Self-distributivity, braces, and the Yang-Baxter equation

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Abstract The theory of the set-theoretic Yang-Baxter equation is reviewed from a purely algebraic point of view. We recall certain algebraic structures called shelves, racks and quandles. These objects satisfy a *self-distributivity* condition and lead to solutions of the Yang-Baxter equation. The quantum algebra as well as the integrability associated to Baxterized involutive set-theoretic solutions is briefly discussed. We then present the theory of the universal algebras associated to rack and general set-theoretic solutions. We show that these are quasi-triangular Hopf algebras and we derive the universal set-theoretic Drinfel'd twist. It is shown that this is an admissible twist allowing the derivation of the universal set-theoretic $\mathscr{R}$-matrix.

1 Introduction

The Yang-Baxter equation (YBE) was first introduced in a purely physical context in [64] as the main mathematical tool for the investigation of quantum systems with many particle interactions, and in [5] for the study of statistical model known as the anisotropic Heisenberg magnet. The idea of set-theoretic solutions to the Yang-Baxter equation was suggested in early 90's by Drinfel'd [27] and since then, set-theoretic solutions have been extensively investigated primarily by means of representations of the braid group, but almost exclusively for the parameter free Yang-Baxter equation (see for instance [30, 39, 55, 56]). The investigation of set-theoretic solutions of the Yang-Baxter equation and the associated algebraic structures is a highly active research field that has been particularly prolific, given that a significant number of related studies has been produced over the past several years (see for instance [6]–[10] [13]–[25],[35]–[38], [41, 42, 49, 50, 51, 59]). The

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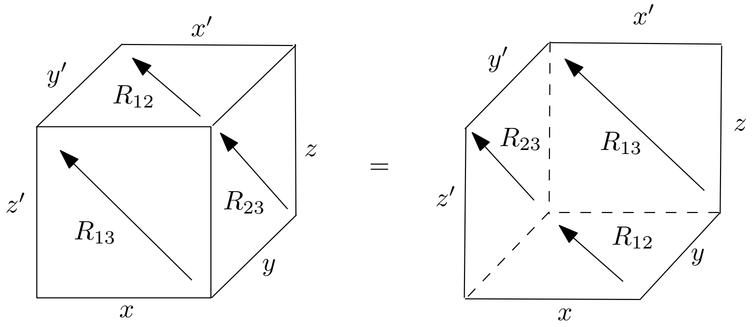
study of the set-theoretic Yang-Baxter equation has produced numerous significant connections to distinct mathematical areas, such as group theory, algebraic number theory, Hopf-Galois extensions, non-commutative rings, knot theory, Hopf algebras and quantum groups, universal algebras, groupoids, heaps and trusses, pointed Hopf algebras, Yetter-Drinfel'd modules and Nichols algebras (see for instance among [3, 4, 6, 8, 9, 10, 30], [41]-[43],[46]-[50], [56, 59]). Moreover, interesting links between the set-theoretic Yang-Baxter equation and geometric crystals [7, 31], or soliton cellular automatons [40, 61] have been shown. Concrete connections with quantum spin-chain like systems were also made in [18, 19].

We note that set-theoretic solution for the parametric Yang-Baxter equation (Yang-Baxter maps) have been primarily studied up to date only in the context of classical discrete and continuous integrable systems connected also to the notion of Darboux-Bäcklund transformation or the discrete zero curvature condition in the Lax pair formulation and the refactorization problem and soliton interactions (see for instance [2, 53, 54, 63], [1, 12, 62]). The refactorization is also synonymous to the so called Bianchi's permutability, which describes the exchange of two consecutive Bäcklund transformations. Specifically, Figure 1 describes graphically the construction of say a two-soliton solution for some integrable non-linear ODE or PDE via two consecutive Bäcklund transformations and the use of Bianchi's permutability.

$$\begin{tikzcd}
 & w_1 \arrow[dr, "\lambda_2"] & \\
w_0 \arrow[ur, "\lambda_1"] \arrow[dr, "\lambda_2"'] & & w_{12} = w_{21} \\
 & w_2 \arrow[ur, "\lambda_1"'] &
\end{tikzcd}$$

1. Bianchi Permutability

The set-theoretic Yang-Baxter equation may be also seen as a cube or 3D consistency condition in classical integrable discrete systems (discrete time evolution), see Figure 2.



2. The 3D consistency condition

In [25] an entirely algebraic analysis for the parametric set-theoretic Yang-Baxter equation was undertaken and purely algebraic solutions were produced. Earlier works on the algebraic structures as well as the associated admissible Drinfel'd twists of the non-parametric, set-theoretic Yang-Baxter equation provided a basic algebraic blueprint [20, 21, 24].

We introduce now the *parameter free* set-theoretic braid equation. Following [27], given a non-empty set X, a map $\check{r} : X \times X \to X \times X$ is said to be a *set-theoretic solution of the braid equation*, if $\check{r}$ satisfies the *braid identity*

$$(\check{r} \times \mathrm{id}_X)\,(\mathrm{id}_X \times \check{r})\,(\check{r} \times \mathrm{id}_X) = (\mathrm{id}_X \times \check{r})\,(\check{r} \times \mathrm{id}_X)\,(\mathrm{id}_X \times \check{r})\,. \tag{1}$$

We call such a map $\check{r}$ simply a *solution* and write $(X, \check{r})$ to denote a solution $\check{r}$ on a set X. Besides, if we write $\check{r}(a,b) = (\sigma_a(b), \tau_b(a))$, with σ_a, τ_a maps from X into itself, then $\check{r}$ is said to be *left non-degenerate* if σ_a is bijective for every $a \in X$, *right non-degenerate* if τ_a is bijective for every $a \in X$, and *non-degenerate* if $\check{r}$ is both left and right non-degenerate. Furthermore, if $\check{r}$ is a solution such that $\check{r}^2 = \mathrm{id}_{X \times X}$, then $\check{r}$ is said to be *involutive*.

