Classically modular points in moduli spaces of Calabi-Yau threefolds

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Abstract We discuss some aspects of the arithmetic modularity of Calabi-Yau threefolds which relate to constructions in supergravity, or superstring theory.

1 Introduction

A Calabi-Yau threefold X with complex structure modulus $\mathbf{z}$ can be viewed as the fibre $X_{\mathbf{z}}$ over the moduli space of complex structures on X. This moduli space can have dimension $h^{2,1}(X)$ greater than one, so that $\mathbf{z}$ is a vector of moduli space coordinates, but we might also consider the case of a one-parameter family X_z.

There can be interesting points within the moduli space of complex structures, where the fibre takes on certain properties or solves a certain problem.

One such set of points $\mathbf{z}_*$ are of interest in number theory. If X is a variety over a number field then, after reducing the equations that define X modulo p, the set of points on X in each finite field $\mathbb{F}_{p^r}$ can be counted. For each fixed p, the numbers $N_{p^r}(X_{\mathbf{z}})$ of these solutions can be collected in the exponential generating function

$$\zeta_p(X_{\mathbf{z}};T) = \exp\left(\sum_{r=1}^{\infty} \frac{1}{r} N_{p^r}(X_{\mathbf{z}}) T^r\right). \tag{1}$$

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This is the local zeta function, which remarkably is a rational function of T. This rationality property is one of the Weil conjectures [41], proved in the works [18, 27, 14],

$$\zeta_p(X_{\mathbf{z}};T) = \frac{R_p(\mathbf{z};T)}{(1-T)(1-pT)^{h^{1,1}}(1-p^2T)^{h^{1,1}}(1-p^3T)}. \tag{2}$$

Here we have specialised the more general formula to the case of a Calabi-Yau threefold with Hodge numbers $h^{1,1}, h^{2,1}$. The numerator $R_p(\mathbf{z};T)$ is a degree $2h^{2,1}+2$ polynomial in T, whose coefficients contain all dependence of ζ_p on the modulus $\mathbf{z}$. This procedure of point-counting only makes sense for varieties defined over number fields, which means in suitable coordinates that the $\mathbf{z}$ are assumed to take values in a number field. The points $\mathbf{z}_*$ that we wish to draw attention to are those such that for every prime p, $R_p(\mathbf{z}_*;T)$ factors over the number field $\mathbf{Q}(\mathbf{z}_*)$ with a quadratic factor. While for any $\mathbf{z}$ there may be finitely many p such that this factorisation occurs, we are specifically interested in cases where the factorisation occurs for all primes minus those in a certain “bad” finite set. This is said to be “persistent factorisation” of the zeta function.

The varieties $X_{\mathbf{z}}$, for $\mathbf{z}$ in a number field, are expected to encode Siegel modular forms in their zeta function numerators $R_p(\mathbf{z};T)$. This is an area of active research, see for instance [23], which moves in step with advances on Siegel modular forms. The special cases where the zeta function persistently factorises are of interest because the associated modular objects are much simpler, being classical modular forms. The persistently factorising $R_p(\mathbf{z};T)$ may contain factors

$$1-\alpha_p pT + p^3T^2, \qquad \text{or} \qquad 1-\beta_p T + p^3T^2. \tag{3}$$

These are explained as factors in the determinant of the Frobenius action on the middle cohomology of $X_{\mathbf{z}}$. They correspond to reducible blocks contained respectively in $H^{2,1}\oplus H^{1,2}$ or $H^{3,0}\oplus H^{0,3}$, where we use the Hodge filtration.

Through the Euler product formula, these factors are combined into a motivic L-function. Conjecturally this motivic L-function equals an automorphic L-function, which is the Mellin transform of a modular form. The α_p associated to a $H^{2,1}\oplus H^{1,2}$ block are the coefficients of q^p in the q-expansion of a weight-two newform. The β_p associated to a $H^{3,0}\oplus H^{0,3}$ block are the coefficients of q^p in the expansion of a weight-four newform. To summarise, there can be interesting $\mathbf{z}_*$ for which $X_{\mathbf{z}_*}$ are related to classical modular forms via point-counting over finite fields.

The connection to physics comes from the fact that certain problems in supergravity require that the complex structure parameter of a Calabi-Yau threefold (equivalently, the complexified Kähler parameter of a mirror Calabi-Yau threefold) is at a special point in moduli space where the Hodge structure degenerates in some way. We explain these supergravity problems below, and discuss their relation to concepts from arithmetic geometry.

1.1 The attractor mechanism

In a IIB supergravity compactification on X, the $\mathbf{z}$ take on a physical role as the $h^{2,1}$ vector multiplet scalars of the resulting matter-coupled 4d $\mathscr{N}=2$ supergravity.

Static, supersymmetric black hole solutions of the 4d theory have an attractor mechanism [21, 22]. Vector multiplet scalars take some boundary values at spatial infinity, and their dependence on the radial coordinate is then governed by the equations of motion. These take the form of a damped oscillator, and solutions have attractor behaviour. The $\mathbf{z}$ approach asymptotic values at the event horizon which extremise the absolute value of the black hole's central charge. If this central charge does not vanish, the limiting value can be given [20, 39] as the solution of

$$Q = \mathrm{Im}[C\mathit{\Pi}(\mathbf{z})], \tag{4}$$

where Q is the 4d black hole's electromagnetic charge vector and $\mathit{\Pi}$ is the integral symplectic period vector of $X_{\mathbf{z}}$. The $h^{2,1}$ complex $\mathbf{z}$ and overall complex scale C furnish $2h^{2,1}+2$ real unknowns, which matches the number of real equations in (4). The functions $\mathbf{z}(Q)$ obtained by inverting (4) were analysed in [8].

A given charge vector Q can be interpreted as the periods of an integral 3-cohomology class on a Calabi-Yau threefold and the equation (4) has a solution at a regular point in moduli space precisely when the cohomology class corresponding to Q has vanishing $(2,1)$ and $(1,2)$ parts in its Hodge decomposition.

