Calabi-Yau threefolds across quadratic singularities

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Abstract We examine aspects of non-Kähler geometry in the context of Calabi-Yau degenerations and resolutions. We begin by reviewing some basics of Calabi-Yau geometry in Section 1, describe topological features of the conifold transition in Section 2, and survey recent developments on the geometrization of conifold transitions in Section 3.

1 Calabi-Yau Threefolds

1.1 Definitions

The topic of this survey is the Calabi-Yau threefold. A central theme is how natural questions about these spaces can lead to non-Kähler outcomes. Arriving at these non-Kähler structures sets a departure point for various new possible directions of research. We begin in Section 1 with a review of the key properties of Calabi-Yau threefolds.

Definition 1. We take the definition of a Calabi-Yau threefold X to be a compact complex manifold with $\dim_{\mathbb{C}} X = 3$ which is projective, satisfies $H^1(X,\mathbb{C}) = 0$, and admits a holomorphic volume form Ω.

Recall that a holomorphic volume form on X is given by a $(3,0)$-form Ω such that

$$\Omega \overset{\text{loc}}{=} f(z)\, dz^1 \wedge dz^2 \wedge dz^3$$

where $f(z)$ is a local nowhere vanishing holomorphic function.

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Remark 1. The definition of a Calabi-Yau threefold is not standardized in the literature, and some setups do not require the vanishing of $H^1(X,\mathbb{C})$ or generalize the existence of a holomorphic volume form Ω to $c_1(X)=0$. A consequence of the Calabi-Yau [13, 113] theorem is that a Kähler manifold X with $c_1(X)=0$ must have K_X holomorphically torsion, but not necessarily trivial. Projectivity in Definition 1 is redundant because $H^1(X,\mathbb{C})=0$. This then implies $h^{0,2}=0$, and projectivity then follows from the Kodaira embedding theorem.

Example 1. A basic example of a Calabi-Yau threefold is the following quintic threefold:

$$X=\left\{\sum_{i=0}^{4} Z_i^5=0\right\}\subset\mathbb{P}^4.$$

This defines a smooth complex manifold by the implicit function theorem, since a hypersurface $\{P=0\}\subseteq\mathbb{P}^n$ with P a homogeneous polynomial is smooth if there is no non-zero point where simultaneously $P=0$ and $DP=0$. The holomorphic volume form in this case is

$$\Omega\overset{\text{loc}}{=}\frac{dw_1\wedge dw_2\wedge dw_3}{w_4^4},\quad w_i=\frac{Z_i}{Z_0}$$

over the open set $U=\{Z_0\neq 0,\ w_4\neq 0\}$. This formula can be verified to glue on overlaps of similar local open sets to define a global section $\Omega\in H^0(X,\Omega^3)$. That X satisfies $H^1(X,\mathbb{C})=0$ follows from the Lefschetz hyperplane theorem.

Example 2. Any homogeneous degree 5 polynomial P will also define a Calabi-Yau threefold $\{P=0\}\subseteq\mathbb{P}^4$ provided the zero locus is smooth.

Example 3. For more examples of Calabi-Yau threefolds beyond quintics $\{P=0\}\subseteq\mathbb{P}^4$, we refer to for example Hübsch's book [60].

The Hodge diamond of a Calabi-Yau threefold has two parameters:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1\\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Mirror symmetry [31, 55, 17, 73, 69, 101] predicts that Calabi-Yau threefolds come in pairs $(X,\check{X})$ exchanging the two parameters $(h^{1,1},h^{2,1})$.

$$h^{1,1}(X)=h^{2,1}(\check{X}),\quad h^{1,1}(\check{X})=h^{2,1}(X).$$

A first indication of the role of non-Kähler manifolds comes from the inherent asymmetry of Calabi-Yau threefolds where $h^{2,1}$ may vanish while $h^{1,1}$ cannot. Indeed, a Kähler metric $\omega>0$ creates a non-zero Kähler class $[\omega]\in H^{1,1}(X)$. Kontsevich

[70] has suggested that the mirror theory of curve enumeration on a threefold X with $h^{2,1}(X) = 0$ should involve Hodge structures on a non-Kähler complex threefold $\check{X}$ with $h^{1,1}(\check{X}) = 0$.

We now review the significance of the two parameters $(h^{1,1}, h^{2,1})$.

1.2 Discussion of $h^{1,1}$

Calabi-Yau threefolds are studied in differential geometry because they support solutions to the Ricci-flat equation. Recall that on a Riemannian manifold (M,g), the Riemann curvature tensor is a second order invariant of g_{ij} given by

$$R_{pq}{}^{k}{}_{j} = \partial_p \Gamma_q{}^{k}{}_{j} + \Gamma_p{}^{k}{}_{r} \Gamma_q{}^{r}{}_{j} - (p \leftrightarrow q),$$

with

$$\Gamma_i^k{}_j = \frac{g^{kp}}{2}(-\partial_p g_{ij} + \partial_i g_{pj} + \partial_j g_{ip}).$$

The Ricci tensor is given by

$$R_{ij} = -R_{ik}{}^{k}{}_{j},$$

and a fundamental equation in geometry and physics is the Ricci-flat equation

$$R_{ij} = 0.$$

On a Calabi-Yau threefold X, one can look for solutions of the following form. Choose a background reference Kähler metric $g_{\mu\bar{\nu}}$ on X such as the pullback of the Fubini-Study metric in the embedding $\iota : X \to \mathbb{P}^N$, and set the ansatz

$$\tilde{g}_{\mu\bar{\nu}} = g_{\mu\bar{\nu}} + \partial_\mu \partial_{\bar{\nu}} u > 0 \tag{1}$$

for an unknown potential function $u \in C^\infty(X)$. Let Greek indices denote holomorphic coordinates $\{z^\alpha\}$ on the complex manifold X. Kähler [64] computed the Ricci tensor for ansatz (1) and derived the equation

$$R_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \log |\Omega|^2_{\tilde{g}}, \quad R_{\alpha\beta} = 0$$

where

$$|\Omega|^2_{\tilde{g}} \overset{\text{loc}}{=} \frac{f(z)\overline{f(z)}}{\det \tilde{g}_{\mu\bar{\nu}}}.$$

We can then look for solutions of the Ricci-flat equation by setting $|\Omega|_{\tilde{g}} = 1$, so that the equation to solve becomes

$$\det(g_{\alpha\bar{\beta}} + u_{\alpha\bar{\beta}}) = e^f \det g_{\alpha\bar{\beta}} \tag{2}$$

where $e^f = |\Omega|^2_g$ is given. This geometric complex Monge-Ampère equation is an adaptation of the fundamental PDE

$$\det D^2 u = \psi > 0,$$

on a domain in $\mathbb{R}^n$. The complex Monge-Ampère equation (2) was solved by Yau.

Theorem 1 (Yau's theorem [113]). *There exists a unique solution* $u \in C^\infty(X)$ *to* (2) *with* $\sup_X u = 0$.

Consequently, there exists Ricci-flat metrics on a Calabi-Yau threefold X. The ansatz (1) is interpreted as finding a unique Kähler Ricci-flat metric in a given Kähler class $[\omega] \in H^{1,1}(X,\mathbb{R})$, where a Kähler metric g defines a form $\omega \in \Omega^{1,1}(X,\mathbb{R})$ via $\omega = i g_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^\nu$ and so by the $\partial\bar{\partial}$-lemma the ansatz (1) is equivalent to setting $[\tilde{\omega}] = [\omega]$.

The construction comes in families parametrized by the choice of background Kähler metric $g_{\mu\bar{\nu}}$, but as noted above really the parameter is the Kähler class $[\omega] \in H^{1,1}(X,\mathbb{R})$. Thus $h^{1,1}$ counts the dimension of the space of Kähler Ricci-flat metrics on X with fixed complex structure.

1.3 Discussion of $h^{2,1}$

Having discussed the parameter $h^{1,1}$, we now give the interpretation of the parameter $h^{2,1}$.

Recall that a family of complex manifolds over a base $B \subseteq \mathbb{C}^k$ with $0 \in B$ is defined as a proper holomorphic submersion

$$\pi : \mathscr{X} \to B$$

where $\mathscr{X}$ is a complex manifold. The fibers $X_b = \pi^{-1}(b)$ for $b \in B$ are then all compact complex manifolds varying holomorphically in the parameter b. Ehresmann's theorem (see e.g. [68] for an exposition) states that after possibly replacing B with a neighborhood of 0, there exists a diffeomorphism

$$\Psi : X_0 \times B \overset{\cong}{\to} \mathscr{X}$$

such that $\pi \circ \Psi = \mathrm{pr}_2$, with pr_2 the projection to the second factor. Therefore all manifolds X_b are diffeomorphic. Each X_b comes with a complex structure tensor J_b, and via the diffeomorphism we obtain a family of complex structures also denoted J_b over the fixed differentiable manifold X_0.