It is also worth recalling the connection between the set-theoretic braid equation (1) and the set-theoretic Yang-Baxter equation (see also e.g. [44, 45]). We introduce the map $r : X \times X \to X \times X$, such that $r = \check{r}\pi$, where $\pi : X \times X \to X \times X$ is the flip map: $\pi(x,y) = (y,x)$. Hence, $r(y,x) = (\sigma_x(y), \tau_y(x))$, and it satisfies the set-theoretic Yang-Baxter equation:

$$r_{12}\, r_{13}\, r_{23} = r_{23}\, r_{13}\, r_{12}, \tag{2}$$

where we denote $r_{12}(y,x,w) = (\sigma_x(y), \tau_y(x), w)$, $r_{23}(w,y,x) = (w, \sigma_x(y), \tau_y(x))$ and $r_{13}(y,w,x) = (\sigma_x(y), w, \tau_y(x))$. If $\check{r}$ is involutive then r satisfies $r_{12}r_{21} = \mathrm{id}_{X \times X}$ and is called *reversible*.

After having introduced the set-theoretic braid or Yang-Baxter equation, which is the main mathematical object in the present work, we may state the main aim of this paper, which is the review of basic algebraic structures associated to solutions of the set-theoretic Yang-Baxter equation. Specifically, the key objectives of each of the subsequent sections of this article are presented below.

- In Section 2 we present basic definitions and examples of fundamental algebraic structures associated to solutions of the set-theoretic Yang-Baxter equation. These structures are self distributive and are known as shelves, racks and quandles [46, 52]. They also satisfy axioms analogous to the Reidemeister moves used to manipulate knot diagrams and are associated to link invariants. Other fundamental structures introduced more recently are the so-called (skew) braces invented precisely for the study of generic solutions of the set-theoretic Yang-Baxter equation [55, 56, 57, 39].
- In Section 3 we focus on Baxterized involutive set-theoretic solutions of the braid equation coming from involutive set-theoretic solutions. We present some basic properties of these solutions and we then construct the associated quantum algebra via the FRT (Faddeev-Reshetikhin-Takhtajan) construction [32]. The Yangian is a special case within this class of quantum algebras. We also present information on the construction of a new class of integrable quantum spin chain-like systems associated to set-theoretic solutions (see also [18, 19] for a more detailed exposition).
- In Section 4 we present the new findings on Drinfel'd twists for involutive, non-degenerate, set-theoretic solutions. We focus on a simple, but characteristic example of set-theoretic solution of the YBE known as Lyubashenko's solution. We show that Lyubashenko's solutions can be obtained from the permutation operator via a simple Drinfel'd twist. We derive the simple twist as well as explicit expressions for the n-fold twist. We also present the action of the twist on the Yangian coproducts and the derivation of the twisted co-products associated to the Baxterized Lyubashenko solution.
- In Section 5, which is divided in three subsections, we focus on the Hopf algebras associated to set-theoretic solutions [20, 21, 22, 24]. Specifically, in the first subsection, we introduce the Yang-Baxter algebras of rack/quandle type as quasi-triangular Hopf algebras (see also related results in [30, 3, 49, 4]). We first introduce and study the rack and quandle algebras $\mathscr{A}$, we then show that these are Hopf algebras and we systematically construct the associated universal invertible (or non-degenerate) $\mathscr{R}$-matrix (i.e. $\mathscr{R}^{-1}$ exists) [24]. In the second subsection, we suitably extend the quandle algebra and present the set-theoretic Yang-Baxter algebra, which is also a Hopf algebra. A suitable universal Drinfel'd twist is introduced in the third subsection and it is shown to be admissible. Then the universal set-theoretic $\mathscr{R}$-matrices are derived as twists, and we conclude that invertible, universal set-theoretic $\mathscr{R}$-matrices are coming from universal rack $\mathscr{R}$-matrices via the admissible Drinfel'd twist. The fundamental representation of the universal $\mathscr{R}$-matrices are the linearized versions of rack and general set-theoretic solutions, and it is consequently shown that all involutive set-theoretic solutions of the braid equation are coming from the permutation operator via the set-theoretic Drinfel'd twist. A detailed analysis on these results is presented in [20, 24].
- In Section 6 using the notion of the admissible Drinfel'd twist as well as a certain algebraic structure associated to any generic set-theoretic solution called the *structure group* we are able to extract explicit invertible set-theoretic solutions

from quandles. We provide several explicit examples of solutions emerging from distinct quandles.

2 Preliminaries

Before we present the fundamental algebraic structures associated to set-theoretic solutions we first introduce some simple examples of finite set-theoretic solutions and the notion of linearization.

Example 2.1

1. **Flip map.** *Let* $X = \{1, 2, \dots, n\}$, *the map* $\check{r} : X \times X \to X \times X$, *such as* $\check{r}(a,b) = (b,a)$ *is set-theoretic solution of the braid equation. This is the simplest set-theoretic solution.*
2. **Lyubashenko's solution.** *Let* $X = \{1, 2, \dots, n\}$, *and the map* $\check{r} : X \times X \to X \times X$, *such as* $\check{r}(a,b) = (b + c, a - c)$, *where the addition is defined mod n and* $c \in \{1, 2, \dots, n-1\}$. *Then* $\check{r}$ *is a solution of the set-theoretic braid equation.*

Notice that both solutions above are involutive, i.e. $\check{r}^2 = id$.

Linearization. *Via the linearization process we will be able to express the maps* $\check{r} : X \times X \to X \times X$ *as* $n^2 \times n^2$ *matrices* $\check{r} \in End(\mathbb{C}X^{\otimes 2})$ *(we slightly abuse the notation and use the same symbol* $\check{r}$ *for the matrices). Specifically, consider a free vector space* $V = \mathbb{C}X$ *of dimension equal to the cardinality of* X. *Let* $\mathbb{B} = \{\hat{e}_a\}_{a \in X}$ *be the basis of* V *and* $\mathbb{B}^* = \{\hat{e}^*_a\}_{a \in X}$ *be the dual basis:* $\hat{e}^*_a \hat{e}_b = \delta_{a,b}$, *also* $e_{a,b} := \hat{e}_a \hat{e}^*_b$. *Then any set-theoretic solution of the braid equation is expressed as an* $n^2 \times n^2$ *matrix:*

$$\check{r} = \sum_{a,b \in X} e_{a,\sigma_a(b)} \otimes e_{b,\tau_b(a)} \tag{1}$$

Specifically, for a set $X = \{x_1, x_2, \dots, x_n\}$, *the canonical basis of the n-dimensional vector space reads as expected:* $\hat{e}_{x_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \hat{e}_{x_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \hat{e}_{x_n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

