K_{num}(X) \to H^{even}(X,\mathbb{Q})$, we have a natural integral structure with a skew symmetric form on $H^{even}(X,\mathbb{Q})$ coming from the structure $(K_{num}(X),\chi(-,-))$.

Definition 3. We define A-structure of a Calabi-Yau manifold X by the nilpotent linear map $L_\kappa : H^{even}(X,\mathbb{Q}) \to H^{even}(X,\mathbb{Q})$ together with the integral structure on $H^{even}(X,\mathbb{Q})$ coming from $(K_{num}(X),\chi(-,-))$.

2.2.2 B-structure

For a Calabi-Yau manifold X, we can define a similar structure to the A-structure defined above, but we need to require some additional assumptions on X. The first assumption is that X has a deformation family $\pi : \mathfrak{X} \to \mathscr{M}$ over a parameter space which is quasi-projective. We assume that there is a compactification $\overline{\mathscr{M}}$ of $\mathscr{M}$ such that the fiber $X_p = \pi^{-1}(p)$ degenerates to a (singular) Calabi-Yau variety $X_{\bar{p}}$

when we take a limit $p \to \bar{p} \in \overline{\mathscr{M}} \setminus \mathscr{M}$. The compactification $\overline{\mathscr{M}}$ is assumed to be a (singular) projective variety whose singular loci is contained in $\overline{\mathscr{M}} \setminus \mathscr{M}$. Also, we assume $\overline{\mathscr{M}} \setminus \mathscr{M}$ is given by a (non-normal crossiong) divisor D in $\overline{\mathscr{M}}$. Let $\bar{o}$ be a singular point on the divisor D and take the following steps:

1. Let $U_{\bar{o}}$ to be an affine neighborhood of $\bar{o}$. By successive blowing-ups starting at $\bar{o}$, we assume the divisor $U_{\bar{o}} \cap D$ is transformed to

$$\hat{U}_{\bar{o}} \setminus U_{\bar{o}}^{sm} = \cup_i D_i,$$

in terms of normal crossing boundary (exceptional) divisors, where $U_{\bar{o}}^{sm}$ represents $U_{\bar{o}} \setminus D$. We focus on a boundary point given by $o = D_{i_1} \cap \cdots \cap D_{i_r} (r = \dim \mathscr{M})$.

2. Recall that for a family of Calabi-Yau manifolds $\pi : \mathfrak{X} \to \mathscr{M}$, we naturally have a locally constant sheaf $R^3\pi_*\mathbb{C}_{\mathfrak{X}}$ over $\mathscr{M}$. This sheaf corresponds to a holomorphic vector bundle whose fibers are $H^3(X_m, \mathbb{C})$ $(m \in \mathscr{M})$ with the holomorphic flat connection (Gauss-Manin connection). Each fiber has (topological) integral structure $H^3(X_m, \mathbb{Z})$ and also a skew symmetric (symplectic) form defined by $(\alpha, \beta) := \int_{X_m} \alpha \cup \beta$. The Gauss-Manin connection is naturally compatible with this symplectic structure.

3. Take a boundary point $o \in \hat{U}_{\bar{o}}$ as described above, and write it $o = D_1 \cap \cdots \cap D_r (r = \dim \mathscr{M})$ by reordering the indices of the (exceptional) divisors. Choose a base point $b_o \in \mathscr{M}$ and fix an integral symplectic basis of $H^3(X_{b_o}, \mathbb{Z})$. With respect to this basis, we represent the monodromy of the Gauss-Manin connection around the boundary divisor D_i by $T_{D_i} : H^3(X_{b_o}, \mathbb{Z}) \to H^3(X_{b_o}, \mathbb{Z})$. We assume that the monodromy matrices T_{D_i} for $i = 1, \cdots, r$ are unipotent, i.e., satisfy

$$(T_{D_i} - \mathrm{id})^{m_i} = O \text{ for some } m_i.$$

Note that this is a condition for the boundary point $o \in \hat{U}_{\bar{o}}$. For each T_{D_i}, we define

