

An overview of Bagger-Witten line bundles

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Abstract We give a brief overview of recent progress in understanding Bagger-Witten line bundles, which are bundles over moduli spaces of two-dimensional $N = 2$ SCFTs whose existence is a consequence of the global $U(1)_R$ symmetry of the theories. Our overview includes a discussion of applications in supergravities coupled to gauge theories, a proposal for a purely geometric interpretation, and explicit examples over moduli spaces of Calabi-Yau manifolds.

1 Introduction

Briefly, Bagger-Witten line bundles are line bundles over moduli spaces¹ of two-dimensional SCFTs, whose existence is implied by symmetries of the theories, and which are closely interrelated with supersymmetry. They were originally discovered in [22], where they resolved a puzzle in four-dimensional $N = 1$ supergravity, and were later given worldsheet realizations in terms of two-dimensional $N = 2$ SCFTs in e.g. [1, 16].

We can understand Bagger-Witten line bundles as follows. Over a moduli space of conformal field theories, there exist bundles whose structure groups coincide with the symmetries of the theories. As one crosses from one coordinate patch to another over the moduli space, across the overlaps, theories in one patch are related to theories in the other by global symmetries. If one has a family of theories with global symmetry G , one is naturally led to expect a principal G bundle over the moduli space, and/or vector bundles associated to such a principal bundle. In the case of

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¹ All moduli ‘spaces’ here will be stacks, usually Deligne-Mumford stacks, as we shall see explicitly in examples, but for simplicity and readability we will usually refer to them as ‘spaces.’

two-dimensional $N = 2$ SCFTs, generically the only global symmetry is the $U(1)_R$ symmetry, hence a moduli space of two-dimensional $N = 2$ SCFTs will naturally carry a line bundle, and as it is an R symmetry, that line bundle should somehow be related to supersymmetry. Bagger-Witten line bundles are of this form.

Although Bagger-Witten line bundles have often been discussed and applied (see e.g. [17]), their properties have historically not been well-understood, and until recently concrete examples were not known.

The purpose of this talk is to quickly review Bagger-Witten line bundles, culminating in some recent developments, a geometric definition, and concrete examples.

2 Appearance in supergravity

In this section we briefly review the original argument of [22], in four-dimensional $N = 1$ supergravity, for the existence of a line bundle over the moduli space², and other supergravity implications discussed since.

Let \mathcal{M} denote the moduli space of scalar field vevs. (In a string compactification, \mathcal{M} is a large patch on the moduli space of SCFTs, corresponding to the weakly-coupled large-radius regime.) Let K denote the Kähler potential on M , which defines the kinetic terms of the supergravity action.

In Kähler geometry, across coordinate patches,

$$K \mapsto K + f + \bar{f}, \quad (1)$$

where f is a holomorphic function. This also defines a symmetry of rigidly symmetric four-dimensional $N = 1$ theories. There, the kinetic terms can be written in superspace as [21, equ'n (22.1)]

$$\int d^4\theta K(\Phi, \Phi^\dagger), \quad (2)$$

and as the $d^4\theta$ annihilates purely holomorphic and antiholomorphic functions of chiral superfields, the rigidly supersymmetric theory is automatically invariant under (1).

Supergravity theories are more complicated, and the supergravity action turns out not to be invariant under (1). For example, the superpotential terms in four-dimensional $N = 1$ supergravity are proportional to [21, equ'n (23.3)]

$$\exp(K) \left[g^{ij^*} (D_i W) (D_j W)^* - 3|W|^2 \right], \quad (3)$$

² It should be noted that supergravity is only meaningful over a large open subset of the full SCFT moduli space, namely the subset 'close' to the weak coupling limit (the large-radius limit, in a Calabi-Yau compactification).

which are clearly not invariant. However, it was noted in [22] that if one combines (1) with an action³ on the graviton ψ_μ and scalar superpartners χ^i ,

$$\psi_\mu \mapsto \exp\left(-\frac{i}{2}\text{Im}f\right)\psi_\mu, \quad \chi^i \mapsto \exp\left(+\frac{i}{2}\text{Im}f\right)\chi^i, \quad (4)$$

along with a transformation of the superpotential

$$W \mapsto \exp(-f)W, \quad (5)$$

then under the combined action, the theory is invariant under (1).