Remark 2.2 *The action of the* $\check{r}$*-matrix on the tensor product of any two elements of the basis* $\hat{e}_x, \hat{e}_y \in V$, *for all* $x, y \in X$ *reads as:*

$$\check{r}\, \hat{e}_{\sigma_x(y)} \otimes \hat{e}_{\tau_y(x)} = \hat{e}_x \otimes \hat{e}_y \quad \& \quad \check{r}^T\, \hat{e}_x \otimes \hat{e}_y = \hat{e}_{\sigma_x(y)} \otimes \hat{e}_{\tau_y(x)}, \tag{2}$$

where ^{T} *denotes total transposition. Or equivalently for any test function* $f(x,y)$, $x, y \in X : (\check{r} f)(x,y) = f(\sigma_x(y), \tau_y(x))$. *Similarly, the action of* $r = \mathscr{P}\check{r}$, *where* $\mathscr{P}$ *is the permutation operator* $\mathscr{P} = \sum_{x,y \in X} e_{x,y} \otimes e_{y,x}$, *on* $\hat{e}_x \otimes \hat{e}_y \in V \otimes V$ *is*

$$r\ \hat{e}_{\sigma_x(y)} \otimes \hat{e}_{\tau_y(x)} = \hat{e}_y \otimes \hat{e}_x \quad \& \quad r^T\ \hat{e}_y \otimes \hat{e}_x = \hat{e}_{\sigma_x(y)} \otimes \hat{e}_{\tau_y(x)}. \tag{3}$$

We consider for the purposes of this article only finite sets. The linerization process can be formally generalized in the case of infinite countable sets as above. In the case of compact sets the use of functional analysis and the study of kernels of integral operators that represent the solution $\check{r}$ is required.

We consider the simple, but non-trivial example of the Lyubashenko solution [27].

Example 2.3 *The Lyubashenko solution may be expressed as a $n^2 \times n^2$ matrix:*

$$\check{r}^{(c)} = \sum_{a,b=1}^{n} e_{a,b+c} \otimes e_{b,a-c}. \tag{4}$$

The addition is defined mod n, and for all $x, y \in \{1,2,\dots,n\}$ and for a fixed $c \in \{1,2,\dots,n-1\}$,

$$\check{r}^{(c)}\ \hat{e}_{y+c} \otimes \hat{e}_{x-c} = \hat{e}_x \otimes \hat{e}_y, \quad r^{(c)}\ \hat{e}_{y+c} \otimes \hat{e}_{x-c} = \hat{e}_y \otimes \hat{e}_x. \tag{5}$$

We focus here on the case $n = 3$ and present the two distinct solutions below as 9×9 matrices:

1. $c = 1$, $\check{r}^{(1)} = \sum_{a,b} e_{a,b+1} \otimes e_{b,a-1}$. *We write all the nine non-zero terms:*

$$\begin{aligned}\check{r}^{(1)} &= e_{1,2} \otimes e_{1,3} + e_{2,3} \otimes e_{2,1} + e_{3,1} \otimes e_{3,2} + e_{1,3} \otimes e_{2,3} + e_{1,1} \otimes e_{3,3} \\ &+ e_{2,1} \otimes e_{3,1} + e_{2,2} \otimes e_{1,1} + e_{3,2} \otimes e_{1,2} + e_{3,3} \otimes e_{2,2}.\end{aligned}$$

2. $c = 2$, $\check{r}^{(2)} = \sum_{a,b} e_{a,b+2} \otimes e_{b,a-2}$, *and explicitly,*

$$\begin{aligned}\check{r}^{(2)} &= e_{1,3} \otimes e_{1,2} + e_{2,1} \otimes e_{2,3} + e_{3,2} \otimes e_{3,1} + e_{1,1} \otimes e_{2,2} \\ &+ e_{1,2} \otimes e_{3,1} + e_{2,2} \otimes e_{3,3} + e_{2,3} \otimes e_{1,3} + e_{3,3} \otimes e_{1,1} + e_{3,1} \otimes e_{2,1}.\end{aligned}$$

The explicit 9×9 matrices are,

$$\check{r}^{(1)} = \begin{pmatrix} 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0&0 \end{pmatrix}, \quad \check{r}^{(2)} = \begin{pmatrix} 0&0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \end{pmatrix}.$$

Notice that $\check{r}^{(2)} = \mathscr{P}\check{r}^{(1)}\mathscr{P}$, where $\mathscr{P}$ is the permutation operator. In general, for any $n \in \{1,2,\dots\}$ we observe that $\check{r}^{(n-k)} = \mathscr{P}\check{r}^{(k)}\mathscr{P}$, $k \in \{1,2,\dots,[\frac{n}{2}]\}$.

2.1 *Self distributivity & special non-involutive solutions*

To describe non-involutive solutions of the braid equation we introduce certain algebraic structures that satisfy a self distributivity condition. Self-distributive structures, such as shelves, racks & quandles [11, 46, 52] satisfy axioms analogous to the Reidemeister moves used to manipulate knot diagrams and are associated to link invariants (see also biracks, biquandles), and the coloring of links, i.e. a knot is tri-colored or not. According to Alexander's theorem all links are closed braids, hence these self distributive structures lead naturally to special non-involutive, set-theoretic solutions of the braid equation. Other algebraic structures naturally connected to bijective, non-degenerate solutions are the *bi-racks*. These are algebraic structures that appear in low-dimensional topology that are associated to link diagrams, and are invariant under the generalized Reidemeister moves for virtual knots and links, see [33]. We provide in what follows some preliminaries on left shelves, racks and quandles. For the first systematic study of shelves, we refer the interested reader to [11]. For recent reviews on self-distributive structures the interested reader is referred to [48, 58, 29] and [34] for potential physical applications.