It may (very rarely) happen that there are two linearly independent charge vectors Q_1 and Q_2 that lead to the same attractor point. Such points are referred to as attractor points of rank two. In this case, there is a rank two sublattice $\Lambda \subset H^3(X,\mathbb{Z})$ with Hodge numbers $(3,0)+(0,3)$. Its symplectically dual lattice then has Hodge number $(2,1)+(1,2)$. When the underlying Calabi-Yau variety at an attractor point of rank two is defined over $\mathbb{Q}$, it has been proven that the Calabi-Yau variety is modular i.e. we expect persistent factorisations of the numerator of the zeta-function and each quadratic factor is determined by the Fourier coefficients of a modular form [24]. This leads to an effective strategy for finding attractor points of rank two [9].

One consequence of modularity is that, at an attractor point of rank two, one may (conjecturally) find exact expressions for the Calabi-Yau periods. These give physically meaningful quantities, such as central charges of D-branes. However, the ultimate role of modularity in Calabi-Yau compactifications is far from understood.

1.2 Supersymmetric flux vacua

We draw on [13, 16], and references therein. The tree-level spacetime superpotential of a 4d $\mathscr{N}=1$ supergravity obtained by reducing IIB supergravity on $X_{\mathbf{z}}$ reads

$$W(\tau,\mathbf{z}) = \int_X (F-\tau H)\wedge\Omega(\mathbf{z}) = (\mathrm{F}-\tau\mathrm{H})\,\Sigma\,\mathit{\Pi}(\mathbf{z}), \tag{5}$$

where τ is the axiodilaton and F, H are the fluxes background values given to the 3-form fields. Let the associated cohomology classes have components F, H in a symplectic basis of $H^3(X, \mathbb{Z})$.

From string theory, there is a requirement for objects of negative tension. These can be provided by O-planes, which necessitate orientifolding the theory. The superpotential remains (5), but only depends on moduli fields that survive the orientifolding. This issue is not important for our present modularity discussions. We will proceed with (5) and discuss implications for the solutions on the threefold, and not on its $\mathbb{Z}_2$ quotient obtained by the orientifolding.

The superpotential defines a potential function that couples all the scalar fields,

$$V(\tau, z_i, t^\alpha) = \mathrm{e}^K \left(G^{AB} (D_A W)(\overline{D_B W}) - 3|W|^2 \right), \tag{6}$$

with K the Kähler potential and G_{AB} the metric on the total moduli space. $D_A = \partial_A + (\partial_A K)$ is the Kähler covariant derivative with respect to the field $A \in \{\tau, z_i, t^\alpha\}$. W has no dependence on the Kähler moduli t^α, i.e. $\partial_{t^\alpha} W = 0$, and therefore $D_{t^\alpha} W = (\partial_{t^\alpha} K) W$ vanishes when $W = 0$. The potential V must vanish in a vacuum.

If the vacuum is to preserve supersymmetry, then the superpotential W itself must also vanish. These two conditions $V = W = 0$ can be brought to the form

$$\mathrm{F}\,\Sigma\,\Pi(\mathbf{z}) = \mathrm{H}\,\Sigma\,\Pi(\mathbf{z}) = 0, \quad (\mathrm{F} - \tau \mathrm{H})\,\Sigma\,\partial_{z_i} \Pi(\mathbf{z}_i). \tag{7}$$

There is moreover a tadpole cancellation condition, requiring that

$$\mathrm{F}\,\Sigma\,\mathrm{H} \neq 0. \tag{8}$$

In particular this implies that the two flux vectors are not a multiple of one another.

Since F and H are integral cohomology classes, the above conditions imply that they span a sublattice of $H^3(X, \mathbb{Z})$ with Hodge numbers $(2,1) + (1,2)$. Just as was the case with attractor points of rank two, this piece of cohomology can be detected as a quadratic factor in the numerator of infinitely many local zeta functions. For Calabi-Yau threefolds with $h^{2,1} = 1$ a rank two attractor is equivalent to a flux vacuum with vanishing superpotential. For $h^{2,1} > 1$, the two are no longer equivalent.

A relation between supersymmetric flux vacua and modularity was conjectured in [34]. [33] makes a striking observation matching the F-theory and modular curves.

These SFV loci are also of use in constructing gauged supergravities, [30]. Here we solely discuss the physical incarnations of modular Calabi-Yau threefolds, for fourfolds see [31, 36].

2 Known 1-parameter examples of split Hodge structures

A seemingly sporadic set has been found by using the methods of [7] to compute the zeta function for a large number of primes and checking for persistent factorisation. We collect as many of these as we are aware of into Table 1. There is not currently a satisfactory explanation as to why any of these varieties are classically modular. We list the AESZ number [1], the special point z_*, the LMFDB labels [40] for the modular forms, and the field K over which the Hodge structure splits[1].

There are many examples of special points with a split Hodge structure that arise from Hadamard products with a symmetry that permutes the singularities at $z = 0$ and $z = \infty$. The special points are typically at apparent singularities, i.e. at points where all the periods and their derivatives are regular, but the Yukawa coupling vanishes. We list examples that are Hadamard squares in Table 1 and a single example is considered in §4, where the method is explained.

Most examples in Table 1 are attractor points of rank two, where the Hodge structure of the middle cohomology of the underlying CY threefold splits as

$$(1,1,1,1) \to (1,0,0,1) + (0,1,1,0) \tag{9}$$

over a real number field. We slightly abuse terminology here when we refer to splittings over non-trivial real number fields (e.g. $\mathbb{Q}(\sqrt{2})$) as rank two attractors. Taken seriously, this would give black holes/D3 branes with charges valued in a number field. The physical meaning here is unclear, but we include them for completeness.