Remark 2. Recall that given a complex manifold X with holomorphic coordinates $\{z^\alpha\}$, the corresponding complex structure tensor $J \in \Gamma(\mathrm{End}\, T_{\mathbb{C}}X)$ is defined by

$$J^\alpha{}_\beta = i\delta^\alpha{}_\beta \quad J^{\bar{\alpha}}{}_{\bar{\beta}} = -i\delta^\alpha{}_\beta, \quad J^\alpha{}_{\bar{\beta}} = J^{\bar{\alpha}}{}_\beta = 0,$$

in a holomorphic coordinate system. Here we use index notation for components of a tensor $A \in \Gamma(\mathrm{End}\, T_{\mathbb{C}}X)$ so that e.g.

$$A\left(\frac{\partial}{\partial z^\beta}\right) = A^\alpha{}_\beta \frac{\partial}{\partial z^\alpha} + A^{\bar\alpha}{}_\beta \frac{\partial}{\partial \bar z^{\bar\alpha}}.$$

In a smooth real coordinate system, a complex structure tensor $J \in \Gamma(\mathrm{End}\, TX)$ is characterized by the property

$$J^2 = -id, \quad N(J) = 0 \tag{3}$$

where $N \in \Omega^2(TM)$ is the Nijenhuis tensor given by

$$N^k{}_{ij} = \frac{1}{4}\left(J^k{}_p \partial_j J^p{}_i + \partial_p J^k{}_j J^p{}_i - (i \leftrightarrow j)\right).$$

The Newlander-Nirenberg theorem (see e.g. [29]) states that the existence of a $J \in \Gamma(\mathrm{End}\, TX)$ satisfying (3) on a smooth manifold X is equivalent to the existence of holomorphic coordinate charts. Such a J splits $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ into the $+i$ and $-i$ eigenvalues of J, and the interpretation of N is that $N = 0$ if and only if $[U,V] \in T^{1,0}X$ for all $U,V \in \Gamma(T^{1,0}X)$.

Given a family of complex structures J_t varying smoothly with a parameter $t \in (-\varepsilon, \varepsilon)$ on a fixed smooth manifold X, differentiation along the path defines the fluctuation tensor

$$\eta = \frac{d}{dt}\Big|_{t=0} J_t.$$

One can verify the following calculations :

1. Differentiating $J_t^2 = -id$ shows that
$$\eta \in \Omega^{0,1}(T^{1,0}X),$$
so that for example $\eta^{\bar\alpha}{}_{\bar\beta} = 0$ in holomorphic coordinates at $t = 0$.
2. Differentiating $N(J_t) = 0$ leads to the constraint
$$\bar\partial \eta = 0,$$
so that $[\eta]$ defines a class in $H^{0,1}(X, T^{1,0}X)$.
3. If the family J_t comes from $J_t = (\Theta_t)_*^{-1} J_0 (\Theta_t)_*$ where $\Theta_t : X \to X$ is a smoothly varying family of diffeomorphisms, then
$$\eta = \bar\partial V, \quad V \in \Gamma(T^{1,0}X)$$
so that $[\eta] = [0]$ in $H^{0,1}(X, T^{1,0}X)$.

In summary, a family of complex structures produces a cohomology class in $H^{0,1}(X, T^{1,0}X)$, and deformations just coming from families of diffeomorphisms are not counted.

Remark 3. On a Calabi-Yau threefold, $K_X \cong \mathcal{O}_X$ and this parameter space has dimension
$$h^{0,1}(T^{1,0}X) = h^{0,1}(T^{1,0}X \otimes K_X) = h^{2,1}.$$
For further analysis of the parameter space of complex structures on a Calabi-Yau threefold, see Candelas-de la Ossa [16].

The natural question that arises is the inverse problem: does a given class $[\eta] \in H^{0,1}(X, T^{1,0}X)$ come from a family $(X, J(t))$ of complex structures with $[\frac{d}{dt}|_{t=0} J] = [\eta]$? It does, and this is known as the BTT theorem (see e.g. [87] for a recent exposition).

Theorem 2 (Bogomolov-Tian-Todorov Theorem [102, 104]). *Let X be a compact Kähler manifold admitting a holomorphic volume form. There is a family of complex manifolds*
$$\pi : \mathscr{X} \to B, \quad \pi^{-1}(0) = X$$
where B is a neighborhood of the origin in $H^{0,1}(X, T^{1,0}X)$. Hence given any $[\eta] \in H^{0,1}(X, T^{1,0}X)$, there exists a family J_t of complex structures varying smoothly with $t \in (-\varepsilon, \varepsilon)$ such that $[\frac{d}{dt}\big|_{t=0} J] = [\eta]$.

We note that there is an analog of this theorem in the non-Kähler setting [87, 112] provided X satisfies the $\partial\bar{\partial}$-lemma in addition to admitting a holomorphic volume form. Later in our discussion, we will see that the $\partial\bar{\partial}$-lemma is preserved throughout the connected web of threefolds [74, 39, 71] even across Kähler to non-Kähler deformations.

1.4 Rational curves

A key topic in the study of Calabi-Yau threefolds is its set of rational curves. These special submanifolds can be studied from various different points of view. For example, the extraordinary work of Candelas-de la Ossa-Green-Parkes [17] shocked the algebraic geometry community by predicting a formula for the number of rational curves of degree d by methods of string theory.

Let X be a compact complex threefold with holomorphic volume form. Let $C \subseteq X$ be a smooth complex submanifold of complex dimension 1 with $C \cong \mathbb{P}^1$. The exact sequence defining the normal bundle is
$$0 \to TC \to TX|_C \to \mathscr{N}_C \to 0.$$
The first Chern class of this sequence satisfies
$$c_1(TX|_C) = c_1(TC) + c_1(\mathscr{N}_C)$$
and so $c_1(\mathscr{N}_C) = -2$. Grothendieck's classification of holomorphic bundles over $\mathbb{P}^1$ implies that the normal bundle of C must be isomorphic to

$$\mathcal{O}(a) \oplus \mathcal{O}(-a-2) \to \mathbb{P}^1.$$

The case of $\mathcal{O}(-1)^{\oplus 2}$ is distinguished by its symmetry and that it is the only case in which the curve is rigid, as the other bundles admit infinitesimal deformations in $H^0(\mathcal{N}_C)$.

Such $(-1,-1)$-curves are expected to exist in abundance on Calabi-Yau threefolds [37, 70]. Work of Clemens [24] and Katz [66] guarantees the existence of $(-1,-1)$-curves on the generic quintic threefold.

Example 4. Here is an explicit example of a $(-1,-1)$-curve following [66]. Consider the quintic threefold given by

$$X = \{f_2 Z_2 + f_3 Z_3 + f_4 Z_4 = 0\} \subseteq \mathbb{P}^4,$$

and the embedded holomorphic curve

$$C = \{Z_2 = Z_3 = Z_4 = 0\} \subseteq X,$$

where we take the specific f_i to be

$$X = \left\{ (Z_0^4 + Z_2^4) Z_2 + (Z_0^2 Z_1^2 + Z_3^4) Z_3 + (Z_1^4 + Z_4^4) Z_4 = 0 \right\} \subseteq \mathbb{P}^4,$$

so that X is smooth. We will compute the normal bundle of $C \subset X$. Recall the definition of the normal bundle: let $p \in C$, and suppose $p \in U \subset X$ is a coordinate chart such that

$$C \cap U = \{y = 0\}$$

where (x_1, y_1, y_2) are holomorphic coordinates on U. Suppose also $p \in \tilde{U}$ with another such submanifold coordinate system $(\tilde{x}_1, \tilde{y}_1, \tilde{y}_2)$. Then on overlaps $U \cap \tilde{U} \cap C$, the transition function

$$\left. \frac{\partial \tilde{y}}{\partial y} \right|_C$$

defines the data of the normal bundle $\mathcal{N}_C$.

- Let U be an open set in $\{Z_0 \neq 0\} \subseteq X$ near C. Here we use local coordinates on X given by (w_1, w_3, w_4) where $w_i = Z_i / Z_0$. The equation of X is

$$(1 + w_2^4) w_2 + (w_1^2 + w_3^4) w_3 + (w_1^4 + w_4^4) w_4 = 0, \tag{4}$$

  and the curve appears as

$$C \cap U = \begin{cases} w_3 = 0 \\ w_4 = 0. \end{cases}$$

  By the inverse function theorem, $w_2(w_1, w_3, w_4)$ is a holomorphic function of the remaining coordinates.