Definition 2.4 *Let X be a non-empty set and $\triangleright$ a binary operation on X. Then, the pair $(X,\triangleright)$ is said to be a* left shelf *if $\triangleright$ is left self-distributive, namely, the identity*

$$a\triangleright(b\triangleright c)=(a\triangleright b)\triangleright(a\triangleright c) \tag{6}$$

is satisfied, for all $a,b,c\in X$. Moreover, a left shelf $(X,\triangleright)$ is called

1. *a* left spindle *if $a\triangleright a=a$, for all $a\in X$;*
2. *a* left rack *if $(X,\triangleright)$ is a* left quasigroup*, i.e., the maps $L_a: X\to X$ defined by $L_a(b):=a\triangleright b$, for all $b\in X$, are bijective, for every $a\in X$.*
3. *a* quandle *if $(X,\triangleright)$ is both a left spindle and a left rack.*

We are mostly interested in racks and quandles here, given that we always require invertible solutions of the Yang-Baxter equation. We provide below some fundamental known cases of quandles and racks (see also [48, 58, 29]):

(a) **Conjugate quandle.** Let $(X,\cdot)$ be a group and define $\triangleright: X\times X\to X$, such that $a\triangleright b=a^{-1}\cdot b\cdot a$. Then $(X,\triangleright)$ is a quandle.
(b) **Core quandle.** Let $(X,\cdot)$ be a group and $\triangleright: X\times X\to X$, such that $a\triangleright b=a\cdot b^{-1}\cdot a$. Then $(X,\triangleright)$ is a quandle.
(c) **Alexander (affine) quandle.** Let Q be a $\mathbb{Z}[t,t^{-1}]$ ring module and $\triangleright: Q\times Q\to Q$, $a\triangleright b=(1-t)a+bt$, then $(Q,\triangleright)$ is a quandle.
Or, let X be a non empty set equipped with two group operations, $+$ and $\circ$, such that $a\circ(b+c)=a\circ b-a+a\circ c$. This is a so-called left skew brace. The precise definition is given later in the text. Then define $\triangleright: X\times X\to X$, such that for a fixed $z\in X$ and for all $a,b\in X$, $a\triangleright b=z-a\circ z+b\circ z-z+a$ and $(a+b)\circ z=a\circ z-z+a\circ b$.

(d) **Rack, but not quandle.** Let $(G,\cdot)$ be a group and define $\triangleright : G\times G \to G$, such that $a\triangleright b = b\cdot a^{-1}\cdot x\cdot a$, where $x\in G$ is fixed. Then $(G,\triangleright)$ is a rack, but not a quandle.

We also present below some concrete examples of finite quandles:

Example 2.5

1. **The dihedral quandle.** *Let $i,j\in X=\mathbb{Z}_n$ and define $\triangleright : X\times X\to X$, such that $i\triangleright j = 2i - j$ mod n : $(X,\triangleright)$ is a quandle. This is a core quandle with an abelian group. An explicit table of the action $\triangleright$ is presented below for $n=3$ and $X=\{x_1=0, x_2=1, x_3=2\}$:*

$\triangleright$	x_1	x_2	x_3
x_1	x_1	x_3	x_2
x_2	x_3	x_2	x_1
x_3	x_2	x_1	x_3

Table 1

Although the table above comes from the dihedral quandle, another simple representation exists, $\rho : X\to End(\mathbb{C}X)$, such that

$$x_1 \mapsto \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \quad x_2 \mapsto \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \quad x_3 \mapsto \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}.$$

$\rho(x\triangleright y) = \rho(x)^{-1}\bullet\rho(y)\bullet\rho(x)$ for all $x,y\in X$ and $\bullet$ is the usual matrix multiplication.

2. **The tetrahedron quandle.** *Let $X=\{1,2,3,4\}$ and define $\triangleright : X\times X\to X$, such that $1\triangleright = (234)$, $2\triangleright = (143)$, $3\triangleright = (124)$ and $4\triangleright = (132)$. This is also a cyclic quandle. We construct below the explicit table of the action $\triangleright$:*

$\triangleright$	x_1	x_2	x_3	x_4
x_1	x_1	x_3	x_4	x_2
x_2	x_4	x_2	x_1	x_3
x_3	x_2	x_4	x_3	x_1
x_4	x_3	x_1	x_2	x_4

Table 2

3. **Example of affine quandle.** *Let $X=U_m$ denote the set of odd integers mod 2^m, $m\in\mathbb{N}$, and the two operation be $+_1$, such that $a+_1 b = a-1+b$ and $\circ$, where $+$ and $\circ$ are the usual addition and multiplication. Then a quandle is obtained as in case (c) (Alexander's quandle), i.e. for a fixed $z\in U_m$ define, $a\triangleright b = z -_1 a\circ z +_1 b\circ z -_1 z +_1 a$. Specifically, the sets are: 1. for $m=1$, $U_1=\{1\}$, 2. for $m=2$, $U_2=\{1,\ 3\}$, 3. for $m=3$, $U_3=\{1,\ 3,\ 5,\ 7\}$, etc.*

Recalling that $a +_1 b = a - 1 + b$ *and that* $(X,+)$ *in an abelian group we conclude that* $a \triangleright b = -a \circ z + b \circ z + a$*. For instance for* $m = 3$ *(*$X = \{x_1 = 1,\ x_2 = 3,\ x_3 = 5,\ x_4 = 7\}$*) and by choosing for example* $z = 3$ *we obtain the following table*

$\triangleright$	x_1	x_2	x_3	x_4
x_1	x_1	x_4	x_3	x_2
x_2	x_3	x_2	x_1	x_4
x_3	x_1	x_4	x_3	x_2
x_4	x_3	x_2	x_1	x_4

Table 3

Notice here the distributivity rule between $+_1$ and $\circ$ in case (3) of Example 2.5 of an affine quandle. Indeed, we observe that for all $a,b,c \in X$ $a \circ (b +_1 c) = a \circ b -_1 a +_1 a \circ c$ (see also later in the text the definition of skew braces, Definition 2.9).

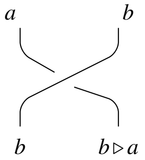
We recall now a fundamental statement regarding shelves and solutions of the set-theoretic Yang-Baxter equation.

Proposition 2.6 *We define the binary operation* $\triangleright : X \times X \to X$, $(a,b) \mapsto a \triangleright b$*. Then* $\check{r} : X \times X \to X \times X$*, such that for all* $a,b \in X$, $\check{r}(a,b) = (b, b \triangleright a)$ *is a solution of the set-theoretic braid equation if and only if* $(X, \triangleright)$ *is a shelf.*

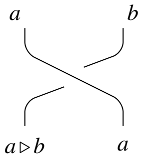
Proof. The proof is straightforward by direct substitution in the Yang-Baxter equation and comparison between LHS and RHS (a graphical depiction of the proof is given below in Figure 4).