We also point out that a "persistent factorisation" of the numerator of the zeta-function will still take place when the Hodge structure splits over a non-trivial number field for infinitely many primes. For example, in the case of $\mathbb{Q}(\sqrt{2})$, we expect a factorisation of the numerator of the zeta function when $x^2 - 2 = 0$ has a solution in the finite field $\mathbb{F}_p$ and the underlying variety is still smooth when reduced mod p.

There are also points where the Hodge structure splits over an imaginary number field. These cannot be rank two attractor points. Instead, the Hodge structure splits as

$$(1,1,1,1) \to (1,0,1,0) + (0,1,0,1) \tag{10}$$

at these points. Such points support flux vacua with non-vanishing superpotential.

3 Periods and quasiperiods

In [4] the periods of hypergeometric one–parameter families of Calabi–Yau threefolds at an attractor point were expressed in terms of the periods and quasiperiods of the associated modular forms. In this section, we apply this to the two families governed by the Picard–Fuchs operators AESZ17 and AESZ22, discussed in detail in [12]. As discussed in Section 2 at an attractor point the Hodge structure splits as

$$H^3(X_{z_*},\mathbb{Q}) = \Lambda \otimes \mathbb{Q} \oplus \Lambda^\perp \otimes \mathbb{Q} \tag{11}$$

where $\Lambda \otimes \mathbb{Q}$ is a Hodge structure of type $(1,0,0,1)$ which corresponds to the Hodge structure of a rigid Calabi–Yau threefold, and $\Lambda^\perp \otimes \mathbb{Q}$ is a Hodge structure of type

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[1] This is not the same as as the field of definition of z_*. For example, AESZ 34 has rank two attractor points defined over $\mathbb{Q}(\sqrt{17})$. Nevertheless the Hodge structure splits over $\mathbb{Q}$ at these points.

Table 1 Known examples of points z_* in one-parameter moduli spaces where a splitting occurs. We give the AESZ number [1] of the Picard-Fuchs operator, the field K over which the zeta numerator factors, LMFDB labels [40] for the weight-four and weight-two newforms f and g respectively in cases where $K = \mathbb{Q}$, and references that further discuss/discover these examples. $A(\cong B)$ indicates that the operators A and B are equivalent by a change of coordinates.

AESZ#	z_*	K	f	g	References
3	-2^{-8}	$\mathbb{Q}(\mathrm{i})$			
3	$(17 \pm 12\sqrt{2})2^{-8}$	$\mathbb{Q}(\sqrt{2})$			
4	$-2^{-3}3^{-6}$	$\mathbb{Q}$	**54.4.a.c**	**54.2.a.a**	[4, 3]
11	$-2^{-4}3^{-3}$	$\mathbb{Q}$	**180.4.a.e**	**36.2.a.a**	[4, 3]
17 $(\cong 290)$	-1	$\mathbb{Q}$	**14.4.a.b**	**14.2.a.a**	[38, 12]
22 $(\cong 118)$	-1	$\mathbb{Q}$	**33.4.a.b**	**11.2.a.a**	[38, 12]
34	-7^{-1}	$\mathbb{Q}$	**14.4.a.a**	**14.2.a.a**	[9]
34	$33 \pm 8\sqrt{17}$	$\mathbb{Q}$	**34.4.b.a**	**34.2.b.a**	[9]
36	-2^{-6}	$\mathbb{Q}$	**96.4.a.e**	**32.2.a.a**	[3]
49	$-2^{-4}5^{-1}$	$\mathbb{Q}$	**400.4.a.o**	**400.2.a.c**	[3]
55	2^{-4}	$\mathbb{Q}$	**60.4.a.a**	**20.2.a.a**	[3]
84	-2^{-4}	$\mathbb{Q}$	**20.4.a.a**	**20.2.a.a**	[3]
84	$2^{-4}3^{-1}$	$\mathbb{Q}$	**12.4.a.a**	**36.2.a.a**	[3]
100	-2^{-3}	$\mathbb{Q}$	**14.4.a.b**	**14.2.a.a**	[3]
101	1	$\mathbb{Q}$	**22.4.a.a**	**11.2.a.a**	[3]
103	-3^{-2}	$\mathbb{Q}$	**180.4.a.d**	**90.2.a.c**	[3]
107	-2^{-5}	$\mathbb{Q}$	**48.4.a.c**	**48.2.a.a**	[3]
111	$-2^{-8}3^{-1}$	$\mathbb{Q}$	**144.4.a.b**	**144.2.a.b**	[3]
115	-2^{-8}	$\mathbb{Q}$	**32.4.a.c**	**32.2.a.a**	[3]
144	$-2^{-3}3^{-2}$	$\mathbb{Q}$	**306.4.a.c**	**306.2.a.c**	[3]
145	-3^{-6}	$\mathbb{Q}$	**108.4.a.d**	**54.2.a.b**	[3]
154	$-2^{-4}3^{-3}$	$\mathbb{Q}$	**216.4.a.d**	**216.2.a.d**	[3]
155	-2^{-12}	$\mathbb{Q}$	**128.4.a.a**	**128.2.a.b**	[3]
165	-3^{-3}	$\mathbb{Q}$	**54.4.a.d**	**27.2.a.a**	[3]
166	$-2^{-8}3^{-6}$	$\mathbb{Q}$	**864.4.a.b**	**864.2.a.g**	[3]
204	-2^{-10}	$\mathbb{Q}(\mathrm{i})$			
204	-2^{-12}	$\mathbb{Q}(\sqrt{5})$			
204	-2^{-20}	$\mathbb{Q}(\sqrt{41})$			
238	-2^{-4}	$\mathbb{Q}$	**88.4.a.a**	**88.2.a.a**	[3]
277	-2^{-14}	$\mathbb{Q}$	**240.4.a.i**	**80.2.a.b**	[3]
2.32	$2^{-4}3^{-3}$	$\mathbb{Q}$	**324.4.a.a**	**324.2.a.d**	[3]