- Let $\tilde{U}$ be an open set in $\{Z_1 \neq 0\} \subseteq X$ near C. Here we use local coordinates on X given by $(\tilde{w}_0, \tilde{w}_2, \tilde{w}_3)$ where $\tilde{w}_i = Z_i/Z_1$. The curve appears as
$$C \cap \tilde{U} = \begin{cases} \tilde{w}_2 = 0 \\ \tilde{w}_3 = 0. \end{cases}$$
- We have covered the curve by open charts: $C \subseteq U \cup \tilde{U}$. We conform to earlier conventions by setting $(x, y_1, y_2) = (w_1, w_3, w_4)$ and $(\tilde{x}, \tilde{y}_1, \tilde{y}_2) = (\tilde{w}_0, \tilde{w}_2, \tilde{w}_3)$. Then we recognize x and $\tilde{x}$ as the coordinates along C and since $\tilde{x} = x^{-1}$ we see $C = \mathbb{P}^1$. The transition function of the normal bundle may be computed by implicit differentiation of (4) which gives
$$\left.\frac{\partial \tilde{y}}{\partial y}\right|_C = \begin{bmatrix} -x & -x^3 \\ x^{-1} & 0 \end{bmatrix}.$$
This transition function is a disguise of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Recall that two sets of transition functions $\{g_{\alpha\beta}, U_\alpha \cap U_\beta\}$ and $\{\hat{g}_{\alpha\beta}, U_\alpha \cap U_\beta\}$ define isomorphic bundles if there exists $\lambda_\alpha : U_\alpha \to GL(k, \mathbb{R})$ with
$$\hat{g}_{\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}.$$
In this example, one can find 2×2 matrices λ_1, λ_2 such that
$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \lambda_1 \begin{bmatrix} -x & -x^3 \\ x^{-1} & 0 \end{bmatrix} \lambda_2^{-1}$$
with the matrix on the left the familiar transition function defining $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$.

We will later use a local model for the holomorphic curve $C \subset X$, but we remark that the holomorphic tubular neighborhood is generally false: if $S \subset X$ is a compact holomorphic submanifold, there is not necessarily a neighborhood $U \subset X$ of S which is biholomorphic to a neighborhood of the zero section in the total space of the normal bundle $\mathcal{N}_S$. When this is true, S is sometimes said to satisfy the formal neighborhood principle. There is much literature on this subject starting with foundational work of Grauert [51] in the case of codimension one submanifolds. In general codimension, there is work of [1] where the condition for a formal neighborhood involves vanishing of $H^1(S, T_S \otimes \mathrm{Sym}^k(\mathcal{N}_S^*))$ and $H^1(S, \mathcal{N}_S \otimes \mathrm{Sym}^{k+1}(\mathcal{N}_S^*))$.

Returning to $(-1,-1)$ curves C on a Calabi-Yau threefold X, it is well-known (e.g. [70]) that there exists a neighborhood $U \subset X$ which is biholomorphic to a neighborhood of the zero section in $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1$.

2 Crossing singularities

This section describes a mechanism known as a conifold transition, and how its study naturally leads to non-Kähler geometry.

2.1 Rolling in the landscape

Consider the set of all possible Calabi-Yau threefolds. It was noticed by string theorists Green-Hübsch and Candelas-Green-Hübsch [20, 53, 54] that there is a way to travel in this landscape by a process known as a conifold transition. This led to the idea of viewing Calabi-Yau threefolds not as isolated objects, but as part of a unified moduli space. We will define a conifold transition momentarily, but let us first state Reid's conjecture.

Conjecture 1. [90] All Calabi-Yau threefolds can be connected by a sequence of conifold transitions.

Early work on the algebro-geometric foundations of conifold transitions goes back to Clemens [25] and Friedman [36]. A conifold transition, which we will denote by

$$\hat{X} \to X_0 \rightsquigarrow X_t,$$

is a two step process. First, N holomorphic $(-1,-1)$-curves on $\hat{X}$ are contracted to points producing a complex analytic space X_0 with N ordinary double point singularities. Second, the complex structure of X_0 is deformed in a family such that X_t are smooth complex manifolds for $t \neq 0$.

Remark 4. One can also study an extension of Reid's conjecture where $\hat{X} \to X_0 \rightsquigarrow X_t$ more generally denotes a birational contraction followed by smoothing. In this generality, X_0 admits more complicated singularities than ordinary double points. We focus our attention on conifold transition in this survey, and refer to Gross [57] for an alternate version of Reid's conjecture on connecting Calabi-Yau threefolds by general geometric transitions.

2.1.1 Topological description

Broadly speaking, a conifold transition is a type of topological surgery of 6-manifolds. We start with the topological implications and return later to the complex analytic definition. At the level of topology, N sets of the form $(D^4 \times S^2)_i$ are removed from one manifold and replaced by $(S^3 \times D^3)_i$ in the other by gluing along the common boundary $S^3 \times S^2$. Call the first manifold $\hat{X}$ and the second X_t. The first Betti number b_1 does not change.

Let $[C_i] \in H_2(D^4 \times S^2)_i$ and $[L_i] \in H_3(S^3 \times D^3)_i$ be generators of homology. By the excision principle for Euler characteristic, [25, 93, 72] the topological change is captured by the formula

$$N = \rho + \mu \tag{5}$$

where:

$$\begin{aligned} N &= \text{number of } (D^4 \times S^2)_i \text{ removed from } \hat{X}, \\ \rho &= \text{number of independent 2-cycles } [C_i] \in H_2(\hat{X}, \mathbb{C}), \\ \mu &= \text{number of independent 3-cycles } [L_i] \in H_3(X_t, \mathbb{C}). \end{aligned}$$

We also have

$$\begin{aligned} b_2 &\mapsto b_2 - \rho \\ b_3 &\mapsto b_3 + 2\mu. \end{aligned}$$

We see that as $\hat{X}$ deforms to X_t across a conifold transition $\hat{X} \to X_0 \rightsquigarrow X_t$, its Betti numbers jump as $b_2 \downarrow$ and $b_3 \uparrow$.

2.1.2 The definition of a conifold transition

We now give the definition of a conifold transition following Friedman [37]. Let $\hat{X}$ be a Calabi-Yau threefold. The deformation process of a conifold transition, denoted $\hat{X} \to X_0 \rightsquigarrow X_t$, is defined as follows:

1. Find disjoint $(-1,-1)$-curves $C_i \subseteq \hat{X}$.
2. Contract the curves C_i to points p_i to form a complex analytic space X_0 with ordinary double point singularities at p_i.
3. Realize X_0 as the central fiber of a smoothing $\pi : \mathscr{X} \to \Delta$.

By a smoothing of X_0, we mean a proper flat map

$$\pi : \mathscr{X} \to \Delta, \quad \pi^{-1}(0) = X_0$$

with $\mathscr{X}$ a smooth complex fourfold, $\Delta \subseteq \mathbb{C}$ the unit disk, and $\pi^{-1}(t) = X_t$ smooth complex manifolds for $t \neq 0$. Furthermore, we require that a neighborhood $\mathscr{U} \subseteq \mathscr{X}$ of each point p_i is locally analytically isomorphic to the local model

$$\pi : \mathscr{V} \to \mathbb{C}, \quad \pi(z,t) = t,$$

where

$$\mathscr{V} = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = t\} \subseteq \mathbb{C}^4 \times \mathbb{C}.$$

Recall that the notion of a flat map $\varphi : X \to Y$ generalizes the notion of a holomorphic submersion by allowing singular fibers. A holomorphic map $\varphi : X \to Y$ between connected complex manifolds is flat if and only if it is an open map. If φ is flat in p and the fiber $X_{\varphi(p)}$ is a manifold at p, then φ is a submersion at p. We refer to Fischer's book [35] for these statements and more.

Remark 5. Realizing step (1) is already part of Reid's conjecture. It is an open problem whether every Calabi-Yau threefold X admits $(-1,-1)$-curves, and Reid-Friedman [90, 37] speculate on whether there is a collection of such curves generating $H_2(X,\mathbb{Z})$.

We will explain Step (2) in §2.2.1, but we note for now that it follows automatically from Step (1). Step (3) requires a global condition on the configuration of the curves C_i. This global condition is Friedman's condition:

$$\sum_{i=1}^{N} \lambda_i [C_i] = 0 \quad \text{in} \quad H^4(\hat{X},\mathbb{C}) \tag{6}$$

with each $\lambda_i \neq 0$. That a linear relation such as this must hold can be deduced from the topological change formula (5) which implies $N > \rho$ since $\mu \geq 1$, and with additional work one can show that all λ_i are non-zero [37, 70, 91]. For extensions of Friedman's condition to nodal Calabi-Yau n-folds in higher dimensions n, see Rollenske-Thomas [91].

The Friedman-Kawamata-Ran-Tian theorem asserts that if Friedman's condition (6) is satisfied, then step (3) can be realized. Namely, X_0 admits a smoothing. The full statement of the theorem takes the form of a singular version of the BTT theorem for ordinary double point singularities. We state the version here taken from Tian ([103] p. 476).

Theorem 3 (Friedman-Kawamata-Ran-Tian Theorem [36, 67, 89, 103]). *Let $\hat{X}$ be a Calabi-Yau threefold. Let C_i be a collection of N disjoint $(-1,-1)$ curves satisfying Friedman's condition* (6). *Let $\mu : \hat{X} \to X_0$ be the holomorphic contraction of the C_i with $\mu(C_i) = p_i$ resulting in a complex analytic space X_0 with ordinary double point singularities p_i.*

Then there exists a proper flat map $\pi : \mathscr{X} \to B$ where $B \subseteq \mathbf{T}$ is a open neighborhood of the origin in $\mathbf{T} = H^1(X_{0,reg}, T^{1,0})$. Here $\mathscr{X}$ is a smooth complex manifold and $\pi^{-1}(0) = X_0$.