Remark 2.7 *If* $\check{r} : X \times X \to X \times X$*, such that for all* $a,b \in X$, $\check{r}(a,b) = (b, b \triangleright a)$ *is an invertible braid solution then* $(X, \triangleright)$ *is a rack (or a quandle).*

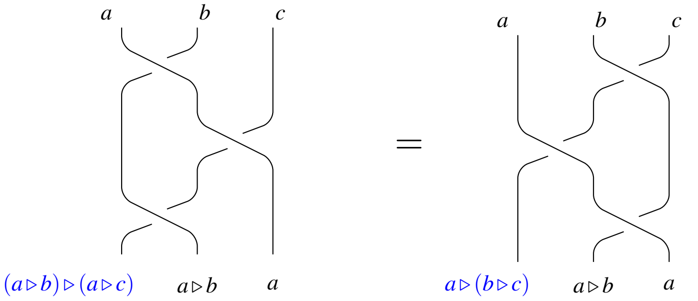
The graphical representation of the shelve solution $\check{r}(a,b) = (b, b \triangleright a)$:



We are interested here in invertible solutions of the braid equation, so we are focusing on rack solutions. We note that the inverse of $\check{r}$ above is $\check{r}^{-1} : X \times X \to X \times X$, $\check{r}^{-1}(a,b) = (a \triangleright^{-1} b, a)$, such that $a \triangleright (a \triangleright^{-1} b) = a \triangleright^{-1} (a \triangleright b) = b$ for all $a,b \in X$. Notice also that a different map denoted as $\check{r}' : X \times X \to X \times X$, such that $\check{r}'(a,b) = (a \triangleright b, a)$ is also a solution of the braid equation. $\check{r}'$ is graphically depicted below in Figure 3 together with the braid relation in Figure 4.
The expressions in blue in Figure 4 indicate the left and right hand side of the self-distributivity condition.



3. The shelve braiding



4. Shelve solution of the braid equation

Example 2.8 *We express the solution of the braid equation associated to the dihedral and tetrahedron quandle of Example 2.5 as* 9×9 *and* 16×16 *matrices respectively. Recall from linearization we obtain,* $\check{r}$ *as a matrix,* $\check{r} = \sum_{x,y \in X} e_{x,y} \otimes e_{y,y \triangleright x}$*, where* $e_{x,y}$ *are the elementary* $n \times n$ *matrices* $e_{x,y} = e_x e_y^T$ *(*^{T} *denotes transposition). In our examples* $n = 3, 4$*.*

1. *(A non-involutive solution from the dihedral quandle). More specifically, from Table 1*

$$\check{r} = \sum_{j=1}^{3} e_{x_j,x_j} \otimes e_{x_j,x_j} + e_{x_1,x_2} \otimes e_{x_2,x_3} + e_{x_2,x_1} \otimes e_{x_1,x_3} + e_{x_2,x_3} \otimes e_{x_3,x_1}$$
$$+ e_{x_3,x_2} \otimes e_{x_2,x_1} + e_{x_1,x_3} \otimes e_{x_3,x_2} + e_{x_3,x_1} \otimes e_{x_1,x_2}.$$

Then $\check{r}$ *is explicitly expressed as a* 9×9 *matrix,*

$$\check{r} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

2. *(A non-involutive solution from the tetrahedron quandle). More specifically, from Table 2*

$$\begin{aligned}\check{r} = &\sum_{j=1}^{4} e_{x_j,x_j} \otimes e_{x_j,x_j} + e_{x_1,x_2} \otimes e_{x_2,x_4} + e_{x_2,x_1} \otimes e_{x_1,x_3} + e_{x_2,x_3} \otimes e_{x_3,x_4}\\ &+ e_{x_3,x_2} \otimes e_{x_2,x_1} + e_{x_1,x_3} \otimes e_{x_3,x_2} + e_{x_3,x_1} \otimes e_{x_1,x_4} + e_{x_2,x_4} \otimes e_{x_4,x_1}\\ &+ e_{x_4,x_2} \otimes e_{x_2,x_3} + e_{x_3,x_4} \otimes e_{x_4,x_2} + e_{x_4,x_3} \otimes e_{x_3,x_1} + e_{x_1,x_4} \otimes e_{x_4,x_3} + e_{x_4,x_1} \otimes e_{x_1,x_2}.\end{aligned}$$

2.2 *Skew braces* & *generic set-theoretic solutions*

It is useful to recall at this point the definition of (skew) braces [55]-[57], [14, 39] as this will allow us to derive generic solutions of the set-theoretic Yang-Baxter equation of the type $r : X \times X \to X \times X$, $r(b,a) = (\sigma_a(b), \tau_b(a))$.

Definition 2.9 *A left skew brace is a set B together with two group operations $+, \circ : B \times B \to B$, the first is called addition and the second is called multiplication, such that for all $a, b, c \in B$,*

$$a \circ (b + c) = a \circ b - a + a \circ c. \tag{7}$$

If $+$ is an abelian group operation, then B is called a left brace. Moreover, if B is a left skew brace and for all $a, b, c \in B$ $(b + c) \circ a = b \circ a - a + c \circ a$, then B is called a two sided skew brace. Analogously if $+$ is abelian and B is a skew brace, then B is called a two sided brace.

The additive identity of a skew brace B will be denoted by 0 and the multiplicative identity by 1. In every skew brace $0 = 1$.

From now on when we say skew brace we mean left skew brace. Some useful examples of braces are presented below:

Example 2.10 (See [9] Corollary 3.14) *Let $\mathrm{U}(\mathbb{Z}/2^m\mathbb{Z})$ denote a set of invertible integers modulo 2^m, for some $m \in \mathbb{N}$. Then a triple $(\mathrm{U}(\mathbb{Z}/2^m\mathbb{Z}), +_1, \circ)$ is a brace, where $a +_1 b = a - 1 + b$, for all $a, b \in \mathrm{U}(\mathbb{Z}/2^m\mathbb{Z})$, $+$ and $\circ$ are addition and multiplication of integer numbers modulo 2^m, respectively.*