$(0,1,1,0)$ which (after tensoring by $\mathbb{Q}(1)$) corresponds to the Hodge structure of an elliptic curve. Conjecturally, it is expected that the periods of the modular Calabi-Yau manifold are related to the critical L–values of a modular form whose weight equals the weight of the Hodge structure plus 1. Hence, the standard conjectures imply that there are integers N_1, N_2 and newforms $f \in S_4(\Gamma_0(N_1)), g \in S_2(\Gamma_0(N_2))$ such that there is a factorization $L(X,s) = L(f,s)L(g,s)$. In other words, there is a basis for $H^3(X_{z_*}, \mathbb{Q})$ and $H_3(X_{z_*}, Q)$ such that period matrix takes the form

$$\mathrm{P} = \begin{pmatrix} \omega_f^+ & \omega_f^- & 0 & 0 \\ \eta_F^+ & \eta_F^- & 0 & 0 \\ 0 & 0 & \tilde{\omega}_g^+ & \tilde{\omega}_g^- \\ 0 & 0 & \tilde{\eta}_G^+ & \tilde{\eta}_G^- \end{pmatrix} \tag{12}$$

where $\omega_f^\pm$, $\eta_F^\pm$, $\tilde{\omega}_g^\pm$, $\tilde{\eta}_G^\pm$ are the periods of f and g and the quasiperiods of their meromorphic partners F and G, respectively. Some of the periods are related to critical values of the L–functions of f and g, respectively. For definitions and explanations of these terms see [4].

At the attractor z_* the CY period matrix $\mathrm{T}_{z_*} = (\Pi(z_*), \Pi'(z_*), \Pi''(z_*), \Pi'''(z_*))^T$ can be expressed in terms of the matrix P through $\mathrm{T}_{z_*}^T = \mathrm{APB}$, where A and B are matrices of rational numbers.

3.1 AESZ 17

Here $N_1 = N_2 = 14$ and we have the following modular forms:

$$\begin{aligned}
f &= q + 2q^2 - 2q^3 + 4q^4 - 12q^5 - 4q^6 + 7q^7 + 8q^8 - 23q^9 + O(q^{10}) && \in S_4(\Gamma_0(14)),\\
F &= q^{-1} + 38q^2 + 199q^3 + 908q^4 + 2529q^5 + 7796q^6 + O(q^7) && \in S_4^!(\Gamma_0(14)),\\
g &= q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + q^9 + O(q^{10}) && \in S_2(\Gamma_0(14)),\\
G &= q^{-1} + 47q^2 + 199q^3 + 329q^4 + 1023q^5 + 1874q^6 + O(q^7) && \in S_2^!(\Gamma_0(14)),
\end{aligned} \tag{13}$$

where $f = f_{14.4.a.b}$, $g = f_{14.2.a.a}$ as in Table 1. The meromorphic modular forms F and G can succinctly be described as

$$\begin{aligned}
F &= \frac{1}{h}\Bigg(\Big(\tfrac{1}{3} - \tfrac{37\sqrt{1969}}{11814}\Big) f_{14.8.a.c} + \Big(\tfrac{1}{3} + \tfrac{37\sqrt{1969}}{11814}\Big)\sigma(f_{14.8.a.c})\\
&\quad + \tfrac{1}{3} f_{7.8.a.a} - \tfrac{16}{3} f_{7.8.a.a}|_8 V_2\Bigg)\\
G &= \frac{1}{h}\Bigg(\Big(-\tfrac{2}{43} - \tfrac{754\sqrt{57}}{2451}\Big) f_{7.6.a.b}|_6 V_2 + \Big(\tfrac{754\sqrt{57}}{2451} - \tfrac{2}{43}\Big)\sigma(f_{7.6.a.b})|_6 V_2\\
&\quad + \Big(-\tfrac{377\sqrt{57}}{9804} - \tfrac{1}{172}\Big) f_{7.6.a.b} + \Big(-\tfrac{1}{172} + \tfrac{377\sqrt{57}}{9804}\Big)\sigma(f_{7.6.a.b}) + \tfrac{87}{86} f_{14.6.a.a}\Bigg)\\
h &= \eta(\tau)^2\eta(2\tau)^2\eta(7\tau)^2\eta(14\tau)^2
\end{aligned} \tag{14}$$

where $V_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and σ denotes Galois conjugation. The indices give the label of the cusp form in the LMFDB. Their periods and quasiperiods are

$$\begin{aligned}
\omega_f^+ &= 3.4463066043046096205\ldots, & \omega_f^- &= 1.0198537741597169074\ldots i,\\
\eta_F^+ &= -12.5614347465463248098\ldots, & \eta_F^- &= -0.77948030622635705217\ldots i,\\
\omega_g^+ &= -0.990670978033441617\ldots, & \omega_g^- &= -1.325491239682486714\ldots i,\\
\eta_G^+ &= 15.68722010803376745967\ldots, & \eta_G^- &= 8.304373900345374755279\ldots i,
\end{aligned} \tag{15}$$

and $\tilde{\omega}_g^\pm = 2\pi i \tilde{\omega}_g^\pm$, $\tilde{\eta}_G^\pm = 2\pi i \eta_G^\pm$. The periods $\omega_f^\pm$, $\omega_g^\pm$ are related to special values of the L–functions of f and g and their twists by

$$\begin{aligned}
\omega_f^+ &= -\frac{3}{28}(2\pi i)^2 L(f,1), & \omega_f^- &= -\frac{1}{7}(2\pi i) L(f,2),\\
\omega_g^+ &= -3L(g,1), & \omega_g^- &= -\frac{1}{2}G(\chi)L(\tilde{g},1),
\end{aligned} \tag{16}$$

where $\chi = \chi_7(6,\cdot)$ with Gauss sum $G(\chi) = \sqrt{-7}$ and $\tilde{g} = g \otimes \chi \in S_2(\Gamma_0(98))$.