The vector space of infinitesimal deformations $\mathbf{T}$ *can be understood by the exact sequence*

$$0 \to H^1(\hat{X}, T^{1,0}) \to \mathbf{T} \to \bigoplus_{i=1}^{N} \mathbb{C}[p_i] \to H^4(\hat{X},\mathbb{C})$$

where the last map is $[p_i] \mapsto [C_i]$. Deformations in direction $H^1(\hat{X}, T^{1,0})$ preserve all singular points. Deformations mapped to a vector

$$\sum_i \lambda_i [p_i] \in \bigoplus_{i=1}^{N} \mathbb{C}[p_i] = \mathbb{C}^N$$

with all $\lambda_i \neq 0$ deform X_0 to a family of smooth complex manifolds.

In summary, one needs to check Friedman's condition (6) on a configuration of curves on an initial threefold, and then general theory produces a conifold transition to a new threefold.

Corollary 1. *[37, 103] Let $\hat{X}$ be a Calabi-Yau threefold, and let $\{C_i\} \subseteq \hat{X}$ be a collection of disjoint $(-1,-1)$-curves satisfying Friedman's condition* (6)*. Then there exists a conifold transition $\hat{X} \to X_0 \rightsquigarrow X_t$.*

We note that realizing the local model $\mathscr{V} \to \Delta$ on the global flat family $\mathscr{X} \to \Delta$ is work of [65].

2.2 Local model

Consider the singular point $0 \in V_0$ where

$$V_0 = \left\{ z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \right\} \subseteq \mathbb{C}^4.$$

This singularity is sometimes called a conifold singularity, nodal singularity, quadratic singularity, or ordinary double point. It is distinguished by the holomorphic Morse lemma [7]: a holomorphic function

$$f : \mathbb{C}^{n+1} \to \mathbb{C}, \quad \text{with } f(0) = 0,\ Df(0) = 0$$

and non-degenerate holomorphic Hessian matrix $f_{\alpha\beta}(0)$ admits local holomorphic coordinates w near the origin such that

$$f(w) \overset{\text{loc}}{=} \sum_{i=1}^{n+1} w_i^2.$$

The singularity $0 \in V_0$ can be desingularized in two distinct ways:

1. **By small resolution:** This is a resolution of singularities

   $$\mu : \hat{V} \to V_0$$

   such that $\hat{V}$ is a smooth complex manifold with $\mu : \hat{V} \backslash E \to V_0 \backslash \{0\}$ a biholomorphism away from the exceptional set

   $$\mu^{-1}(0) = \mathbb{P}^1.$$

   The space $\hat{V}$ is biholomorphic to the total space of $\mathscr{O}(-1)^{\oplus 2} \to \mathbb{P}^1$. We will give further details on this small resolution in §2.2.1 below.
2. **By smoothing:** The complex analytic space V_0 can be deformed into a smooth complex manifold by adding a parameter t:

   $$V_t = \left\{ z_1^2 + z_2^2 + z_3^2 + z_4^2 = t \right\} \subseteq \mathbb{C}^4.$$

The spaces V_t for $t \neq 0$ are smooth complex manifolds diffeomorphic to T^*S^3. We can insert V_0 as the central fiber of a family

$$\mathscr{V} \xrightarrow{\pi} \mathbb{C}, \quad \pi(z,t) = t$$

where

$$\mathscr{V} = \left\{ (z,t) \in \mathbb{C}^4 \times \mathbb{C} : z \in V_t \right\}.$$

We note that $\|z\|^2 \geq |t|$ on V_t, and the set $L_t = \{\|z\|^2 = |t|\} \subseteq V_t$ is called the vanishing cycle. It is diffeomorphic to a 3-sphere, so that $L_t \cong S^3$ and these collapse to a point as $t \to 0$.

The local model of the conifold transition is then

$$\hat{V} \xrightarrow{\mu} V_0 \rightsquigarrow V_t. \tag{7}$$

It should be checked that this description is compatible with the local topological surgery of replacing a neighborhood of the form $D^4 \times S^2$ with one of the form $S^3 \times D^3$. This is done in [25] Lemma 1.11, where in particular the following diffeomorphisms are shown:

$$\hat{V} \cong \mathbb{R}^4 \times S^2, \quad V_0 \backslash \{0\} \cong (0,\infty) \times (S^3 \times S^2), \quad V_t \cong S^3 \times \mathbb{R}^3.$$

See also [93] or [72] for alternate expositions of these diffeomorphisms. There is a classic diagram depicting the local model of the conifold transition which one can find in Figure 1 of Candelas-Green-Hübsch [20] (see also Figure D.1. of [60]).

2.2.1 More on small resolutions

We now provide more details on the small resolution $\hat{V} \xrightarrow{\mu} V_0$. We define

$$\hat{V} = \mathscr{O}(-1)^{\oplus 2} \to \mathbb{P}^1.$$

The space $\hat{V} = U \cup \tilde{U}$ is covered by two coordinate charts with coordinates (u,v,λ) satisfying the coordinate transformation

$$\tilde{\lambda} = \lambda^{-1}, \quad \tilde{u} = \lambda u, \quad \tilde{v} = \lambda v. \tag{8}$$

The coordinate λ is in the $\mathbb{P}^1$-direction while the u,v are along the fibers of the vector bundle. Let $E \cong \mathbb{P}^1$ be the zero section $\{u = v = 0\}$. There is a biholomorphism

$$\mu : \hat{V} \backslash E \to V_0 \backslash \{0\},$$

where

$$V_0 = \left\{ z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \right\} \subseteq \mathbb{C}^4.$$

This can be constructed as follows. First, by unitary change of coordinates we identify V_0 with

$$\hat{V}_0 = \{w_1 w_2 - w_3 w_4 = 0\} \subseteq \mathbb{C}^4.$$

Next, we desingularize this space by blowing-up $\mathbb{C}^4$ along $\{w_2 = w_4 = 0\}$ and taking the proper transform of $\hat{V}_0$. Recall that this means

$$\mathrm{Bl}\,\mathbb{C}^4_{\{w_2=w_4=0\}} = \left\{ ((w_1,w_2,w_3,w_4),[u,v]) \in \mathbb{C}^4 \times \mathbb{P}^1 : (w_2,w_4) \in [u,v] \right\}.$$

The proper transform of $\hat{V}_0$ is biholomorphic to $\hat{V}$. Indeed, over the chart $U = \{u = 1\}$ there are coordinates (v, w_2, w_3), while over the chart $V = \{v = 1\}$ there are coordinates (u, w_1, w_4), and the change of coordinate relation is

$$u = v^{-1}, \quad w_1 = v w_3, \quad w_4 = v w_2,$$

which can be identified with (8). The blow-up map of $\hat{V}_0$ induced by $\mu(w,[u]) = w$, is denoted

$$\mu : \hat{V} \to \hat{V}_0,$$

with exceptional set $\mu^{-1}(0) = E \cong \mathbb{P}^1$ so that $\mu : \hat{V} \backslash E \to \hat{V}_0 \backslash \{0\}$ is a biholomorphism.

Remark 6. The resolution of singularities

$$\mu : \hat{V} \to V_0, \quad \mu^{-1}(0) = \mathbb{P}^1$$

is called a small resolution. It is not unique [7] as we could have alternatively blown-up $\hat{V}_0 \subseteq \mathbb{C}^4$ along $\{w_2 = w_3 = 0\}$.

Remark 7. If X_0 is a complex analytic space with ordinary double point singularities, then a neighborhood of each singularity may be identified with a neighborhood of $0 \in V_0$ and this local procedure defines a small resolution $\mu : \hat{X} \to X_0$. This construction also defines the contraction of a $(-1,-1)$ curve C on a Calabi-Yau threefold $\hat{X}$ to a singular point of the local form $0 \in V_0$. Indeed, let $C \subseteq \hat{X}$ be a $(-1,-1)$-curve. Then there exists a neighborhood $U \subseteq \hat{X}$ of C biholomorphic to a neighborhood of the zero section in $\hat{V}$, and the local construction defines a contraction $\mu : \hat{X} \to X_0$.

2.3 Examples

Complete intersection Calabi-Yau threefolds can all be connected by conifold transitions; see [53, 54] for work in the string theory literature and [111] in the mathematics literature. From another perspective, rather than connecting known examples, conifold transitions can also be used to construct new examples of projective Calabi-Yau threefolds; see e.g. [8, 14]. We list here some simple explicit examples of conifold transitions.

Example 5. This example can be found in Candelas-Green-Hübsch [20]. Consider the singular quintic

$$X_0 = \left\{ Z_3 G(Z_0, \dots, Z_4) + Z_4 H(Z_0, \dots, Z_4) = 0 \right\} \subseteq \mathbb{P}^4$$

with $G = Z_3^4 + Z_2^4 - Z_0^4$ and $H = -Z_4^4 - Z_1^4 - Z_0^4$. There are 16 singular points. Near each singularity there exists holomorphic local coordinates such that the local model of the singularity is

$$\{uv - xy = 0\} \subseteq \mathbb{C}^4,$$

and this is biholomorphic to $V_0 = \{\sum_i z_i^2 = 0\} \subseteq \mathbb{C}^4$. We can desingularize X_0 in two different ways:

- By small resolution $\hat{X} \to X_0$. Define $\hat{X} \subseteq \mathbb{P}^4 \times \mathbb{P}^1$ by

  $$UZ_4 - VZ_3 = 0, \quad UG(Z) + VH(Z) = 0,$$

  with $[U,V] \in \mathbb{P}^1$ and $[Z_0, \dots, Z_4] \in \mathbb{P}^4$.
- By smoothing

  $$X_t = \{Z_3 G + Z_4 H = t Z_0 Z_1 Z_2 Z_3 Z_4\} \subseteq \mathbb{P}^4.$$

  The X_t are smooth quintics for $t \neq 0$.