$$\begin{aligned} \log T_{D_i} &= \log(\mathrm{id} + (T_{D_i} - \mathrm{id})) \\ &= (T_{D_i} - \mathrm{id}) - \frac{1}{2}(T_{D_i} - \mathrm{id})^2 + \cdots + \frac{(-1)^{m_i - 1}}{m_i - 1}(T_{D_i} - \mathrm{id})^{m_i - 1} \end{aligned}$$

and denote this by $N_i = \log T_{D_i}$. By definition, N_i determines a nilpotent endomorphism in $\mathrm{End}(H^3(X_{b_o}, \mathbb{Q}))$. Moreover, since the boundary is a normal crossing divisor, we have $N_i N_j - N_j N_i = O$. Now we define $N_\lambda := \sum_i \lambda_i N_i$ in terms of real parameters $\lambda_i > 0$, which gives a nilpotent endomorphism acting on $H^3(X_{b_o}, \mathbb{R})$ due to the commuting relation. The following result is due to Cattani-Kaplan [8, Thm.2]:

Theorem 1. *For general $\lambda_i > 0 (\lambda_i \in \mathbb{Q})$, the nilpotent matrix $N_\lambda := \sum_i \lambda_i N_i$ defines a monodromy weight filtration of the same form on $H^3(X_{b_o}, \mathbb{Q})$.*

As a general result of monodromy theorem, the nilpotent matrix N_λ satisfies $N_\lambda^k = O$ for $k > \dim X (= 3)$. When we have $N_\lambda^3 \neq O$, $N_\lambda^4 = O$, we call the boundary point $o \in \hat{U}_{\bar{o}}$ maximally degenerated. If a boundary point $o \in \hat{M}_{\bar{o}}$ is maximally degenerated and moreover the monodromy weight filtration by N_λ has the form

$$0 \subset W_0 = W_1 \subset W_2 = W_3 \subset W_4 = W_5 \subset W_6 = H^3(X_{b_o}, \mathbb{Q}),$$

then we call o a LCSL (Large Complex Structure Limit) point.

Definition 4. If a Calabi-Yau manifold X has a deformation family $\pi : \mathfrak{X} \to \mathscr{M}$ which gives rise to a LCSL boundary point $o \in \hat{U}_{\bar{o}}$, then we define B-structure of X from o by the nilpotent action $N_\lambda : H^3(X_{b_o}, \mathbb{Q}) \to H^3(X_{b_o}, \mathbb{Q})$ together with the integral structure $(H^3(X_{b_o}, \mathbb{Z}), (\ ,\ ))$.

Remark 1. It is not clear whether every Calabi-Yau manifold admits a family $\mathfrak{X} \to \mathscr{M}$ where we can extract a B-structure defined above. However, for Calabi-Yau hypersurfaces or complete intersections in toric varieties, we can construct explicitly their families where we verify the B-structures [15, 16].

Remark 2. B-structure is not necessarily unique for a Calabi-Yau manifold X. That is, we often observe that many B-structures of X arise from different LCSL points of a family $\mathfrak{X} \to \mathscr{M}$. Such cases are of main interest in this article.

2.2.3 Mirror symmetry

In contrast to A-structures, describing B-structures of Calabi-Yau manifolds are difficult as well as showing their existence. Mirror symmetry of Calabi-Yau manifolds is a conjecture that says for a given Calabi-Yau manifold X, there exists another Calabi-Yau manifold $\check{X}$ such that the A-structure of X is interchanged by the B-structure of $\check{X}$.

To describe the symmetry in more precise, let $\mathscr{K}_X$ be the Kähler cone of X. The Kähler cone is an open convex cone in $H^2(X, \mathbb{R})$ and coincides with the ample cone when $\dim X = 3$ and satisfies $\mathscr{K}_X \cap H^2(X, \mathbb{Z}) \neq \phi$. The shape of the ample cone is complicated in general and is related to the classification problem of algebraic varieties. Take integral elements $\kappa_1, \cdots, \kappa_r \in \overline{\mathscr{K}}_X \cap H^2(X, \mathbb{Z})$ so that these generate the group $H^2(X, \mathbb{Z})$ and define a cone $\sigma_A := \mathbb{R}_{\geq 0}\kappa_1 + \cdots + \mathbb{R}_{\geq 0}\kappa_r$. Cones of this property may be constructed by decomposing the cone $\overline{\mathscr{K}}_X$ into (possibly infinitely many) simplicial cones. We determine a cone σ_A and introduce a Kähler class κ by a general linear combination