Example 2.11 (See [10] Example 5.7) *Consider a ring $\mathbb{Z}/8\mathbb{Z}$. A triple*

$$\left(\mathrm{OM} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,d \in \{1,3,5,7\},\ b,c \in \{0,2,4,6\} \right\}, +_{\mathbb{I}}, \circ \right)$$

is a brace, where $(A,B) \xmapsto{+_{\mathbb{I}}} A - \mathbb{I} + B$, $(A,B) \xmapsto{\circ} A \cdot B$, *and* $+, \cdot$ *are addition and multiplication of two by two matrices over* $\mathbb{Z}/8\mathbb{Z}$*, respectively.*

Example 2.12 (See [10] Example 5.6 or [9] Example 3.15) *Consider a set* $\mathrm{Odd} := \left\{ \frac{2n+1}{2k+1} \mid n,k \in \mathbb{Z} \right\}$ *together with two binary operations* $(a,b) \xmapsto{+_1} a - 1 + b$ *and* $(a,b) \xmapsto{\circ} a \cdot b$*, where* $+, \cdot$ *are addition and multiplication of rational numbers, respectively. The triple* $(\mathrm{Odd}, +_1, \circ)$ *is a brace.*

We recall now the basic conditions associated to any generic solution of the set-theoretic Yang-Baxter equation, as they will be used in our analysis here.

Proposition 2.13 *Let* X *be a nonempty set and define for all* $a,b \in X$*, the maps* $\sigma_a, \tau_b : X \to X$, $b \mapsto \sigma_a(b)$ *and* $a \mapsto \tau_b(a)$*. Then* $r : X \times X \to X \times X$*, such that for all* $a,b \in X$, $r(b,a) = (\sigma_a(b), \tau_b(a))$ *is a solution of the set-theoretic Yang-Baxter equation if and only if for all* $a,b,c \in X$,

$$\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c)) \tag{8}$$

$$\tau_c(\tau_b(a)) = \tau_{\tau_c(b)}(\tau_{\sigma_b(c)}(a)) \tag{9}$$

$$\sigma_{\tau_{\sigma_b(c)}(a)}(\tau_c(b)) = \tau_{\sigma_{\tau_b(a)}(c)}(\sigma_a(b)). \tag{10}$$

Proof. Let r be a solution of the Yang-Baxter equation. We explicitly compute the LHS and RHS of the parametric Yang-Baxter equation. The LHS of the Yang-Baxter equation gives, $a,b,c \in X$,

$$r_{12}\, r_{13}\, r_{23}(c,b,a) = \big(\sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c)),\ \tau_{\sigma_{\tau_b(a)}(c)}(\sigma_a(b)),\ \tau_c(\tau_b(a))\big), \tag{11}$$

whereas the RHS gives

$$r_{23}\, r_{13}\, r_{12}(c,b,a) = \big(\sigma_a(\sigma_b(c)),\ \sigma_{\tau_{\sigma_b(c)}(a)}(\tau_c(b)),\ \tau_{\tau_c(b)}(\tau_{\sigma_b(c)}(a))\big). \tag{12}$$

By equating (11) and (12) we arrive at (8)-(10). And conversely, if conditions (8)-(10) are satisfied then r automatically satisfies the Yang-Baxter equation.

We may now prove a key proposition on generic set-theoretic solutions coming from skew braces [55, 56, 14, 39].

Proposition 2.14 *Let* $(X, +, \circ)$ *be a skew brace and let* $\sigma_a(b) := -a + a \circ b$ *and* $a \circ b = \sigma_a(b) \circ \tau_b(a)$ *for all* $a,b \in X$*. Then the map* $r : X \times X \to X \times X$, $r(b,a) = (\sigma_a(b), \tau_b(a))$ *is a solution of the set theoretic Yang-Baxter equation.*

Proof. In order to prove that r is a solution of the Yang-Baxter equation, it suffices to show that the three conditions (8)-(10) hold.

Before we proceed with the proof, we observe that the distributivity condition in skew braces for all $a,b,c \in X$, $a \circ (b + c) = a \circ b - a + a \circ c$ is equivalent to $a \circ (b - c + d) = a \circ b - a \circ c + a \circ d$ (see also [22]) for a detailed proof).

We first show condition (8), indeed for $a,b,c \in X$,

$$\begin{aligned}\sigma_a(\sigma_b(c)) &= -a+a\circ\sigma_b(c) = -a+a\circ(-b+b\circ c) = -a\circ b+a\circ b\circ c = \sigma_{a\circ b}(c)\\ &= \sigma_{\sigma_a(b)\circ\tau_b(a)}(c) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c)).\end{aligned}$$

We show now condition (9),

$$\tau_c(\tau_b(a)) = \sigma_{\tau_b(a)}(c)^{-1}\circ\tau_b(a)\circ c = \sigma_{\tau_b(a)}(c)^{-1}\circ\sigma_a(b)^{-1}\circ a\circ b\circ c.$$

But,

$$\begin{aligned}\sigma_a(b)\circ\sigma_{\tau_b(a)}(c) &= \sigma_a(b)\circ(-\tau_b(a)+\tau_b(a)\circ c) = \sigma_a(b)-a\circ b+a\circ b\circ c\\ &= -a+a\circ b\circ c = \sigma_a(b\circ c)\end{aligned} \tag{13}$$

hence,

$$\tau_c(\tau_b(a)) = \tau_{b\circ c}(a) = \tau_{\sigma_b(c)\circ\tau_c(b)} = \tau_{\tau_c(b)}(\tau_{\sigma_b(c)}(a)). \tag{14}$$

Condition (10) follows from (8), (9) using $a\circ b = \sigma_a(b)\circ\tau_b(a)$, for all $a,b\in X$. And, this concludes our proof.

The group $(X,\circ)$, such that for all $a,b\in X$, $a\circ b = \sigma_a(b)\circ\tau_b(a)$ is called the *structure group* of a set-theoretic solution. In the case that $(X,+,\circ)$ is a brace we obtain involutive solutions [55, 56, 57]. According to Rump all involutive solutions are obtained from braces, therefore in this article we often call the involutive set-theoretic solutions *brace solutions*.

The important fact that will be discussed in Section 5, is that all involutive solutions are obtained from the permutation operator by a suitable Drinfel'd twist (see also [20, 60, 50]), whereas all the non-involutive but invertible solutions are obtained from rack solutions via a Drinfel'd twist (degenerate solutions are coming from shelves) [22, 24].

3 Involutive solutions & Baxterization

We focus in this section on involutive set-theoretic solutions and derive Baxterized solutions of the braid and Yang-Baxter equations. We then identify the quantum algebras associated to these solutions (see also [18, 19]).