This provides an explicit example of a conifold transition

$$\hat{X} \to X_0 \rightsquigarrow X_t.$$

One can verify that the topological change formula (5) becomes $16 = 1 + 15$ by direct calculation of the Betti numbers of $\hat{X}$ and X_t, which results in $b_2 \mapsto b_2 - 1$ and $b_3 \mapsto b_3 + (2)(15)$. See [93] for a detailed analysis of this example.

Example 6. The following example was found by Friedman [36]. Let $\hat{X} \subseteq \mathbb{P}^4$ be a smooth quintic threefold. Then $b_2(\hat{X}) = 1$, so a pair of disjoint $(-1,-1)$-curves C_1, C_2 satisfy Friedman's condition (6). Choose C_1, C_2 to generate $H_2(\hat{X}, \mathbb{Z})$; see [24, 37] for such curves. General theory (Theorem 3) gives the existence of a conifold transition

$$\hat{X} \to X_0 \rightsquigarrow X_t.$$

We notice that something has gone wrong. The topological change formula (5) implies $b_2(X_t) = 0$ and so the complex analytic manifolds X_t cannot admit a Kähler structure. In fact, X_t is a simply connected 6-manifold with $H_2(X_t, \mathbb{Z}) = 0$ and $H_3(X_t, \mathbb{Z})$ torsion-free. By a theorem of Wall [110], the X_t are diffeomorphic to connected sums of $S^3 \times S^3$.

Example 7. We note that Lu-Tian [77, 78] has constructed examples of conifold transitions from Kähler Calabi-Yau threefolds to non-Kähler complex manifolds diffeomorphic to $\#_k(S^3 \times S^3)$ for $k \geq 2$. We refer to their work for these explicit examples.

Remark 8. The above examples show that a conifold transition may deform a Kähler Calabi-Yau threefold into a non-Kähler complex manifold. We will see in the sequel that these complex analytic threefolds inherit some properties of Kähler Calabi-Yau manifolds. It is argued by Reid [90], Friedman [37], Kontsevich [70], Yau [114, 41], Tian [103], etc that these analytic threefolds ought to be included into the mathematical theory developed for the study of projective Calabi-Yau threefolds. Friedman [36] and Reid [90] propose a handle-body decomposition for Calabi-Yau threefolds by contraction and smoothing of rational curves spanning $H_2(X,\mathbb{Z})$ so that the resulting threefold is non-Kähler and diffeomorphic to a connected sum $\#_k(S^3\times S^3)$. This program would lead to a sort of uniformization theorem for complex threefolds.

2.3.1 Reversing the arrow

The directionality of a conifold transition is not standardized in the literature. In all cases, the main feature is the desingularization of ordinary double point singularities by two mechanisms: either small resolution or smoothing of complex structure. Following the conventions established here, we refer to a reverse conifold transition as a deformation where the first step is degeneration of complex structure followed by small resolution:

$$X_t \rightsquigarrow X_0 \to \hat{X}.$$

This direction of travel may also potentially connect a projective Calabi-Yau threefold to a non-Kähler threefold. Indeed, it is well-known that Moishezon manifolds [83] are not necessarily Kähler.

Example 8. Here is a classic example of a quintic with a single ordinary double point. We take the explicit coefficients from §1.8 in [20], and this particular example can also be found in Hübsch's book ([60] Ch. D, section D.3.3). Consider the family

$$X_t = \left\{ Z_5^3\left(\sum_{i=1}^4 Z_i^2\right) + \sum_{i=1}^4 a_i Z_i^5 + tZ_5^5 \right\} \subseteq \mathbb{P}^4$$

where a_i are non-zero generic constants, and $t\in\mathbb{C}$ is a parameter. As

$$X_t \overset{t\to 0}{\rightsquigarrow} X_0$$

the family of smooth quintics degenerates to a singular space X_0 with a single ordinary double point at $P=[0,0,0,0,1]$. Let

$$\mu:\hat{X}\to X_0$$

be a small resolution with $\mu^{-1}(P)=C\cong\mathbb{P}^1$. The formula $N=\rho+\mu$ becomes $1=0+1$, and we see that

$$[C]=0\in H_2(\hat{X},\mathbb{R}).$$

The existence of a Kähler metric ω on $\hat{X}$ would lead to a contradiction:

$$0 < \int_C \omega = \int_{\partial\Omega} \omega = 0.$$

We see that the process of degeneration and resolution intertwines Kähler and non-Kähler manifolds.

2.4 Summary

A conifold transition is a mechanism to connect two distinct Calabi-Yau threefolds. We have noted that a conifold transition starting from a Kähler manifold may lead to a non-Kähler manifold. We may explore the ways to transfer mathematical structures from one manifold to the other. The following section, Section §3, will build on this theme from the point of view of differential geometry. One can hope that these extra structures can be used to constrain the space of non-Kähler manifolds connected to projective Calabi-Yau and potentially bound these threefolds in a suitable sense. For studies on how algebro-geometric or symplectic structures deform through a conifold transition, we refer to e.g. Li-Ruan [75], Lee-Lin-Wang [72], Lin-Wang [76], and references therein.

3 Geometric structures across singularities

3.1 Geometry of local conifold transitions

A theme in differential geometry is to understand metric constraints preserved through surgery. We will study conifold transitions in this context. Our starting point is the geometrization of the local model of a conifold transition, and afterwards we will move on to global compact geometries. On the non-compact local model, Ricci-flat metrics were constructed by Candelas-de la Ossa [15] and we now review their construction. For more details in the presentation style similar to the one taken here, we refer to e.g. [23, 27, 41].

3.1.1 Small resolution

On the total space

$$\hat{V} = \mathscr{O}(-1)^{\oplus 2} \to \mathbb{P}^1,$$

the Candelas-de la Ossa ansatz [15] with parameter $a > 0$ is

$$\omega_{co,a} = i\partial\bar{\partial} f_a(r^3) + 4a^2 \omega_{\mathrm{FS}},$$

where r is a power of the distance function to the zero section E given by

$$r^3(u,v) = |u|^2_{\omega_{\rm FS}} + |v|^2_{\omega_{\rm FS}},$$

with u,v fiber coordinates as before. The power of r will be motivated later in (9) below. This ansatz is substituted into the Ricci-flat equation where the equation becomes the following ODE

$$x(f_a'(x))^3 + 6a^2(f_a'(x))^2 = 1, \quad f_a(x) = f_a(r^3).$$

The parameter a measures the volume of $\{r=0\} = E \cong \mathbb{P}^1$, so that

$$\mathrm{Vol}(\mathbb{P}^1, \omega_{co,a}) \to 0$$

as $a \to 0$. A holomorphic volume form $\hat{\Omega}$ on $\hat{V}$ is given by extending $\mu^*\Omega_0$ across E by Hartog's theorem, where $\mu : \hat{V} \to V_0$ is a small resolution and Ω_0 is given below in (11).

3.1.2 The singular space

Setting $a = 0$, we find the following explicit solution on $\hat{V} \backslash E$:

$$\omega_{co,0} = \frac{3}{2} i\partial\bar{\partial} r^2, \quad f_0(x) = \frac{3}{2} x^{2/3}.$$

We noted earlier that $\hat{V}\backslash E$ is biholomorphic to the complement of the origin in

$$V_0 = \left\{ \sum_{i=1}^4 z_i^2 = 0 \right\} \subseteq \mathbb{C}^4,$$

and the correspondence identifies the radius as $r^3 = \|z\|^2$ on $V_0 \subseteq \mathbb{C}^4$. After rescaling the metric to neglect the factor of 3, the limit of $\omega_{co,a}$ as $a \to 0$ can be identified with the Ricci-flat geometry

$$(V_0, \omega_{co,0}), \quad \omega_{co,0} = \frac{i}{2}\partial\bar{\partial} r^2.$$

We note that $V_0 \subseteq \mathbb{C}^4$ comes with a scaling action $z \mapsto \lambda z$ and is diffeomorphic to a cone

$$V_0\backslash\{0\} \cong (0,\infty) \times \Sigma, \quad \Sigma = V_0 \cap \{r=1\}$$

where the link is $\Sigma = S^2 \times S^3$ [15]. Indeed, if we write $z = x + iy$ then the equation for V_0 becomes

$$\|x\|^2 = \frac{\|z\|^2}{2}, \quad \|y\|^2 = \frac{\|z\|^2}{2}, \quad \langle x,y \rangle = 0.$$

We interpret the link $\Sigma = \{\|z\| = 1\}$ as a sphere bundle $S(TS^3)$ associated to $TS^3 \to S^3$, and as TS^3 is a trivial bundle we have $\Sigma = S^2 \times S^3$.