Now, let us examine these across elements of an open cover $\{U_\alpha\}$ of \mathcal{M} (formally treating \mathcal{M} as a manifold), following [22]. Let K_α denote the Kähler potential on coordinate patch U_α , and let $f_{\alpha\beta}$ denote the holomorphic function appearing in the transformation (1) across patches U_α, U_β on the intersection $U_\alpha \cap U_\beta$. Then, for example, on the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, one finds

$$\begin{aligned} f_{\alpha\beta} + \bar{f}_{\alpha\beta} + f_{\beta\gamma} + \bar{f}_{\beta\gamma} + f_{\gamma\alpha} + \bar{f}_{\gamma\alpha} &= (K_\beta - K_\alpha) + (K_\gamma - K_\beta) + (K_\alpha - K_\gamma), \\ &= 0, \end{aligned} \quad (6)$$

hence

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = h_{\alpha\beta\gamma}, \quad (7)$$

which are pure imaginary. On quadruple overlaps, it is straightforward to demonstrate that the Čech coboundary $\delta h = 0$.

If there is no gauge symmetry, one can further examine the transformations of the scalar superpartners χ^i , following [22], to argue that on triple overlaps,

$$\exp\left(\frac{i}{2}\text{Im}h_{\alpha\beta\gamma}\right) = 1, \quad (8)$$

hence $h_{\alpha\beta\gamma} = 4\pi i n_{\alpha\beta\gamma}$ for integers $n_{\alpha\beta\gamma}$. One then argues that $\exp(-f_{\alpha\beta}/2)$ are transition functions for a line bundle, the ‘‘Bagger-Witten line bundle’’ \mathcal{L}_{BW} , where the $(h_{\alpha\beta\gamma})$ are a Čech representative of c_1 (which is then even).

To be clear, this only defines a line bundle up to a tensor product with a flat line bundle. Consider transition functions, across which the Kähler potential transforms as $K \mapsto K + f + f^*$. Suppose one defines the transition functions to be $\exp(-f/2)$, ala Bagger-witten. Now, K is invariant if one replaces f by $f + g$ for g pure imaginary (hence constant, as follows from holomorphy). After all, $g + g^* = 0$. Hence, we could replace the transition functions $\exp(-f/2)$ by $\exp(-(f + g)/2)$, which in general will define a different line bundle, differing by a flat line bundle. Hence, the supergravity analysis above does not uniquely define a line bundle. However, given a Bagger-Witten line bundle, the analysis above describes how it ties into four-dimensional $N = 1$ supergravity.

³ The action is chiral, and so in the quantum theory, anomalies must be taken into account. We will focus on other aspects here.

From the transformation law (5), we also see that the spacetime superpotential W is a (meromorphic) section of $\mathcal{L}_{\text{BW}}^{\otimes 2}$. Similarly, positivity of the kinetic terms of the scalars and their superpartners, which couple to the pullback of the metric, was used in [22] to argue that $\mathcal{L}_{\text{BW}}^{-1}$ must be ample.

More recently, the Bagger-Witten line bundle has been studied in the context of four-dimensional gauge theories in supergravity. To define the gauge theory, in supergravity, one must specify an action of the gauge group on the scalars and their superpartners – hence, one must specify an action on the moduli space. Now, a group action on a space does not necessarily lift to a line bundle over the same space, and if it does lift, the action will not be unique in general. It was observed in [2, 10] that in the present case, in a gauge theory, if the action of the gauge group on the moduli space does not lift to the Bagger-Witten line bundle, then the action is not (classically) invariant, and hence the gauge theory is well-defined. Furthermore, if a lift does exist, the choice of lift is encoded physically in the Fayet-Iliopoulos parameter, which as a result, must be quantized (as choices of lifts are also quantized).