$$\kappa = \alpha_1 \kappa_1 + \cdots + \alpha_r \kappa_r$$

with $\alpha_i > 0$. The nilpotent operator L_κ defines an A-structure of X with parameters α_i, which we denote by

$$\big(L_{\kappa(\alpha)}, (K_{num}(X), \chi(-,-)), \Sigma_A\big), \tag{2}$$

where $\Sigma_A := \{\alpha_1 L_{\alpha_1} + \cdots + \alpha_r L_{\alpha_r} \mid \alpha_i > 0\}$ is a cone in $\mathrm{End}(H^{even}(X, \mathbb{R}))$ and corresponds to the cone $\sigma_A \in H^2(X, \mathbb{Z})$.

The nilpotnet matrix $N_\lambda = \sum_i \lambda_i N_i$ to describe B-structure may be regarded as an element in a cone $\Sigma_o := \mathbb{R}_{\geq 0} N_1 + \cdots + \mathbb{R}_{\geq 0} N_m$ $(m = \dim H^1(X, TX))$ for a LCSL boundary point o. This cone Σ_o is called monodromy nilpotent cone in Hodge theory. We denote the B-structure coming from a LCSL boundary point o by

$$\big(N_\lambda, (H^3(X_{b_o},\mathbb{Z}), (\ ,\ )), \Sigma_o\big). \tag{3}$$

Here $X_{b_o} = \pi^{-1}(b_o)$ represents a smooth fiber over a base point $b_o \in \mathscr{M}$, but this will often be simplified by X.

Definition 5. A-structure of a Calabi-Yau manifold X and B-structure of another Calabi-Yau manifold Y are said isomorphic if the following two properties holds:

(1) There exists an isomorphism

$$\begin{array}{cccc} \varphi: & H^{even}(X,\mathbb{R}) & \xrightarrow{\sim} & H^3(Y,\mathbb{R}) \\ & \cup & & \cup \\ & (K_{num}(X), \chi(-,-)) & & (H^3(Y,\mathbb{Z}), (\ ,\ )), \end{array}$$

which preserves the integral and skew symmetric forms.

(2) When we set $\alpha_i = \lambda_i$, the nilpotent matrices N_λ and $L_{\kappa(\alpha)}$ are related by the isomorphism in (1) as

$$N_\lambda = \varphi \circ L_{\kappa(\alpha)} \circ \varphi^{-1}.$$

Now we are ready to define mirror symmetry (of Calabi-Yau threefolds):

Definition 6. Two Calabi-Yau manifolds X and $\check{X}$ are said mirror to each other if they have A- and B- structures which are exchanged in the following sense:

$$\text{(i) } \varphi: \big(L_{\kappa(\alpha)}, (K_{num}(X), \chi(-,-)), \Sigma_A\big) \xrightarrow{\sim} \big(N_\lambda, (H^3(\check{X},\mathbb{Z}), (\ ,\ )), \Sigma_o\big),$$
$$\text{(ii) } \check{\varphi}: \big(L_{\kappa(\check{\alpha})}, (K_{num}(\check{X}), \chi(-,-)), \Sigma_{\check{A}}\big) \xrightarrow{\sim} \big(N_{\check{\lambda}}, (H^3(X,\mathbb{Z}), (\ ,\ )), \Sigma_{\check{o}}\big).$$

Remark 1. From the isomorphism between A-, B- structures, it holds

$$h^{1,1}(X) = h^{2,1}(\check{X}),\ h^{1,1}(\check{X}) = h^{2,1}(X) \tag{4}$$

when X and $\check{X}$ are mirror symmetric.

Remark 2. When X and $\check{X}$ are mirror symmetric, we have families $\mathfrak{X} \to \mathscr{M}_X$ and $\check{\mathfrak{X}} \to \check{\mathscr{M}}_{\check{X}}$ describing their B-structures. We call, for example, the family $\check{\mathfrak{X}} \to \check{\mathscr{M}}_{\check{X}}$ a mirror family of X.