The metric $\omega_{co,0} = \frac{i}{2}\partial\bar{\partial}r^2$ is a conical metric (see e.g. [98]) as it can be written in polar coordinates

$$g_{co,0} = dr \otimes dr + r^2 g_\Sigma, \tag{9}$$

where g_Σ is a metric on the link Σ. Thus the Candelas-de la Ossa metric $\omega_{co,0}$ is an example of a Calabi-Yau cone metric.

Remark 9. The power of $r = \|z\|^{2/3}$ for the Calabi-Yau cone metric can be anticipated by scaling: the Kähler-Ricci flat equation is

$$\omega_{co,0}^3 = i\Omega_0 \wedge \bar{\Omega}_0 \tag{10}$$

where

$$\Omega_0 \overset{\text{loc}}{=} \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_4}. \tag{11}$$

The scaling $z \mapsto \lambda z$ leaves (10) invariant.

3.1.3 Smoothing

Next, we equip the smoothings

$$V_t = \left\{ \sum_{i=1} z_i^2 = t \right\} \subseteq \mathbb{C}^4,$$

with Kähler Ricci-flat metrics. The holomorphic volume form Ω_t has the same expression as (11). The Candelas-de la Ossa ansatz [15] for the Calabi-Yau metric is

$$\omega_{co,t} = i\partial\bar{\partial} f_t(r^3), \quad r^3 = \|z\|^2,$$

and the Ricci-flat equation on V_t leads to the following ODE for the potential $f_t(x) = f_t(r^3)$:

$$(f_t'(x))^3 x + (f_t'(x))^2 f_t''(x)(x^2 - |t|^2) = 1/6.$$

Setting $t = 0$ recovers the solution $f_0(x) = c_0 x^{2/3}$. The submanifold

$$L_t = \{\|z\|^2 = |t|\} \cong S^3$$

is the vanishing cycle $L_t \subseteq V_t$, and it is special Lagrangian with respect to $(V_t, \omega_{co,t}, \Omega_t)$. Assuming $t > 0$ for simplicity, the special Lagrangian equations take the form

$$\omega|_L = 0, \quad \mathrm{Im}\,\Omega|_L = 0.$$

The parameter t measures the volume of the special Lagrangian 3-spheres, so that

$$\mathrm{Vol}(L_t, \omega_{co,t}) \to 0$$

as $t \to 0$.

3.1.4 Summary

The local model of a conifold transition is thus geometrized by Kähler Ricci-flat metrics. We have

$$(\hat{V}, g_{co,a}) \to (V_0, g_{co,0}) \leftarrow (V_t, g_{co,t})$$

where convergence is uniform as $a \to 0$, $t \to 0$ on compact sets away from the singularities. The convergence is also continuous in the Gromov-Hausdorff sense [40]. The process replaces a holomorphic $\mathbb{P}^1$ by a special Lagrangian S^3. For a uniqueness result on Kähler Ricci-flat metrics asymptotic to the cone $(V_0, g_{co,0})$, see [28].

3.2 Global Kähler geometry

Next, we consider the global setting of a conifold transition of compact Calabi-Yau manifolds. Let $\hat{X}$ be a smooth compact projective Calabi-Yau threefold, and let us degenerate a collection of holomorphic curves to deform $\hat{X}$ by conifold transition

$$\hat{X} \to X_0 \rightsquigarrow X_t.$$

As a first step, we insert the additional hypothesis into our setup that the smoothings X_t happen to be projective. Then both $\hat{X}$ and X_t admit Kähler Ricci-flat metrics by Yau's theorem [113]. The question becomes understanding these metrics and their degenerations in families. The study of this problem was instigated by Ruan-Zhang [94] and Rong-Zhang [92]. Let us recall their setup.

- Let
$$\pi : \mathscr{X} \to \Delta, \quad X_t = \pi^{-1}(t)$$
be a projective smoothing of Calabi-Yau threefolds such that X_t is smooth for $t \neq 0$ and the singular set of X_0 consists of finitely many ordinary double points. Let $\mathscr{L} \to \mathscr{X}$ be an ample line bundle. We consider the family of Calabi-Yau metrics (X_t, ω_t) defined by
$$\omega_t \in c_1(\mathscr{L})|_{X_t}, \quad t \neq 0$$
where ω_t is Kähler Ricci-flat.
- Let
$$\mu : \hat{X} \to X_0$$
be a small resolution. Let $\hat{\omega}$ be a reference Kähler on $\hat{X}$. We consider the family of Calabi-Yau metrics $(\hat{X}, \hat{\omega}_a)$ defined by
$$\hat{\omega}_a \in a[\hat{\omega}] + \mu^* c_1(\mathscr{L})|_{X_0}, \quad a \neq 0$$
where $\hat{\omega}_a$ is Kähler Ricci-flat.

The theorem of Ruan-Zhang [94], Rong-Zhang [92], and J. Song [97] is that

$$(\hat{X}, \hat{\omega}_a) \to (X_0, d_0) \leftarrow (X_t, \omega_t)$$

in the Gromov-Hausdorff sense, (X_0, d_0) is a compact metric length space induced by a singular Kähler Ricci-flat metric ω_0, and the metrics converge smoothly to ω_0 on compact sets away from the singularities.

The significance of this result is the manifestation of continuity in a process which is plainly discontinuous topologically as the Betti numbers b_i jump across the conifold transition. Nevertheless, string theory [100, 56] realizes passing through the conifold singularity in the moduli space as a continuous process. The theorem of [94, 92, 97] is a mathematical counterpart, where the continuity is realized by using families of Kähler Ricci-flat metrics and Gromov-Hausdorff convergence.

The singular Calabi-Yau metric (X_0, ω_0) exists by Eyssidieux-Guedj-Zeriahi [34]. Better estimates for the singular metric in this context were derived by Hein-Sun [59], who showed that the limit (X_0, g_0) inducing the metric space (X_0, d_0) is a singular Calabi-Yau metric with conical singularities, meaning that

$$\sup_{\{r<\delta\}} |g_0 - c g_{co,0}|_{g_{co,0}} \leq C r^{\lambda}$$

in a small neighborhood of an ordinary double point singularity identified with $V_0 \subseteq \mathbb{C}^4$ and $(V_0, r, \omega_{co,0})$ are as defined above. This is a kind of local uniqueness, where the global metric ω_0 is well-approximated by the explicit Candelas-de la Ossa local model near the singular points. For more results on convergence of global Calabi-Yau metrics to local models, see [22, 115].

A consequence of [59] is that the vanishing 3-cycles can be given the structure of a special Lagrangian 3-sphere with respect to the global Calabi-Yau structure $(X_t, \omega_t, \Omega_t)$. Therefore global Calabi-Yau conifold transitions exchange holomorphic $\mathbb{P}^1$s with a special Lagrangian S^3s.

3.3 Global non-Kähler geometry

We have seen that despite taking initial data $\hat{X}$ to be a projective Calabi-Yau threefold, when deforming across a quadratic singularity

$$\hat{X} \to X_0 \rightsquigarrow X_t$$

the complex threefold X_t will not necessarily remain Kähler. We conclude that when considering Calabi-Yau threefolds in families, we are forced to enlarge the space of objects considered by including appropriate limiting manifolds. These limiting objects are certain non-Kähler complex manifolds.

We look for geometric structures on X_t to help understand it. The Kähler-Ricci flat (2) equation is no longer applicable. Instead, the results of Fu-Li-Yau [41] and joint work with Collins and Yau [27] find the following structure:

Theorem 4. *[41, 27] Let $\hat{X}$ be a Calabi-Yau threefold, and let*

$$\hat{X} \to X_0 \rightsquigarrow X_t$$

be a conifold transition. For small enough t, the complex manifold X_t admits the structure (Ω_t, g_t, h_t) solving

$$d(|\Omega_t|_{\omega_t}\, \omega_t^2) = 0 \tag{12}$$

$$F_{h_t} \wedge \omega_t^2 = 0 \tag{13}$$

$$d\Omega_t = 0, \tag{14}$$

where:

- *Ω_t is a holomorphic volume form*
- *(g_t, h_t) is a pair of hermitian metric on $T^{1,0}X$*
- *$\omega_t = g_t(J_t \cdot, \cdot)$*
- *$F_h = \bar{\partial}(h^{-1}\partial h)$ is the Chern curvature.*

Furthermore, near the vanishing cycles $(S^3)_i$, the global metrics (g_t, h_t) satisfy

$$|g_t - a_i\, g_{co,t}|_{g_{co,t}} \leq C|t|^{\lambda},$$
$$|h_t - b_i\, g_{co,t}|_{g_{co,t}} \leq C|t|^{\lambda}$$

for constants $C > 1$ and $\lambda > 0$ and scaling constants a_i, b_i. Here $g_{co,t}$ are the Kähler Ricci-flat metrics obtained by Candelas-de la Ossa on the local model V_t.