We can see this more explicitly, following [2, 10]. Under a gauge transformation, the supergravity scalars ϕ transform as

$$\delta\phi^i = \varepsilon^{(a)}X^{(a)i}, \quad (9)$$

where $\varepsilon^{(a)}$ are gauge transformation parameters and $X^{(a)i}$ the components of a set of holomorphic Killing vectors, the gauge field A_μ^a transforms as

$$\delta A_\mu^a = \partial_\mu \varepsilon^{(a)} + f^{abc} \varepsilon^{(b)} A_\mu^c \quad (10)$$

for f^{abc} the structure constants of the Lie algebra, and the Kähler potential transforms as

$$\delta K = \varepsilon^{(a)} F^{(a)} + \varepsilon^{(a)} \bar{F}^{(a)}, \quad (11)$$

where

$$F^{(a)} = X^{(a)}K + iD^{(a)}. \quad (12)$$

These Kähler potential transformations $F^{(a)}$ also appear in the gauge transformations of the fermions, as for example

$$\delta\chi^i = \varepsilon^{(a)} \left(\frac{\partial X^{(a)i}}{\partial \phi^j} \chi^j + \frac{i}{2} \text{Im} F^{(a)} \chi^i \right), \quad (13)$$

$$\delta\psi_\mu = -\frac{i}{2} \varepsilon^{(a)} \text{Im} F^{(a)} \psi_\mu, \quad (14)$$

reflecting the fact that if the Kähler potential transforms, then the spinors must also pick up a corresponding phase in order for the theory to remain invariant. This means that the infinitesimal lift of the group action on the moduli space to the Bagger-Witten line bundle is encoded by

$$\frac{i}{2} \varepsilon^{(a)} \text{Im} F^{(a)}. \quad (15)$$

Physically, from (12), we see that shifts in the imaginary parts of $F^{(a)}$ are the Fayet-Iliopoulos parameters, so we see that the Fayet-Iliopoulos parameters encode a choice of lift of the group action to the Bagger-Witten line bundle.

Let

$$g = \exp\left(i\varepsilon^{(a)}T^a\right) \quad (16)$$

be an element of the Lie group acting on the scalars, where T^a is a Lie algebra generator, then from the discussion above, the lift of g to a Bagger-Witten line bundle can be described as

$$\tilde{g} = \exp\left(\frac{i}{2}\varepsilon^{(a)}\text{Im}F^{(a)}\right). \quad (17)$$

We can modify the lift by changing $\tilde{g} \mapsto \tilde{g}\exp(i\theta_g)$ for some collection of phases, subject to the condition that the lift represents the group honestly:

$$\tilde{g}\tilde{h} = \widetilde{gh} \quad (18)$$

for all $g, h \in G$, the gauge group. Such lifts might not always exist – it is not guaranteed that a set $\{\tilde{g}\}$ exist which satisfy (18) even after adding phases. (More formally, if \tilde{G} is the group generated by the \tilde{g} , then a priori it is merely an extension of G by $U(1)$, and we need that extension to split in order to be able to satisfy (18).)

Furthermore, if a lift exists satisfying (18), then there are multiple lifts, obtained by deforming by phases $\exp(i\theta_g)$ satisfying

$$\exp(i\theta_g)\exp(i\theta_h) = \exp(i\theta_{gh}), \quad (19)$$

for all $g, h \in G$. A collection of such θ_g defines a group homomorphism $G \rightarrow U(1)$, and so we see that if lifts exist, they are classified by $\text{Hom}(G, U(1))$. This means the possible lifts are quantized: for example, if $G = U(1)$, then $\text{Hom}(G, U(1)) = \mathbb{Z}$.

We have already seen that the choice of lift of group action to the Bagger-Witten line bundle is encoded by the Fayet-Iliopoulos parameters. Since those lifts are quantized, we see that the Fayet-Iliopoulos parameters are quantized.

In passing, it was noted in [2] that this story is closely related to the geometric invariant theory (GIT) description of quotients. There, the mathematical analogue of the Fayet-Iliopoulos parameters are also encoded by a choice of group action on a line bundle, which (partially) defines the GIT quotient, in the same way that a choice of point in \mathfrak{g}^* partially determines a symplectic reduction.