Recall the Yang-Baxter equation in the braid form in the presence of spectral parameters $\lambda_1,\ \lambda_2$ ($\delta = \lambda_1-\lambda_2$):

$$\check{R}_{12}(\delta)\,\check{R}_{23}(\lambda_1)\,\check{R}_{12}(\lambda_2) = \check{R}_{23}(\lambda_2)\,\check{R}_{12}(\lambda_1)\,\check{R}_{23}(\delta), \tag{1}$$

where $\check{R}: V\otimes V\to V\otimes V$, ($V$ is an n dimensional space) and let in general $\check{R} = \sum_j a_j\otimes b_j$, then in the index notation $\check{R} = \sum_j a_j\otimes b_j\otimes 1_V$, $\check{R}_{23} = \sum_j 1_V\otimes a_j\otimes b_j$ and $\check{R}_{13} = \sum_j a_j\otimes 1_V\otimes b_j$.

We focus here on Baxterized solutions of the form

$$\check{R}(\lambda) = \lambda \check{r} + 1_V^{\otimes 2}, \tag{2}$$

where $\check{r}$ is an involutive solution of the braid equation. Let also, $R = \mathscr{P}\check{R}$, then

$$R(\lambda) = \lambda r + \mathscr{P}, \tag{3}$$

and R is a solution of the Yang-Baxter equation,

$$R_{12}(\delta)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\delta). \tag{4}$$

Remark 3.1 *It would be useful for the following Proposition to define the partial transposition. Let $A \in End\big(\mathbb{C}^n \otimes \mathbb{C}^n\big)$ expressed as: $A = \sum_{i,j,k,l=1}^n A_{ij,kl}\ e_{i,j} \otimes e_{k,l}$. We define the partial transposition as follows (in the index notation):*

$$A_{12}^{t_1} = \sum_{i,j,k,l=1}^{n} A_{ij,kl}\ e_{i,j}^t \otimes e_{k,l}, \quad A_{12}^{t_2} = \sum_{i,j,k,l=1}^{n} A_{ij,kl}\ e_{i,j} \otimes e_{k,l}^t \tag{5}$$

where $e_{i,j}^t = e_{j,i}$.

Proposition 3.2 *The brace R-matrix satisfies the following fundamental properties:*

$$R_{12}(\lambda)\ R_{21}(-\lambda) = (-\lambda^2+1)1_V^{\otimes 2}, \qquad \textit{Unitarity} \tag{6}$$

$$R_{12}^{t_1}(\lambda)\ R_{12}^{t_2}(-\lambda-n) = \lambda(-\lambda-n)1_V^{\otimes 2}, \qquad \textit{Crossing-unitarity} \tag{7}$$

$$R_{12}^{t_1 t_2}(\lambda) = R_{21}(\lambda), \tag{8}$$

where $^{t_{1,2}}$ denotes transposition on the first, second space respectively.

Proof. The proof follows immediately after a straightforward computation [19].

3.1 The quantum algebra associated to braces

We recall the definitions of two quadratic algebras $\mathscr{A}$ and $\mathscr{Q}$ associated to solutions of the Yang-Baxter equation, which arise from the FRT (Faddeev, Reshetikhin and Takhtajan) construction [32]. Indeed, from the FRT construction we recall that given a solution of the braid equation $\check{r} : V \times V \to V \times V$ (henceforth we consider $V = \mathbb{C}^n$) the associated quantum algebra $\mathscr{A}$ is a quotient of a free associative $\mathbb{C}$-algebra, generated by $\{L_{z,w} | \ x, w \in X\}$, and relations

$$\check{r}_{12}\ L_1\ L_2 = L_1\ L_2\ \check{r}_{12}, \tag{9}$$

where $L = \sum_{x,y \in X} e_{x,y} \otimes L_{x,y} \in \mathrm{End}(V) \otimes \mathscr{A}$. Recall the *index notation*: $\check{r}_{12} = \check{r} \otimes 1_{\mathscr{A}}$ and $L_1 = \sum_{z,w \in X} e_{z,w} \otimes 1_V \otimes L_{z,w}$, $L_2 = \sum_{z,w \in X} 1_V \otimes e_{z,w} \otimes L_{z,w}$.

From the fundamental relation (9) and by considering the set-theoretic solution of the braid equation

$$\check{r} = \sum_{x,y \in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}, \tag{10}$$

we obtain [30]:

$$L_{x,\hat{x}} L_{y,\hat{y}} = L_{\sigma_x(y),\sigma_{\hat{x}}(\hat{y})} L_{\tau_y(x),\tau_{\hat{y}}(\hat{x})}. \tag{11}$$

Recall that sometimes we call the involutive, set-theoretic solutions, brace solutions, because they are all obtained from braces [55, 56]. Given a solution of the braid equation $\check{r} : V \otimes V \to V \otimes V$, the quadratic algebra $\mathscr{Q}$ is generated by $\{q_x |\ x \in X\}$ and relations

$$\check{r}_{12}\ q_1\ q_2 = q_1\ q_2, \tag{12}$$

where $q = e_x \otimes q_x \in V \otimes \mathscr{Q}$. Also, $\check{r}_{12} = \check{r} \otimes 1_{\mathscr{A}},\ q_1 = \sum_{z,w \in X} e_x \otimes 1_V \otimes q_x,\ q_2 = \sum_{x \in X} 1_V \otimes e_x \otimes q_x$. The quadratic relation (12) for the set-theoretic solution implies

$$q_x q_y = q_{\sigma_x(y)} q_{\tau_y(x)}, \tag{13}$$

also obtained in [30].

We now consider the Baxterized solution $\check{R}(\lambda) = \lambda \check{r} + 1_{V \otimes V}$, where $\check{r}$ in our analysis here is the set-theoretic solution of the braid equation (10). Given a parametric solution of the Yang-Baxter equation, the quantum algebra is defined via the fundamental relation [32]:

$$\check{R}_{12}(\lambda_1 - \lambda_2)\ L_1(\lambda_1)\ L_2(\lambda_2) = L_1(\lambda_2)\ L_2(\lambda_1)\ \check{R}_{12}(\lambda_1 - \lambda_2). \tag{14}$$

$\check{R}(\lambda) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n),\ L(\lambda) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathfrak{A}$, where $\mathfrak{A}$[1] is the quantum algebra defined by (14). We shall focus henceforth on Baxterized solutions coming from involutive, set-theoretic solutions. The defining relations of the corresponding quantum algebra were derived in [18, 19].