3.2 AESZ 22

Here $N_1 = 33$, $N_2 = 11$ and we have the following modular forms:

$$\begin{aligned}
f &= q - q^2 - 3q^3 - 7q^4 - 4q^5 + 3q^6 - 26q^7 + 15q^8 + 9q^9 + O(q^{10}) & &\in S_4(\Gamma_0(33)),\\
F &= q^{-1} - 4q^2 - 15q^3 - 44q^4 - 126q^5 - 228q^6 - 486q^7 - O(q^8) & &\in S_4^!(\Gamma_0(33)),\\
g &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + O(q^{10}) & &\in S_2(\Gamma_0(11)),\\
G &= q^{-1} + 116q^2 + 361q^3 + 962q^4 + 2676q^5 + 6172q^6 + O(q^7) & &\in S_2^!(\Gamma_0(11)),
\end{aligned} \tag{17}$$

with periods and quasiperiods

$$\begin{aligned}
\omega_f^+ &= 34.669451430295846076\ldots, & \omega_f^- &= 2.3806231256299333180\ldots i,\\
\eta_F^+ &= -15.6465119259448742023\ldots, & \eta_F^- &= 2.5029717810597002614\ldots i,\\
\omega_g^+ &= -0.63460465213977671084\ldots, & \omega_g^- &= 1.4588166169384952293\ldots i,\\
\eta_G^+ &= 11.733403441655492016\ldots, & \eta_G^- &= 17.071571363716644932\ldots i,
\end{aligned} \tag{18}$$

and $\tilde{\omega}_g^\pm = 2\pi i \tilde{\omega}_g^\pm$, $\tilde{\eta}_G^\pm = 2\pi i \eta_G^\pm$. In this example the expressions for F and G in terms of cusp forms with LMFDB label is too lengthy and is available from the authors on request. The periods $\omega_f^\pm$, $\omega_g^\pm$ are related to special values of the L–functions of f and g. In this case $L(f,2) = 0$ hence we have to replace f with an appropriate twist. Let $\chi = \left(\frac{-3}{\bullet}\right)$ with Gauss sum $G(\chi) = \sqrt{-3}$ and consider

$$\tilde{f} = f \otimes \chi \in S_4(\Gamma_0(99)), \qquad \tilde{g} = g \otimes \chi \in S_2(\Gamma_0(99)). \tag{19}$$

Then we have the relations

$$\begin{aligned} \omega_f^+ &= \frac{5}{6}(2\pi i)^2 L(f,1), & \omega_f^- &= -\frac{1}{132} G(\chi)(2\pi i)^2 L(\tilde{f},1), \\ \omega_g^+ &= -\frac{5}{2} L(g,1), & \omega_g^- &= \frac{1}{2} G(\chi) L(\tilde{g},1). \end{aligned} \tag{20}$$

4 Hadamard Products and Symmetries

As explained in previous sections, there are few systematic methods for finding special members of families of Calabi-Yau threefolds where the Hodge structure splits like a rank two attractor (equivalently, in the one-parameter case, like a supersymmetric flux vacuum). At the time of writing, most known examples are Hadamard products of two second order Picard-Fuchs equations, where each second order operator admits a symmetry that permutes its singularities. The Hadamard product will typically inherit this symmetry, and the resulting smooth fixed points will be apparent singularities that are attractor points of rank two. For a geometric explanation and an explicit example, the reader may consult [19].

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two second order Picard-Fuchs equations. We say that an operator $\mathscr{L}_1 * \mathscr{L}_2$ is a Hadamard product of $\mathscr{L}_1$ and $\mathscr{L}_2$ if

$$\mathscr{L}_i \left(\sum_{n=0}^{\infty} a_n^{(i)} u^n \right) = 0, \qquad (\mathscr{L}_1 * \mathscr{L}_2) \left(\sum_{n=0}^{\infty} a_n^{(1)} a_n^{(2)} z^n \right) = 0. \tag{21}$$

Note that $\mathscr{L}_1 * \mathscr{L}_2$ is not uniquely defined in this way. There is an entire ideal of differential operators with this property. However, in practice, there is an operator of minimal degree that we refer to as the Hadamard product.

Many second order operators can be found in [2], which can be used to construct attractor points of rank two. In their choice of labeling, operators $\mathscr{L}_a$ to $\mathscr{L}_j$ all admit a symmetry that permutes its singularities. We may take Hadamard products of these operators and produce many examples of fourth order operators with symmetries and smooth fixed points.

We illustrate this approach with the example of AESZ 175 and leave a more thorough search for future work. The operator AESZ 175 has the Riemann symbol

$$\mathscr{P} \left\{ \begin{array}{ccccccc} 0 & \frac{1}{81} & \frac{1}{72} & \frac{1}{9} & \frac{1}{8} & \pm\frac{1}{8\sqrt{2}} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 3 & 1 \\ 0 & 2 & 2 & 2 & 2 & 4 & 1 \end{array} \right\}. \tag{22}$$

It is the Hadamard product of two operators $\mathscr{L}_c$ and $\mathscr{L}_g$. These have symbols

$$\mathscr{L}_c : \mathscr{P} \left\{ \begin{array}{cccc} 0 & \frac{1}{9} & 1 & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right\} \qquad \text{and} \qquad \mathscr{L}_g : \mathscr{P} \left\{ \begin{array}{cccc} 0 & \frac{1}{9} & \frac{1}{8} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right\}. \tag{23}$$

AESZ 175 is is invariant under the involution $z \to \frac{1}{648z}$, and we see that both fixed points $\pm\frac{1}{18\sqrt{2}}$ are apparent singularities. We compute the periods at this point and numerically confirm a splitting of Hodge structure of the form (9) over the number field $\mathbb{Q}(\sqrt{2})$. We may fix a basis of periods with rational monodromy by assuming the standard Gamma class formula (see, for example, the appendix of [4]) and search for a choice of the triple intersection number H^3, second Chern-class $c_2 \cdot H$ and Euler characteristic of a smooth mirror manifold that leads to monodromy in $Sp(4,\mathbb{Z})$. Unfortunately, a numerical search suggests that there is no choice of topological data that leads to monodromy in $Sp(4,\mathbb{Z})$ when used with the the standard Gamma class formula. The results below are quoted for the the topological data $(H^3, c_2 \cdot H, \chi) = (72, 72, 216)$. This leads to monodromy matrices in $Sp(4,\mathbb{Q})$, which is good enough for testing splitting of Hodge structure.