Finally, before going on, in cases such as these, when one gauges a group action, the moduli space is best understood as a stack, rather than a space, following e.g. [10, 14]. Revisiting other details of the argument of [22], in [10] it was also pointed out that the transition functions for the bundles in question only close on triple overlaps up to a gauge transformation. The resulting quantities can still be understood as bundles on the corresponding stack, but are not equivalent to ordinary bundles on a space.

3 SCFT description

Bagger-Witten line bundles can also be understood more directly over SCFT moduli spaces, see for example [1, 16]. Consider a family of conformal field theories with some global symmetry G . As one moves across coordinate patches on that family, on overlaps, conformal field theories in one patch are related to those on the overlapping patch by an action of G . This overlap data defines transition functions for a G bundle over the parameter space.

In the case of moduli spaces of two-dimensional $N = 2$ SCFTs, since there is always at least a global $U(1)_R$ symmetry (which exists as part of the $N = 2$ superconformal algebra), the argument above implies that there exists a line bundle over the moduli space.

We can then read off how various CFT operators transform across the moduli space from their $U(1)_R$ charges. Those operators then transform globally as sections of line bundles associated to an underlying principal $U(1)$ bundle by the representations defined by $U(1)_R$ charges.

Two particularly important examples (in the notation of [11]) are as follows:

- The spectral operator \mathcal{U}_1 . This has the same $U(1)_R$ charge as a holomorphic top-form, and we identify the corresponding line bundle over the moduli space with the Hodge line bundle \mathcal{L}_H of holomorphic top-forms.
- The spectral flow operator $\mathcal{U}_{1/2}$. This has half the $U(1)_R$ charge of \mathcal{U}_1 , and indeed, $(\mathcal{U}_{1/2})^2 = \mathcal{U}_1$. The corresponding line bundle over the moduli space corresponds to a Bagger-Witten line bundle \mathcal{L}_{BW} .

The SCFT description illuminates an important relationship. Since $(\mathcal{U}_{1/2})^2 = \mathcal{U}_1$, it is natural to expect that Hodge line bundles and Bagger-Witten line bundles are related by

$$\mathcal{L}_{BW}^{\otimes 2} = \mathcal{L}_H. \quad (20)$$

Indeed, this turns out to be the case.

4 Geometric description

Over moduli spaces of Calabi-Yau's, a proposal for a purely geometric description of Bagger-Witten line bundles was presented in [3, 4]. Briefly, it was argued that Bagger-Witten line bundles can be interpreted as bundles of covariantly constant spinors, in the same way that the Hodge line bundle is the bundle of holomorphic top-forms, essentially because of their association with the spectral flow operator $\mathcal{U}_{1/2}$.

On Calabi-Yau threefolds, however, there is a potential issue. A Calabi-Yau threefold has two nowhere-zero covariantly constant spinors. Given the description above, one is led to ask how it can be consistent with the original Bagger-Witten de-

scription in four-dimensional $N = 1$ supergravity, as only one bundle was discussed there.

To resolve this puzzle, we utilize the fact that the Hodge and Bagger-Witten line bundles over moduli spaces of Calabi-Yau's are believed to be flat. Flatness of the Hodge line bundle over moduli spaces of Calabi-Yau's has been discussed in e.g. [19], [20, theorem 42, corollary 53], and over moduli spaces of two-dimensional SCFTs in⁴ [7]. Since Bagger-Witten line bundles are square roots of Hodge line bundles, their flatness follows from that of the Hodge line bundle.

Now, we return to the puzzle of consistency of the four-dimensional $N = 1$ supergravity description of Bagger-Witten line bundles with the idea that a Calabi-Yau threefold will host two over its moduli space. As we noted earlier in section 2, the original Bagger-Witten paper [22] only defines the Bagger-Witten line bundle up to tensoring with a flat bundle. Given that the Bagger-Witten line bundle itself is flat, the four-dimensional $N = 1$ supergravity description of [22] is completely ambiguous. In particular, in this case one can (and will) have multiple Bagger-Witten line bundles, all of which are consistent with the four-dimensional $N = 1$ supergravity description in [22].