Remark 3. A quintic hypersurface $X = (5)$ in $\mathbb{P}^4$ is an example of Calabi-Yau threefold. In this case, we have $h^{1,1}(X) = 1, h^{2,1}(X) = 101$ for the Hodge numbers of X. A Calabi-Yau manifold $\check{X}$ having Hodge numbers $h^{1,1}(\check{X}) = 101, h^{2,1}(\check{X}) = 1$ has been constructed in the pioneering work by Candelas et. al. in 1991 [7], and surprising implications of mirror symmetry have been found there. In this case, while the isomorphism (i) has been shown, verifying the isomorphism (ii) seems difficult since the dimension of the deformation space is large, i.e., $h^{2,1}(X) = 101$. Beyond quintic hypersurfaces in $\mathbb{P}^4$, we can construct pairs $(X, \check{X})$ of Calabi-Yau hypersurfaces or complete intersections satisfying the relation (4) in toric varieties [2, 3]. If the dimensions of the deformation spaces are small enohgh, we can verify either the isomorphism (i) or (ii) [15, 16]. However verifying both relations (i) and (ii) is difficult in general, since the relation (4) implies that the smaller dimensions of the deformation space for X, the larger dimensions for $\check{X}$ when the Euler numbers $e(X) = -e(\check{X})$ have large absolute values. It is believed that both isomorphisms (i) and (ii) hold for the pairs of Calabi-Yau manifolds given in [2, 3].

3 Mirror symmetry from families of Calabi-Yau manifolds

Mirror symmetry is a conjecture which asserts that for a Calabi-Yau manifold X there exists another Calabi-Yau manifold $\check{X}$ which is mirror symmetric in the sense of Definition 6. If $\check{X}$ exists, by definition, it comes with its deformation family $\check{\mathfrak{X}} \to \mathscr{M}_{\check{X}}$. The symmetry of $\check{X}$ to X is described by the B-structure from a LCSL boundary point of $\overline{\mathscr{M}}_{\check{X}}$. By studying families over $\overline{\mathscr{M}}_{\check{X}}$ globally for many examples of X and $\check{X}$, we often observe multiple LCSL boundary points in $\overline{\mathscr{M}}_{\check{X}}$ and recognize the following correspondence:

$$\text{LCSL boundary points in } \overline{\mathscr{M}}_{\check{X}} \quad \leftrightarrow \quad \text{birational geometry and Fourier-Mukai partners of } X.$$

We present two interesting examples where we see this correspondence.

3.1 *Birational geometry related to determinantal quintics in* $\mathbb{P}^4$

Consider the complete intersection of five $(1,1)$ divisors in $\mathbb{P}^4 \times \mathbb{P}^4$,

$$X = \cap_{i=1}^{5}(1,1) \subset \mathbb{P}^4 \times \mathbb{P}^4.$$

X is a smooth Calabi-Yau manifold if we take the five $(1,1)$ divisors general. We can determine its Hodge numbers as $h^{1,1}(X) = 2, h^{2,1}(X) = 52$ [19]. After some works, it turns out that we have a mirror Calabi-Yau manifold $\check{X}$ from a special family X_{sp} of X with parameters a, b;

$$X_{sp} := \{z_i w_i + a z_i w_{i+1} + b z_{i+1} w_i = 0 \ (i = 1, .., 5)\} \subset \mathbb{P}^4_z \times \mathbb{P}^4_w.$$

As described in the following proposition, X_{sp} is singular for general parameters a, b, but there exists a crepant ($c_1 = 0$) resolution of the singularity which gives rise to a mirror Calabi-Yau manifold [19, Thm.5.11,Thm.5.17].

Proposition 1. *When $a, b \in \mathbb{C}$ are general, the following properties hold:*

(1) *X_{sp} is singular along 20 $\mathbb{P}^1$ with the A_1-singularities.*

(2) *There exists a crepant resolution $\check{X} \to X_{sp}$ which gives a Calabi-Yau manifolds with the Hodge numbers $h^{1,1}(\check{X}) = 52$, $h^{2,1}(\check{X}) = 2$.*

Taking the symmetry of the defining equations of X_{sp} into account, we see that the special family is parametrized by a^5 and b^5. Moreover, these affine parameters are projectivised naturally by $[a^5, b^5, 1] \in \mathbb{P}^2$. The family parametrized by $[a^5, b^5, 1]$ has been studied in detail in [21, Prop.3.11].