We find that

$$\begin{aligned} \mathit{\Pi}\left(\frac{1}{18\sqrt{2}}\right) &= \alpha_1 \mathit{\Gamma}_1 + \alpha_2 \mathit{\Gamma}_2, & \mathit{\Gamma}_1 &= \left(-\frac{3}{2}, 9+3\sqrt{2}, 1, 0\right), \\ D_z \mathit{\Pi}\left(\frac{1}{18\sqrt{2}}\right) &= \beta_1 \overline{\mathit{\Gamma}}_1 + \beta_2 \overline{\mathit{\Gamma}}_2, & \mathit{\Gamma}_2 &= \left(9-3\sqrt{2}, -18, 0, 1\right), \end{aligned} \tag{24}$$

and $\overline{\mathit{\Gamma}}_i$ is the Galois conjugate of $\mathit{\Gamma}_i$ in $\mathbb{Q}\left(\sqrt{2}\right)$. We compute the coefficients to be

$$\begin{aligned} \alpha_1 &= 224.9708237697899355\ldots - \mathrm{i}215.5343428532531063\ldots, \\ \alpha_2 &= 165.5115435704756999\ldots + \mathrm{i}44.6386239834950043\ldots, \\ \beta_1 &= 3036.246857171535074\ldots + \mathrm{i}812.733738853537295\ldots, \\ \beta_2 &= 802.4731812316557819\ldots + \mathrm{i}981.0564074692015020\ldots\ . \end{aligned} \tag{25}$$

Note that we normalise the periods so that the holomorphic solution around $z = 0$ has leading term equal to $(2\pi i)$. We find similar results at $z = -\frac{1}{18\sqrt{2}}$,

$$\begin{aligned} \mathit{\Pi}\left(-\frac{1}{18\sqrt{2}}\right) &= \alpha_1 \mathit{\Gamma}_1 + \alpha_2 \mathit{\Gamma}_2, & \mathit{\Gamma}_1 &= \left(-9, 18+6\sqrt{2}, 1, 0\right), \\ D_z \mathit{\Pi}\left(-\frac{1}{18\sqrt{2}}\right) &= \beta_1 \overline{\mathit{\Gamma}}_1 + \beta_2 \overline{\mathit{\Gamma}}_2, & \mathit{\Gamma}_2 &= \left(18-6\sqrt{2}, -36, 0, 1\right). \end{aligned} \tag{26}$$

As before, $\overline{\mathit{\Gamma}}_i$ is the Galois conjugate of $\mathit{\Gamma}_i$ in $\mathbb{Q}\left(\sqrt{2}\right)$ and

$$\begin{aligned} \alpha_1 &= -\mathrm{i}165.0515735127136317\ldots, \quad \beta_1 = \mathrm{i}972.9410328711081378\ldots, \\ \alpha_2 &= 107.07556468687835349\ldots - i82.52578675635681589\ldots, \\ \beta_2 &= -316.5064782118421090\ldots + i486.4705164355540689\ldots. \end{aligned} \tag{27}$$

In each case we find a two dimensional subspace of $H^3\left(X, \mathbb{Q}\left(\sqrt{2}\right)\right)$ with Hodge numbers $(3,0)+(0,3)$, with Galois conjugate subspaces of type $(2,1)+(1,2)$.

5 Supersymmetric flux vacua from $\mathbb{Z}_2$ permutation symmetry

We discuss a solution of the SFV equations (7) from [11], distinct from the method of [15] used in the works [34, 33] that put forward the flux modularity conjecture: the fibres $X_{\mathbf{z}_*}$ over solutions $\mathbf{z}_*$ of the SFV equations are weight-two modular.

The integral symplectic period vector $\Pi(\mathbf{z})$ is a vector of $2h^{2,1}+2$ functions that solve the Picard-Fuchs system of $X_{\mathbf{z}}$. It is known [28, 29] that in an expansion about the large complex structure point $\mathbf{z}=0$ there are $h^{2,1}$ single-log periods Π^i and $h^{2,1}$ double log periods Π_i. Now assume that there exists a pair of moduli z_I, z_J such that

$$\Pi_I(\mathbf{z}) = \Pi_J(\mathbf{z}') \quad \text{and} \quad \Pi^I(\mathbf{z}) = \Pi^J(\mathbf{z}'), \tag{28}$$

where the argument $\mathbf{z}'$ is obtained from $\mathbf{z}$ by permuting z_I and z_J. If a manifold $X_{\mathbf{z}}$ can be found, and coordinates such that these conditions on Π are met, then we can obtain a solution of (7) by choosing flux vectors such that[2]

$$F\,\Sigma\,\Pi = \Pi^I - \Pi^J, \qquad H\,\Sigma\Pi = \Pi_I - \Pi_J. \tag{29}$$

With this choice made, all bar two of the equations (7) are automatically satisfied by restricting $\mathbf{z}$ to the codimension-1 locus $z_I = z_J$. The only condition that is not automatically met is $(F-\tau H)\Sigma\partial_{z_I}\Pi = 0$, which is equivalent to $(F-\tau H)\Sigma\partial_{z_J}\Pi = 0$. This last equation can then be satisfied by constraining the axiodilaton via

$$\tau(\hat{\mathbf{z}}) = -\left.\frac{F\,\Sigma\,\partial_{z_I}\Pi(\mathbf{z})}{H\,\Sigma\,\partial_{z_I}\Pi(\mathbf{z})}\right|_{z_I=z_J}, \qquad \hat{\mathbf{z}} = \mathbf{z}|_{z_I=z_J}. \tag{30}$$