5 Example: elliptic curves

Let us begin with elliptic curves. The moduli space of elliptic curves, as frequently advertised, is

$$\mathcal{M}_0 = [\mathfrak{h}/PSL(2, \mathbb{Z})], \tag{21}$$

where $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$, and \mathfrak{h} denotes the upper half plane. This space is the space of τ parameters, which transform as

$$\tau \mapsto \tau' = \frac{a\tau + b}{c\tau + d}, \tag{22}$$

for

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}). \tag{23}$$

The \mathbb{Z}_2 center of $SL(2, \mathbb{Z})$, namely diagonal matrices $\text{diag}(\pm 1, \pm 1)$, act trivially, and so they are modded out in the description above.

Now, the Hodge line bundle is not defined over \mathcal{M}_0 . The reason for this is as follows. If we let z denote a local coordinate on T^2 , then under $SL(2, \mathbb{Z})$, at the same time that τ transforms [9, section 2.3]

$$z \mapsto \frac{z}{c\tau + d}. \tag{24}$$

⁴ In fact, the reference [7] claimed triviality, but as discussed in [8], their arguments only really imply flatness, and indeed, [8] gives examples which are flat but nontrivial. See also [5] for comments on higher-dimensional cases.

As consistency checks, let us consider a pair of special values of z , namely 1 and τ . In principle, they should map to themselves. It is straightforward to compute that under the transformation above,

$$1 \mapsto \frac{1}{c\tau+d} = 1 - c \left(\frac{a\tau+b}{c\tau+d} \right) + (a-1) \sim 1, \quad (25)$$

$$\tau \mapsto \frac{\tau}{c\tau+d} = \frac{a\tau+b}{c\tau+d} + (d-1) \left(\frac{a\tau+b}{c\tau+d} \right) - b \sim \frac{a\tau+b}{c\tau+d}, \quad (26)$$

where \sim denotes the equivalence $z \sim z + m\tau' + n$ for $m, n \in \mathbb{Z}$.

Returning to the actions (22), (24), we see that although τ is invariant under the action of the center $\pm I$, z is not, and in particular, under $\pm I$, $dz \mapsto \pm dz$.

Thus, we see that holomorphic top-forms on T^2 are not invariant under the action of the center of $SL(2, \mathbb{Z})$.

As a result, the Hodge line bundle is defined over a different space, namely

$$\mathcal{M}_1 = [\mathfrak{h}/SL(2, \mathbb{Z})]. \quad (27)$$

The brackets $[\]$ indicate that one takes the quotient as a stack, and stacks distinguish quotients by trivial group actions, so even though the difference is merely a trivially-acting group, we see $\mathcal{M}_1 \neq \mathcal{M}_0$.

(As an aside, in physics also, gauging a trivially-acting group results in a different theory than not gauging at all. This was discussed at length in two-dimensional orbifolds and gauge theories in [12, 13, 14], and formed a key component of making sense of string propagation on stacks. In more modern language, a gauge theory in which a subgroup of the gauge group acts trivially has a one-form symmetry, not possessed by the theory in which that same subgroup is not gauged.)

Mathematically, \mathcal{M}_1 is a \mathbb{Z}_2 gerbe over \mathcal{M}_0 . The Hodge line bundle of \mathcal{M}_1 is the generator of $\text{Pic}(\mathcal{M}_1) = \mathbb{Z}_{12}$.

To define either Bagger-Witten bundle, we must work harder. From the theory of orbifolds, in a supersymmetric sigma model, given a non-R symmetry \mathbb{Z}_2 that maps $z \mapsto -z$, its superpartner $\psi \mapsto -\psi$, implying that the Ramond sector vacua map as

$$|\pm\rangle \mapsto \exp(\pm i\pi/2) |\pm\rangle = \pm i |\pm\rangle. \quad (28)$$

Because of the factors of i , more precisely because $i^2 = -1$ and not $+1$, we see that the center of \mathbb{Z}_2 cannot be represented on the Ramond vacua. More generally, under an $SL(2, \mathbb{Z}_2)$ transformation, applying standard rules for orbifolds,

$$|\pm\rangle \mapsto \pm \frac{|\pm\rangle}{\sqrt{c\tau+d}}, \quad (29)$$

the same transformation as \sqrt{dz} .