Proposition 2. *We have a family of Calabi-Yau manifolds $\check{\mathfrak{X}} \to \mathscr{M}_{\check{X}} \subset \mathbb{P}^2$ with the fibers being crepant resolutions $\check{X} \to X_{sp}$. The parameter space $\mathscr{M}_{\check{X}}$ satisfies $\overline{\mathscr{M}}_{\check{X}} = \mathbb{P}^2$, and the fibers over $\mathscr{M}_{\check{X}}$ extend to over $\overline{\mathscr{M}}_{\check{X}} \setminus \mathscr{M}_{\check{X}} = D_1 \cup D_2 \cup D_3 \cup Dis_0$ with some degeneration. Here D_1, D_2, D_3 are the toric divisors in $\mathbb{P}^2$ and Dis_0 represents the discriminant of X_{sp} which has degree 5.*

Since we have the relations $h^{1,1}(X) = h^{2,1}(\check{X}) = 2, h^{2,1}(X) = h^{1,1}(\check{X}) = 52$ for the Hodge numbers, we may expect that the family $\check{\mathfrak{X}} \to \mathscr{M}_{\check{X}}$ is a mirror family of X. In fact, it turns out that not only X but also its birational models arise naturally from the family [21, Prop.3.12].

Proposition 3. *The boundary points $o_1 = D_1 \cap D_2, o_2 = D_2 \cap D_3, o_3 = D_3 \cap D_1$ are all LCSLs, and the B-structures from each point are isomorphic to the A-structures of $X =: X_1$ and its birational models X_2, X_3.*

The birational models X_2, X_3 in the above proposition are described in the following diagram which we can complete starting from $X = X_1$:

$$
\begin{array}{ccccccccccccc}
X_1 & \overset{\rho_{12}}{\dashleftarrow\dashrightarrow} & & & X_2 & \overset{\rho_{23}}{\dashleftarrow\dashrightarrow} & & & X_3 & \overset{\rho_{31}}{\dashleftarrow\dashrightarrow} & & & X_1 \\
 & {\scriptstyle \pi_{21}} \searrow & & \swarrow {\scriptstyle \pi_{22}} & & {\scriptstyle \pi_{32}} \searrow & & \swarrow {\scriptstyle \pi_{33}} & & {\scriptstyle \pi_{13}} \searrow & & \swarrow {\scriptstyle \pi_{11}} & \\
 & & Z_2 & & & & Z_3 & & & & Z_1 & &
\end{array}
\tag{5}
$$

Here, Z_2 represents a determinantal quintic hypersurface $(5) \subset \mathbb{P}^4_w$ obtained from X_1 by projecting to the second factor $\pi_{21} : \mathbb{P}^4_z \times \mathbb{P}^4_w \to \mathbb{P}^4_w$. Similarly, we obtain a determinantal quintic hypersurface $Z_1 \subset \mathbb{P}^4_z$ by projecting to the first factor. Both Z_2 and Z_1 are singular quintic hypersurface with 50 ordinary double points. The birational models X_2 and X_3 are small resolutions of Z_2 and Z_1, respectively, see [19] for details. The A-structures of these birational models X_1, X_2, X_3 can be glued together in $H^2(X, \mathbb{R})$ and determines the so-called movable cone. Corresponding to the gluing A-structures, it was found in [21] that the B-structures from the LCSL boundary points o_1, o_2, o_3 can also be glued together by the global analysis of the local solutions over the parameter space $\mathscr{M}_{\check{X}}$. In [21], this gluing B-structures is called gluing of monodromy nilpotent cones contrasting the resulting large cone to the movable cone.

Remark 1. The claim in Proposition 3 is based on the isomorphism of the A-structure of X with the B-structure of $\check{X}$. Verifying the both properties (i) and (ii) in Definition 4 is difficult in this case since the large dimension of the deformation space,

$\dim \mathscr{M}_X = 52$. It should be noted that the Calabi-Yau manifolds in the next section provide examples where we can verify both (i) and (ii).