A large set of examples are informed by the expressions for period functions of mirrors of complete intersection Calabi-Yau manifolds given in [28, 29]. This gives

$$\varpi_0 = \sum_{n_i\geq 0} c(n_1,\ldots,n_h) z_1^{n_1}\ldots z_h^{n_h}, \qquad h = h^{2,1}(X), \tag{31}$$

we refer to [29, eq.(3.8)] for $c(\mathbf{n})$ in terms of the matrix that defines the mirror Y of X as a complete intersection in a product of projective spaces. We restrict to cases where the number of projective space factors in the ambient space equals $h^{1,1}(Y) = h^{2,1}(X) = h$. All remaining Frobenius periods can be found from

$$\varpi_\varepsilon = \sum_{n_i\geq 0} \frac{c(n_1+\varepsilon_1,\ldots,n_h+\varepsilon_h)}{c(\varepsilon_1\ldots\varepsilon_h)} z_1^{n_1+\varepsilon_1}\ldots z_h^{n_h+\varepsilon_h}, \tag{32}$$

with Gamma functions replacing factorials, and then forming the combinations

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[2] See [11, §3] for specific components of F and H, as well as other details of the solution that we omit presently. As discussed in [11], there is the possibility to rescale the flux vectors by integers, which also provides a solution on the same locus in $\mathbf{z}$-space but scales the axiodilaton by a rational number. This is not reflected in the zeta function, but this makes sense because the point counts of isogeneous elliptic curves (related by scaling the modulus τ) are equal.

$$\varpi_{1,i} = \partial_{\varepsilon_i}\varpi_{\varepsilon}\Big|_{\varepsilon=0}, \quad \varpi_{2,i} = \frac{1}{2}\sum_{j,k}\kappa_{ijk}\partial_{\varepsilon_j}\partial_{\varepsilon_k}\varpi_{\varepsilon}\Big|_{\varepsilon=0}, \quad \varpi_3 = \frac{1}{6}\sum_{i,j,k}\kappa_{ijk}\partial_{\varepsilon_i}\partial_{\varepsilon_j}\partial_{\varepsilon_k}\varpi_{\varepsilon}\Big|_{\varepsilon=0}. \tag{33}$$

Here we introduce the triple intersection numbers $\kappa_{ijk} = \int_Y e_i e_j e_k$ of the mirror Y, where the e_i generate $H^2(Y,\mathbb{Z})_{\text{Free}}$. In expansions about the point $\mathbf{z} = 0$ one has

$$\begin{aligned}
\varpi_0 &= 1 + O(z), \qquad \varpi_i^{(1)} = \varpi_0 \log(z_i) + f_i^{(1)}, \\
\varpi_i^{(2)} &= \frac{\varpi_0}{2}\kappa_{ijk}\log(z_j)\log(z_k) + \kappa_{ijk}f_j^{(1)}\log(z_k) + f_i^{(2)}, \\
\varpi_3 &= \frac{\varpi_0}{6}\kappa_{ijk}\log(z_i)\log(z_j)\log(z_k) + \frac{\kappa_{ijk}}{2}f_i^{(1)}\log(z_j)\log(z_k) + f_k^{(2)}\log(z_k) + f^{(3)},
\end{aligned} \tag{34}$$

where each function f is a power series in $\mathbf{z}$ with constant term 0.

If the configuration matrix is symmetric under exchanging the I^{th} and J^{th} rows, then (34) will have a $z_I \leftrightarrow z_J$ symmetry. Also, the topological data of the intersection [29, eq.(2.5)] will respect this symmetry. So, the integral symplectic period vector Π will satisfy (28) and an SFV solution is found by setting $z_I = z_J$.

The axiodilaton is constant throughout spacetime, but is a function of the $\mathbf{z}$ on the locus $z_I = z_J$. Using the expressions (34), we can write the axiodilaton from (30) as

$$\begin{aligned}
\tau(\hat{\mathbf{z}}) &= -\frac{1}{2\pi\mathrm{i}}\frac{\partial_{z_I}(\varpi_{2,I} - \varpi_{2,J})}{\partial_{z_I}(\varpi_{1,I} - \varpi_{1,J})}\bigg|_{z_I=z_J} + \frac{\kappa_{IJJ} - \kappa_{III}}{2} \\
&= -\frac{1}{2\pi\mathrm{i}}(\kappa_{IIk} - \kappa_{IJk})\log(z_k)\Big|_{z_I=z_J} \\
&\qquad - \frac{1}{2\pi\mathrm{i}}\frac{(\kappa_{IIk} - \kappa_{IJk})f_k^{(1)} + z_I\partial_{z_I}\left(f_I^{(2)} - f_J^{(2)}\right)}{\varpi_0 + z_I\partial_I\left(f_I^{(1)} - f_J^{(1)}\right)}\Bigg|_{z_I=z_J} + \frac{\kappa_{IJJ} - \kappa_{III}}{2}.
\end{aligned} \tag{35}$$

The rightmost term originates in the change of basis from Frobenius ϖ to integral symplectic Π. Note that shifting τ by an integer is a symmetry of the theory, and of the function $j(\tau)$. It was recognised in [33] that this axiodilaton, which defines the F-theory fibre, is also the complex structure modulus of the elliptic curve whose point-counts are reflected in the zeta numerator of the modular threefold $X_{\mathbf{z}_*}$.

Along with explicit checks of the zeta function as computed by the methods of [7, 37, 10], $j(\tau(\mathbf{z}_*))$ was found in a number of examples in [11] to be a rational function of $\mathbf{z}_*$, similarly to the example of [34, 33].

These algebraicity properties of flux vacua were more closely analysed in [26], where families of elliptic curves were identified such that ratios of their period functions gave the axiodilatons of the examples in [11].