To construct a Bagger-Witten line bundle, we work over the moduli space [8]

$$\mathcal{M}_2 = [\mathfrak{h}/Mp(2, \mathbb{Z})], \quad (30)$$

where $Mp(2, \mathbb{Z})$ is the metaplectic group, the unique nontrivial \mathbb{Z}_2 central extension of $SL(2, \mathbb{Z})$:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Mp(2, \mathbb{Z}) \longrightarrow SL(2, \mathbb{Z}) \longrightarrow 1, \quad (31)$$

whose elements can be thought of as pairs

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}), \pm\sqrt{c\tau + d} \right\}, \quad (32)$$

with product of the form

$$(A, f(-)) \cdot (B, g(-)) = (AB, f(B(-))g(-)), \quad (33)$$

for $A, B \in SL(2, \mathbb{Z})$.

The moduli space \mathcal{M}_2 is a \mathbb{Z}_2 gerbe over \mathcal{M}_1 , just as \mathcal{M}_1 was itself a \mathbb{Z}_2 gerbe over \mathcal{M}_0 .

The Picard group $\text{Pic}(\mathcal{M}_2) = \mathbb{Z}_{24}$, whose generator g and its inverse g^{-1} can be interpreted as Bagger-Witten line bundles.

This may seem like a rather abstract result, but it has concrete implications. For example, implicitly in the result above is an extension of T-duality for supersymmetric sigma models on T^2 , from $SL(2, \mathbb{Z})$ to $Mp(2, \mathbb{Z})$. More generally, in string duality groups, $SL(2, \mathbb{Z})$ is typically extended to $Mp(2, \mathbb{Z})$, after one takes into account sign flips on fermions. This is explored in more detail in [15] (see also [18]).

6 Examples: Calabi-Yau threefolds

Moduli spaces of smooth Calabi-Yau threefolds, constructed from orbifolds of tori, are discussed in [3, 4]. Specifically, these papers studied moduli spaces of toroidal orbifolds by products of \mathbb{Z}_2 's, classified in [6], focusing especially on cases in which the orbifolds have $h^{2,1} = 3$.

At some level, the idea of the construction is to utilize results for elliptic curves, in orbifolds of products of the form $E_1 \times E_2 \times E_3$, where each of the E_i is an elliptic curve. For a G orbifold, the moduli spaces are of the form $[\mathfrak{h}^3/H']$, where $H' = H/G$ for H the normalizer of the image of the orbifold group G in the maximal automorphism group of $E_1 \times E_2 \times E_3$ (which includes both $SL(2, \mathbb{Z})^3$ and S_3 from exchanging the three factors).

In most of the cases with $h^{2,1} = 3$, the Picard group contains elements of infinite order; however, in all cases, the Hodge line bundle is of finite order. This is a strong consistency check on the general claim that the Hodge line bundle should always be flat – as the Picard group contains elements of infinite order, not all line bundles are flat, unlike the example of elliptic curves. Also, as a consequence of the finite-order property, a globally-defined Kähler potential exists over the moduli space, which is of the form

$$\log |\eta(\tau_1)\eta(\tau_2)\eta(\tau_3)|^2, \quad (34)$$

where the τ_i are modular parameters associated with the three elliptic curve factors.

Finally, as described in [3], Bagger-Witten bundles exist, and are defined on a \mathbb{Z}_2 gerbe over the moduli space which hosts the Hodge line bundle, much as in the case of elliptic curves.

7 Conclusions

We have briefly reviewed progress in understanding Bagger-Witten line bundles over moduli spaces of SCFTs, including recent developments such as applications in supergravities coupled to gauge theories, a proposal for a purely geometric interpretation, and the construction of explicit examples over moduli spaces of Calabi-Yau manifolds.

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