Remark 2. We came to the above Clabi-Yau manifold $X = \cap_{i=1}^{5}(1,1)$ when studying the three dimensional Reye congruence in [18]. Historically Reye congruence stands for an Enriques surface [9], which we can describe as a $\mathbb{Z}_2$ quotient of complete intersection of four symmetric $(1,1)$ divisors in $\mathbb{P}^3 \times \mathbb{P}^3$. Three dimensional Reye congruence is its generalization [20, 19], which we can describe by a $\mathbb{Z}_2$ quotient of $X_s = \cap_{i=1}^{5}(1,1)_s$ with five symmetric divisors $(1,1)_s$. For symmetric divisors, the determinantal quintic Z_3 in the diagram (5) describes the so-called symmetroid which is singular along a curve. It was found in [18, 20] that the double cover of Z_3 branched along the singular locus defines a Calabi-Yau manifod which is a Fourier-Mukai partner of the three dimensional Reye congruence $X_s/\mathbb{Z}_2$.

3.2 Calabi-Yau manifolds fibered by abelian surfaces

Calabi-Yau manifolds related to determinantal quintics show us interesting implications of mirror symmetry to the birational geometry of Calabi-Yau manifolds. Recall that homological mirror symmetry describes the symmetry in terms of the derived categories of coherent sheaves. Therefore if two Calabi-Yau manifolds are derived equivalent, then these two should have the same mirror Calabi-Yau manifold. As for the derived equivalence, the following result is due to Bridgeland [6]:

Theorem 2. *In dimension three, birational Calabi-Yau manifolds are derived equivalent.*

Combined with homological mirror symmetry, it is natural we have encountered the birational models of X from the study of mirror family $\check{\mathfrak{X}} \to \mathscr{M}_{\check{X}}$. At the same time, Calabi-Yau manifolds which are not birational to a Calabi-Yau manifold but derived equivalent to it come into our interest. Such Calabi-Yau manifolds are called Fourier-Mukai partners of a Calabi-Yau manifold, and the so-called Grassmannian and Phaffian duality provides an interesting example of Fourier-Muaki partners [25, 5, 4, 14]. (For K3 surfaces, we have a general result for the numbers of Fourier-Mukai partners, see [17] and reference therein.) The double cover of Z_3 in the preceding subsection (Remark 2), provides another interesting example of Fourier-Mukai partner [18, 20].

Our second example of this article turns out to be more interesting because we observe there that both birational models and Fourier-Mukai partners arise from its mirror family. To describe it briefly, let us consider an abelian surface A with its polarization $\mathscr{L}$ of $(1,d)$ type. Such an abelian surface A can be embedded into $\mathbb{P}^{d-1}$ by the linear system $|\mathscr{L}|$. Gross and Popescu have studied the ideal $\mathscr{I}(A)$ of the image of this embedding [10]. In particular, when the polarization is of type $(1,8)$, it was found that the ideal contains four quartics of the form;

$$f_1 = \frac{w_0}{2}(x_0^2 + x_4^2) + w_1(x_1x_7 + x_3x_5) + w_2x_2x_6, \;\; f_{i+1} = \sigma^i f_1 \;\; (i = 1,2,3),$$

where $\sigma : x_i \mapsto x_{i+1}, \tau : x_i \mapsto \xi^{-i}x_i\,(\xi^8 = 1)$ represent the actions of the Heisenberg group $\mathscr{H}_8 = \langle \sigma, \tau \rangle$ on the homogeneous coordinates x_i of $\mathbb{P}^7$. The parameters $w_i = w_i(A)$ are determined by A and represent a point $[w_0, w_1, w_2] \in \mathbb{P}^2$. Thus the correspondence A $\mapsto w(A)$ defines a (rational) map from an open set of the moduli space $\mathscr{A}^{(1,8)}$ of $(1,8)$-polarized abelian surfaces to $\mathbb{P}^2$. The following theorem is due to Gross and Popescu [10]:

Theorem 3. *$\mathscr{A}^{(1,8)}$ is birational to a conic bundle over $\mathbb{P}^2$.*

While the ideal $\mathscr{I}(A)$ is related to the moduli space $\mathscr{A}^{(1,8)}$ as above, it defines a complete intersection of four quadrics in $\mathbb{P}^7$ which is a singular Calabi-Yau threefold. We denote this Calabi-Yau variety by V_w. The following properties of V_w has been studied in [10].