We include the two $h^{2,1} = 2$ examples of [11, 26], together with more $\mathbb{Z}_2$ symmetric two-parameter examples that we are aware of, in Table 2. Five of these are mirrors of CICYs [25, 6], and a further two are CICY quotients [5] with $h^{1,1} = 2$, for which the period formulae of [28, 29] can be applied. A further example uses a manifold due to [35], to which we refer for the Picard-Fuchs equations that can be used to obtain the series f in (34) without applying the Frobenius method.

Our last example is mirror to the Gulliksen-Negård threefold, for which no mirror construction is known at present. The PF operators are available from GLSM techniques exploited in [32]. The coordinates therein do not manifestly have a $\mathbb{Z}_2$ symmetry, but the GW invariants found suggest there should be one. By taking their operators [32, eq.(5.21)] and making a coordinate transformation[3] $z_{\text{there}} = z_1^{-2}$, $w_{\text{there}} = -\frac{z_2}{z_1}$, we get a symmetric fundamental period with $\varpi_0^{GN}(z_1,z_2) = \varpi_0^{GN}(z_2,z_1)$. We find an SFV solution, with axiodilaton given in Table 2.

There may be more to understand about this choice of coordinates. We note that $\varpi_0^{GN}(\sqrt{z},\sqrt{z})$ is annihilated by a degree 8 operator whose instanton expansion we do not recognise, while $\varpi_0^{GN}(\mathrm{i}\sqrt{z},-\mathrm{i}\sqrt{z})$ is annihilated by AESZ18.

Table 2 $h^{1,1} = 2$ CY threefolds whose mirrors support supersymmetric flux vacua on the locus $z_1 = z_2 = z$. A number of rational functions $j(\tau(z))$ repeat across examples, sometimes exactly and sometimes only with a coordinate change $z \to z^2$ or $z \to -z^2$.

#	Mirror CY3	κ_{111}	κ_{112}	$j(\tau(z))$
1	$\begin{matrix}\mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\begin{bmatrix}3\\3\end{bmatrix}$	0	3	$-\frac{(1+24z)^3}{z^3(1+27z)}$
2	$\begin{matrix}\mathbb{P}^2\\ \mathbb{P}^2\end{matrix}\begin{bmatrix}3\\3\end{bmatrix}/\mathbb{Z}_3$	0	1	$-\frac{(1-216z)^3}{z(1+27z)^3}$
3	$\begin{matrix}\mathbb{P}^3\\ \mathbb{P}^3\end{matrix}\begin{bmatrix}1&1&2\\1&1&2\end{bmatrix}$	2	6	$\frac{(1+16z+16z^2)^3}{z^4(1+16z)}$
4	$\begin{matrix}\mathbb{P}^3\\ \mathbb{P}^3\end{matrix}\begin{bmatrix}1&2&1\\1&1&2\end{bmatrix}$	2	7	$-\frac{(1+12z+14z^2-12z^3+z^4)^3}{z^5(1+11z-z^2)}$
5	$\begin{matrix}\mathbb{P}^4\\ \mathbb{P}^4\end{matrix}\begin{bmatrix}1&1&1&1&1\\1&1&1&1&1\end{bmatrix}$	5	10	$-\frac{(1+12z+14z^2-12z^3+z^4)^3}{z^5(1+11z-z^2)}$
6	$\begin{matrix}\mathbb{P}^4\\ \mathbb{P}^4\end{matrix}\begin{bmatrix}1&1&1&1&1\\1&1&1&1&1\end{bmatrix}/\mathbb{Z}_5$	1	2	$-\frac{(1-228z+494z^2+228z^3+z^4)^3}{z(1+11z-z^2)^5}$
7	$\begin{matrix}\mathbb{P}^4\\ \mathbb{P}^4\end{matrix}\begin{bmatrix}2&0&1&1&1\\0&2&1&1&1\end{bmatrix}$	4	12	$\frac{(1-16z^2+16z^4)^3}{z^8(1-4z)(1+4z)}$
8	Pfaffian fibration Y_1 of [35, §4.3.2]	0	5	$-\frac{(1+12z+14z^2-12z^3+z^4)^3}{z^5(1+11z-z^2)}$
9	Guliksen-Negård Threefold, $z_1 = z_2 = z$	8	24	$\frac{(1-16z^4+16z^8)^3}{z^{16}(1-2z)(1+2z)(1+4z^2)}$

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[3] JM thanks Johanna Knapp for discussions on these lines.

We end this section by giving the axiodilaton profiles for two $h^{2,1} = 3$ families, whose fully symmetric locus $z_1 = z_2 = z_3$ has been analysed in [17]. We go to the SFV locus $z_1 = z_2 = z$, $z_3 = y$.

$$\text{Mirror of} \quad \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \!\! \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{h^{1,1}=3,\, h^{2,1}=48}, \qquad \kappa_{ijk} = \begin{cases} 0, & i = j = k, \\ 6, & i, j, k \text{ distinct}, \\ 3, & \text{otherwise}, \end{cases}$$

$$j\left(\tau(z,y)\right) = -\frac{(1-y+z)^3 \left(1-3y+3z+3y^2+18yz+3z^2-y^3+3y^2z-3yz^2+z^3\right)^3}{y^3z^3\left(1-3y+3y^2-y^3+3z+21yz+3y^2z+3z^2-3yz^2+z^3\right)}. \tag{36}$$

$$\text{Mirror of} \quad \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \!\! \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{h^{1,1}=3,\, h^{2,1}=18}_{/\mathbb{Z}_3}, \qquad \kappa_{ijk} = \begin{cases} 0, & i = j = k, \\ 2, & i, j, k \text{ distinct}, \\ 1, & \text{otherwise}, \end{cases}$$

$$j\left(\tau(y,z)\right) = -\frac{(1-y+z)^3 \left(1-3y+3z+3y^2-222yz+3z^2-y^3+3y^2z-3yz^2+z^3\right)^3}{yz\left(1-3y+3y^2-y^3+3z+21yz+3y^2z+3z^2-3yz^2+z^3\right)^3}. \tag{37}$$

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