We now provide some context for these equations.

Remark 10. On a complex manifold of dimension n, a dual notion to the Kähler condition $d\omega = 0$ is the balanced metric condition $d \star \omega = 0$, or

$$d\omega^{n-1} = 0. \tag{15}$$

The geometry of such metrics was studied by Michelsohn [82]. A theorem of Alessandrini-Bassanelli [2] states that the existence of balanced metrics is invariant under modifications. Here X_t is not birational to $\hat{X}$, and Fu-Li-Yau's [41] result is the construction of balanced metrics on X_t. A conformal change of the metric can be used to go between (15) and (12).

Remark 11. The equation

$$F_h \wedge \omega^{n-1} = 0$$

is the Hermitian-Yang-Mills equation on the tangent bundle $T^{1,0}X$. This equation was solved for general stable holomorphic vector bundles $E \to X$ over a Kähler manifold by Donaldson-Uhlenbeck-Yau [33, 109]. On the complex manifolds $\#_k S^3 \times S^3$ from Example 6, [11] proved stability of the tangent bundle by algebraic methods.

In principle, this equation could also be solved through conifold transitions for bundles other than the tangent bundle, though it is not evident how a conifold transition creates a stable holomorphic vector bundle $E_t \to X_t$. There is a proposal for this mechanism in the string theory literature [3]. In the mathematics literature, Chuan [23] solved the Hermitian-Yang-Mills equation with the simplifying assumption that the bundle to carry through the transition is holomorphically trivial in a neighborhood of the contracted curves.

Remark 12. Though $(X_t, \omega_t, \Omega_t)$ is non-Kähler, the vanishing cycles in X_t are also special Lagrangian [26] in the sense that they are calibrated cycles with respect to a conformal change of metric and the closed 3-form calibration

$$\mathrm{Re}\, e^{-i\hat{\theta}} \Omega$$

for a constant angle $\hat{\theta}$. The concept of such special cycles on (not necessarily Kähler) complex manifolds was introduced by Harvey-Lawson [58]. Thus from the perspective of special submanifolds, conifold transitions exchange holomorphic $\mathbb{P}^1$s with special Lagrangian S^3s regardless of whether X_t is Kähler or not.

Remark 13. Strominger [99] derived equations (12), (13), (14) from constraints of supersymmetry [10]. A possible interpretation of the theorem is that supersymmetry is preserved through conifold transitions, even though the Kähler condition is relaxed.

Remark 14. The system (12) (13) (14) is not expected to be the final word on geometrization of conifold transitions. To rigify this structure and for example obtain a finite dimensional moduli space, or alternatively a sort of uniqueness theorem, one needs to impose an additional equation. For example, in heterotic string theory, the heterotic Bianchi identity is coupled to the supersymmetric equations (12) (13) (14), and this gives the finite dimensionality of the moduli space [18, 19, 30, 46, 48, 86].

3.3.1 The geometric system

We may wonder whether X_t admits Ricci-flat metrics, even though it is non-Kähler. From the perspective of string theory, this is not the right equation once the 3-form flux H is non-zero [62]. Candelas-Horowitz-Strominger-Witten [21] found the Kähler Ricci-flat equation by setting $H = 0$, but in general the equations of motion couple the Ricci tensor to a 3-form H and a scalar function Φ (see e.g. [79] for a modern exposition).

In the setting of conifold transitions, computing the Ricci tensor of the metric g satisfying (12) with $\omega = i g_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^\nu$ leads to (see e.g. [4, 85] for this calculation)

$$R_{mn} + 2\nabla_m \nabla_n \Phi - \frac{1}{4} H_{mpq} H_n{}^{pq} = \frac{1}{2} (dH)^p{}_{pmn}, \tag{16}$$

where $H = i(\partial - \bar{\partial})\omega$ and $\Phi = -\frac{1}{2} \log |\Omega|_\omega$.

Although Theorem 4 constructs solutions of (16) through conifold transitions, the system of equations of heterotic string theory [61, 99] require an additional equation on dH, the heterotic Bianchi identity, to ensure the cancellation of anomalies [52], and this additional equation has not yet been solved through conifold transitions.

S.-T. Yau has conjectured that the complete set of equations appearing in Strominger's paper [99] is solvable through conifold transitions.

Conjecture 2 (Yau's conjecture [114]). Let $\hat{X}$ be a Calabi-Yau threefold, and let

$$\hat{X} \to X_0 \rightsquigarrow X_t$$

be a conifold transition. For small enough t, there exists a triple $(\Omega_t, \omega_t, h_t)$ solving (12), (13), (14) and

$$i\partial\bar{\partial}\omega_t = \alpha_t'(\mathrm{Tr}\,R_{\omega_t} \wedge R_{\omega_t} - \mathrm{Tr}\,F_{h_t} \wedge F_{h_t}) \tag{17}$$

where R_ω is the Chern curvature of ω and $\alpha_t' > 0$.

Remark 15. The conjecture is stated here in its simplest form where the unknown to solve for is a pair of metrics (g_t, h_t) on $T^{1,0}X_t$. Conceivably there could be a setup where h_t is a metric on an auxiliary bundle $E_t \to X_t$.

Remark 16. There are other proposed versions of (17). In string theory, the equation is a formal expansion about the parameter α' which is only valid in certain regimes, and at higher order in α' the curvature R_ω should be computed using the Hull connection [61, 79, 81]. There is an alternate version of (17) compatible with the formalism of generalized geometry where the equation is

$$i\partial\bar{\partial}\omega = \langle F_h \wedge F_h \rangle \tag{18}$$

where the right-hand side generalizes the special case of $\langle \cdot, \cdot \rangle = \mathrm{Tr}_{V_0} - \mathrm{Tr}_{V_1}$ with F_h the curvature of $h = h_0 \oplus h_1$ on $V_0 \oplus V_1$ for a pair of holomorphic vector bundles V_0, V_1. For the interpretation of this equation as a natural structure in generalized geometry, we refer to [43, 44, 45, 47].

Assuming this additional equation, either (17) or (18), can be solved, what are the implications? Returning to the theme of understanding Kähler to non-Kähler conifold transitions, a major direction for future work in this area is to understand what we learn about X_t from (17). By associating to X_t the moduli space $\mathscr{M}(X_t)$ of solutions to the equations, we may hope to learn about X_t from $\mathscr{M}(X_t)$. For the implications of these equations in string theory, see e.g. [5, 6, 18, 30, 80] and references therein.

3.3.2 Degenerations of the Hermitian-Yang-Mills equation

We compare this setup to the compact Kähler case described in Section §3.2. The analogous non-Kähler theorem is best understood as a result on degenerations of

Hermitian-Yang-Mills metrics. The background geometry is set by the conformally balanced metrics ω constructed by Fu-Li-Yau [41] along the conifold transition $\hat{X} \to X_0 \rightsquigarrow X_t$. These satisfy

$$\mathrm{Vol}(C_i, \hat{\omega}_a) \overset{a\to 0}{\to} 0, \quad \mathrm{Vol}(L_t, \omega_t) \overset{t\to 0}{\to} 0$$

and the geometry causes submanifolds to degenerate: holomorphic curves $C_i \subseteq \hat{X}$ and special Lagrangian cycles $L_t \subseteq X_t$ are tending to zero volume. The task is to solve the Hermitian-Yang-Mills equation on this degenerating background and analyze its limiting singularities. Combining joint work with T. Collins and S.-T. Yau [27], and B. Friedman and C. Suan [40], we have the following theorem:

Theorem 5. *[27, 40] Let*

$$(\hat{X}, \hat{\omega}_a) \to (X_0, \omega_0) \leftarrow (X_t, \omega_t)$$

be the path of reference Fu-Li-Yau metrics. Then there exists a unique normalized sequence of Hermitian-Yang-Mills metrics h on $T^{1,0}X$ with respect to this degenerating geometry such that

$$(\hat{X}, \hat{h}_a) \to (X_0, h_0) \leftarrow (X_t, h_t)$$

in the Gromov-Hausdorff sense. Furthermore the limit (X_0, ω_0, h_0) is a singular solution to the Hermitian-Yang-Mills equation with conical singularities, in the sense that

$$\sup_{\{r<\delta\}} |h_0 - c g_{co,0}|_{g_{co,0}} \leq C r^{\lambda} \tag{19}$$

near the singularities.