Proposition 4. *$V_w = V(f_1, \cdots, f_4) \subset \mathbb{P}^4$ has the following properties:*

(1) *V_w is a pencil of $(1,8)$-polarized abelian surfaces which has 64 base points.*
(2) *Only at the 64 base points, V_w is singular with ordinary double points (ODPs). These singularities are resolved by a blow-up along an abelian surface A, giving a small resolution $X' \to V_w$.*
(3) *By flopping exceptional curves of the small resolution, $X' \to V_w \leftarrow X$, we have a Calabi-Yau threefold with fibers $(1,8)$-polarized abelian surfaces over $\mathbb{P}^1$.*
(4) *Both Calabi-Yau manifolds X and X' admit free actions of the Heisenberg group $\mathscr{H}_8$ (which actually acts as $\mathbb{Z}_8 \times \mathbb{Z}_8$ on X and X').*

We can determine the Hodge numbers of X as follows [10]: First we obtain $h^{2,1}(X') = h^{2,1}(X) = 2$ by studying the deformation spaces. Since the singular fibers of X turns out to be translation scrolls of elliptic curves, we see that the topological Euler number of X is zero. This entails that $h^{1,1}(X) = h^{1,1}(X') = 2$.

Using free actions of the Heisenberg group $\mathscr{H}_8$ (which acts on X as $\mathbb{Z}_8 \times \mathbb{Z}_8$), we have quotient Calabi-Yau manifolds

$$\check{X} := X/\mathbb{Z}_8,\ Y := X/\mathbb{Z}_8 \times \mathbb{Z}_8$$

(and also $\check{X}' := X'/\mathbb{Z}_8$, $Y' := X'/\mathbb{Z}_8 \times \mathbb{Z}_8$). All these are Calabi-Yau manifolds with $h^{1,1} = h^{2,1} = 2$. The following interesting properties are shown in [26, 1].

Proposition 5. *The fiber-wise dual of the abelian-fibered Calabi-Yau manifold $X \to \mathbb{P}^1$ is isomorphic to $Y = X/\mathbb{Z}_8 \times \mathbb{Z}_8$, and X and Y are derived equivalent.*

These properties naturally motivate us studying mirror symmetry of X. Regarding this, Gross and Pavanelli [11] made a conjecture:

Conjecture 1. Mirror of X is $\check{X} = X/\mathbb{Z}_8$, and mirror of $\check{X}$ is $Y = X/\mathbb{Z}_8 \times \mathbb{Z}_8$.

This conjecture came from their study on the Brauer groups and fundamental groups of these Calabi-Yau manifolds. Recently, by constructing families of $\check{X}$ and Y and studying LCSL boundary points carefully, we verified the conjecture affirmatively [22] as follows:

Proposition 6. *If we take a subgroup $\mathbb{Z}_8 \simeq \langle \tau \rangle \subset \mathscr{H}_8$ to define $\check{X} = X/\mathbb{Z}_8$, then there exists a family $\check{\mathfrak{X}} \to \mathscr{M}_{\check{X}}$ which describes a mirror family of X, i.e., the B-structure coming from a LCSL boundary point $o \in \overline{\mathscr{M}}_{\check{X}}$ is isomorphic to the A-structure of X. Also, there is another LCSL boundary point $\tilde{o}$ in $\overline{\mathscr{M}}_{\check{X}}$, the B-structure from which is isomorphic to the A-structure of the Fourier-Mukai partner Y of X.*

The parameter space of the family $\check{\mathfrak{X}} \to \mathscr{M}_{\check{X}}$ is given by a toric variety. Studying the boundary points in $\overline{\mathscr{M}}_{\check{X}} \setminus \mathscr{M}_{\check{X}}$ in more detail, we actually find more LCSL boundary points other than $o, \tilde{o}$ above, and find that the B-structures from them are isomorphic to the birational models X' and Y' [22]. Namely, we observe that the birational models as well as Fourier-Mukai partners of X emerge from the LCSL boundary points in the parameter space of the family $\check{\mathfrak{X}} \to \mathscr{M}_{\check{X}}$. We refer to the recent work [22] for more details.

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