This theorem shows that the Candelas-de la Ossa local model described in §3.1 still holds in the global compact case. Namely, the compact metrics h are well-approximated by the Kähler Ricci-flat local model $g_{co,0}$ near the singularities, though globally it receives non-Kähler corrections. This is an analog of the Hein-Sun theorem [59] for the Hermitian-Yang-Mills equation in the non-Kähler case. The two main steps of proof in [27] are:

1. Obtain uniform estimates

$$|\hat{h}_a|_{g_{co,a}} + |\hat{h}_a^{-1}|_{g_{co,a}} \leq C$$

near the exceptional curves by adapting the Uhlenbeck-Yau method [109]. This estimate requires the global condition that $T^{1,0}\hat{X}$ is a stable vector bundle over $\hat{X}$. Taking a limit then yields

$$C^{-1} g_{co,0} \leq h_0 \leq C g_{co,0}.$$

2. Upgrade uniform equivalence to polynomial decay (19). That is, show that $g_{co,0}^{-1} h_0$ decays to $c\mathrm{Id}_{T^{1,0}X}$. The toy model for this sort of phenomenon in PDE

is the following: suppose u is a scalar function on a cone V_0 with $\Delta_{g_{cone}} u = 0$ and $|u| \leq C_1$, and show the decay

$$|u - u(0)| \leq C_2 r^{\lambda}.$$

This is elementary for harmonic functions, and that the analogous sort of statement holds for the nonlinear Hermitian-Yang-Mills equation depends on a certain Poincaré inequality invoking a stability condition on $T^{1,0}V_0$. We refer to [27] for details, and see [63] for another instance of this technique.

3.3.3 Alternate setups

We now note some of the alternate approaches to the geometrization of conifold transitions.

- The equations (12), (13), (14) have not yet been solved in the reverse direction.

$$X_t \rightsquigarrow X_0 \to \hat{X}$$

  Here X_t is a degenerating family of projective threefolds and $\hat{X}$ is a small resolution of singularities. The complex manifold $\hat{X}$, though possibly non-Kähler, admits balanced metrics [49, 50].
- There are alternate non-Kähler equations which may be relevant. One of these is the balanced Chern-Ricci flat equations

$$d\omega^2 = 0, \quad |\Omega|_\omega = const.$$

  The analysis of these equations was developed in [42, 106, 107]. The balanced Chern-Ricci flat equations were solved in [50] across reverse conifold transitions. They are still unsolved in the direction $\hat{X} \to X_0 \rightsquigarrow X_t$.
- There are other options inspired by Type IIB string theory with flux [108] [105].

$$d\omega^2 = 0, \quad i\partial\bar{\partial}\omega = \rho_B.$$

  As noted in [43], it is not clear how a conifold transition creates a 4-form ρ_B which is Poincaré dual to a linear combination of holomorphic curves for a forward conifold transition. In the reverse direction, the small resolution creates holomorphic curves satisfying Friedman's relation and the Type IIB equation may be well-suited. See also [45] for more on this idea.
- As X_t cannot be simultaneously complex analytic and symplectic, another alternative is to let go of the complex analytic structure of X_t but preserve the symplectic structure. This point of view was developed by Smith-Thomas-Yau [96]. In other words, although the initial projective threefold solves $d\omega = 0$ and $d\Omega = 0$, across the singularity we may choose to either: preserve $d\omega = 0$ but allow $d\Omega \neq 0$, or preserve $d\Omega = 0$ and allow $d\omega \neq 0$.

In all cases, the fundamental question is how to use the above geometric structures to constrain the possible manifolds appearing on the other side of a degeneration and resolution.

3.4 Departure from Kähler geometry

3.4.1 Complex analytic threefolds

We have seen that conifold transitions can take us out of Kähler geometry. Generally speaking, there are sometimes advantages to working with complex analytic threefolds instead of restricting to projective threefolds or manifolds admitting a Kähler structure. One such example is J. Pardon's resolution of the MNOP conjecture [84]. Pardon's theory of enumeration of holomorphic curves is formulated in the analytic category. His proof of the MNOP conjecture uses the generality of complex analytic families rather than algebraic geometry. The central object is Pardon's Grothendieck group

$$H^*_c(\mathscr{Z}, \mathrm{Cpx}_3)$$

which is the total homology of the double complex

$$C_{-p}(\mathrm{Cpx}_3, C^q_c(\mathscr{Z})) = \bigoplus_{\mathscr{X}\to\Delta^p} C^q_c(\mathscr{Z}(\mathscr{X}/\Delta^p))$$

with direct sum over all complex analytic families $\mathscr{X} \to \Delta^p$ of threefolds over a p-simplex $\Delta^p \subseteq \mathbb{R}^p$. The two differentials are the differential of cohomology with compact support $C^*_c(Z)$ and the other is an alternating sum of restriction to the boundary of the simplex. Here $\mathscr{Z}(\mathscr{X}/\Delta^n)$ is the space of compact 1-cycles lying entirely in the fibers of $\mathscr{X} \to \Delta^n$.

In Pardon's formalism [84], curve enumeration theories such as Gromov-Witten theory a la Behrend-Fatechi [9] are homomorphisms out of the Grothendieck group

$$\mathrm{GW} : H^*_c(\mathscr{Z}, \mathrm{Cpx}_3) \to \mathbb{Q}((u))$$

where powers of u keep track of the genus of the curve. Given a projective threefold X and denoting by $Z(X,\beta)$ the space of curves C with $[C] = \beta$, the constant function $\mathbf{1}_{X,Z(X,\beta)}$ defines an element in $[\alpha] \in H^0_c(\mathscr{Z}, \mathrm{Cpx}_3)$. This produces an enumerative invariant

$$\mathrm{GW}([\alpha]).$$

Deformation invariance comes from connecting a pair of threefolds by a family $\mathscr{X} \xrightarrow{\pi} \Delta^1$ to define the same class in $H^0_c(\mathscr{Z}, \mathrm{Cpx}_3)$. Indeed, equivalence in Pardon's homology is a way to package a vast generalization of deformation invariance.

This formalism for studying curve enumeration invariants uses cohomology with compact support, and enables the exploration of possibly non-compact spaces of curves on general complex analytic threefolds. In Pardon's application to the MNOP conjecture, the framework allows him to extract open neighborhoods of holomorphic curves and separately deform them to break the curve into a union of isolated rigid curves, and then deduce the general conjecture from the case of local curves [12].

3.4.2 The web of threefolds

Consider the set $\mathscr{W}$ of all complex threefolds connected to projective Calabi-Yau threefolds by conifold transitions. It is unknown what is the precise category of complex threefolds constituting this set. We have seen in Example 6 that a conifold transition may connect a Kähler Calabi-Yau threefold to a non-Kähler complex threefold. It is not necessary to collapse b_2 to zero to do this: contract for example only two curves on $\hat{X}$ in Example 5 to obtain a non-Kähler X_t with $b_2(X_t) = 1$.

An open question is whether there is a sense in which $\mathscr{W}$ is bounded. One could hope to constrain this space of complex threefolds by equipping them with special geometric structures. Currently, we know that the threefolds X_t with small t linked to a projective Calabi-Yau threefold $\hat{X}$ by conifold transition $\hat{X} \to X_0 \rightsquigarrow X_t$ satisfy the following properties:

- There exists a holomorphic volume form Ω [37].
- There holds $h^{1,0} = h^{0,1} = 0$ [37].
- The $\partial\bar{\partial}$-lemma is satisfied [74, 39, 71]. Moreover, the Hodge-Riemann bilinear relation holds on $H^{2,1}$.
- The tangent bundle $T^{1,0}$ is stable with respect to a balanced metric ω [41, 27].

We see that the analytic threefolds in the web of conifold transitions $\mathscr{W}$ inherit some of the properties of Kähler geometry, even though they may or may not actually be Kähler.

Remark 17. The general theory of balanced Calabi-Yau $\partial\bar{\partial}$-manifolds and their deformation theory has been developed by Wu [112], Popovici [88], and Lee [71].

A consequence of Yau's theorem [113] is that Kähler Calabi-Yau manifolds have stable tangent bundle. The result of [41, 27] is a generalization of Yau's theorem to the web of threefolds in regions where Kähler Ricci-flat metrics cannot exist. As a corollary, we may rule out complex threefolds with unstable $T^{1,0}X$ from appearing as limits of Kähler to non-Kähler conifold transitions. Recall that given a hermitian metric ω satisfying $d\omega^2 = 0$ on a complex threefold X, stability of $T^{1,0}X$ is the property

$$\int_X c_1(F) \wedge \omega^2 < 0$$

for all torsion-free coherent subsheaves $F \subseteq T^{1,0}X$ of rank 1,2. It is possible in this generality that $[\omega^2] = 0$; this is for example the case in Example 6 and it was shown in [11] that for this particular example $T^{1,0}$ has no holomorphic subbundles.

It is well-known (e.g. [38]) that stability implies

$$H^0(\mathrm{End}\,T^{1,0}X) = \{\lambda\,\mathrm{Id} : \lambda \in \mathbb{C}\}. \tag{20}$$

Stability is also relevant when associating a moduli space to a holomorphic bundle, which can illuminate the topology of the underlying manifold [32]. It is hoped that the geometric structures presented in this survey will help improve our understanding of the possible complex analytic threefolds appearing as limits of degenerations and resolutions of Calabi-Yau threefolds.

Acknowledgements I owe much of my understanding of Calabi-Yau threefolds to frequent discussions with J. Bryan, T. Collins, and S.-T. Yau. I thank the MATRIX-Simons Scholarship program for the opportunity to travel and participate in the MATRIX research program on "The geometry of moduli spaces in string theory". Thanks to C. Suan, B. Friedman, R. Friedman, T.-J. Lee, T. Hübsch and C.-L. Wang, for discussions and comments. Thank you to the reviewer for helpful suggestions.

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