The quantum algebra associated to Baxterized solutions coming from braces is defined by generators $L^{(m)}_{z,w},\ z, w \in X$, and defining relations (see also [19])

$$\begin{aligned} L^{(n)}_{z,w} L^{(m)}_{\hat{z},\hat{w}} - L^{(m)}_{z,w} L^{(n)}_{\hat{z},\hat{w}} = {} & L^{(m)}_{z,\sigma_w(\hat{w})} L^{(n+1)}_{\hat{z},\tau_{\hat{w}}(w)} - L^{(m+1)}_{z,\sigma_w(\hat{w})} L^{(n)}_{\hat{z},\tau_{\hat{w}}(w)} \\ & - L^{(n+1)}_{\sigma_z(\hat{z}),w} L^{(m)}_{\tau_{\hat{z}}(z),\hat{w}} + L^{(n)}_{\sigma_z(\hat{z},)w} L^{(m+1)}_{\tau_{\hat{z}}(z),\hat{w}}. \end{aligned} \tag{15}$$

The proof is based on the fundamental relation (14) and the form of the Baxterized brace R-matrix (for the detailed proof see [18, 19]). Recall also that in the index notation we define $\check{R}_{12} = \check{R} \otimes \mathrm{id}_{\mathfrak{A}}$:

$$L_1(\lambda) = \sum_{z,w \in X} e_{z,w} \otimes 1_V \otimes L_{z,w}(\lambda), \quad L_2(\lambda) = \sum_{z,w \in X} 1_V \otimes e_{z,w} \otimes L_{z,w}(\lambda). \tag{16}$$

The exchange relations among the various generators of the affine algebra are derived below via (14). Let us express L as a formal power series expansion

---

[1] Notice that in L in addition to the indices 1 and 2 in (14) there is also an implicit "quantum index" n associated to $\mathfrak{A}$, which for now is omitted, i.e. one writes $L_{1n},\ L_{2n}$.

$L(\lambda) = \sum_{n=0}^{\infty} \frac{L^{(n)}}{\lambda^n}$. Substituting expressions (2), and the λ^{-1} expansion in (14) we obtain the defining relations of the quantum algebra associated to a brace R-matrix (we focus on terms $\lambda_1^{-n}\lambda_2^{-m}$):

$$\begin{aligned}&\check{r}_{12}L_1^{(n+1)}L_2^{(m)} - \check{r}_{12}L_1^{(n)}L_2^{(m+1)} + L_1^{(n)}L_2^{(m)}\\ &= L_1^{(m)}L_2^{(n+1)}\check{r}_{12} - L_1^{(m+1)}L_2^{(n)}\check{r}_{12} + L_1^{(m)}L_2^{(n)}.\end{aligned} \tag{17}$$

The latter relations immediately lead to the quantum algebra relations (15), after recalling: $L_1^{(k)} = \sum_{x,y\in X} e_{x,y} \otimes 1_V \otimes L_{x,y}^{(k)}$, $L_2^{(k)} = \sum_{x,y\in X} 1_V \otimes e_{x,y} \otimes L_{x,y}^{(k)}$, and $\check{r}_{12} = \check{r} \otimes \mathrm{id}_{\mathfrak{A}}$, $L_{x,y}^{(k)}$ are the generators of the associated quantum algebra. The quantum algebra is also equipped with a co-product $\Delta : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ [32, 28]. Indeed, we define

$$(\mathrm{id} \otimes \Delta)L(\lambda) := L_{13}(\lambda)L_{12}(\lambda), \tag{18}$$

which satisfies (14) and is expressed as $(\mathrm{id} \otimes \Delta)L(\lambda) = \sum_{x,y\in X} e_{x,y} \otimes \Delta(L_{x,y}(\lambda))$.

Remark 3.3 *In the special case $\check{r} = \mathscr{P}$ the $\mathscr{Y}(\mathfrak{gl}_n)$ algebra is recovered:*

$$\Big[L_{i,j}^{(n+1)},\, L_{k,l}^{(m)}\Big] - \Big[L_{i,j}^{(n)},\, L_{k,l}^{(m+1)}\Big] = L_{k,j}^{(m)}L_{i,l}^{(n)} - L_{k,j}^{(n)}L_{i,l}^{(m)}. \tag{19}$$

The next natural step is the classification of solutions of the fundamental relation (14), for the brace quantum algebra. A first step towards this goal will be to examine the fundamental object $L(\lambda) = L_0 + \frac{1}{\lambda}L_1$, and search for finite and infinite representations of the respective elements. The classification of L-operators will allow the identification of new classes of quantum integrable systems, such as the analogues of Toda chains or deformed boson models. A first obvious example to consider is associated to Lyubashenko's solution, which is further discussed later in the manuscript.

3.2 *Integrability: local Hamiltonians*

Given any solution of the Yang-Baxter equation we define the so-called monodromy matrix $T_{0,12\ldots N}(\lambda) \in \mathrm{End}\big(\mathbb{C}^n \otimes (\mathbb{C}^n)^{\otimes N}\big)$, which is a tensor representation of the quantum group (14), [32]

$$T_{0,12\ldots N}(\lambda) := R_{0N}(\lambda)\ldots R_{02}(\lambda)\, R_{01}(\lambda), \tag{20}$$

recall $R = \mathscr{P}\check{R}$. We define also the transfer matrix $\mathfrak{t}_{12\ldots N}(\lambda) = tr_0\big(T_{0,12\ldots N}(\lambda)\big) \in \mathrm{End}\big((\mathbb{C}^n)^{\otimes N}\big)$. The monodromy matrix T satisfies (14), and hence one can show that the transfer matrix provides mutually commuting quantities [32]: ($\mathfrak{t}(\lambda) = \lambda^N \sum_k \frac{\mathfrak{t}^{(k)}}{\lambda^k}$)

$$\Big[\mathfrak{t}(\lambda),\, \mathfrak{t}(\mu)\Big] = 0 \;\Rightarrow\; \Big[\mathfrak{t}^{(k)},\, \mathfrak{t}^{(l)}\Big] = 0. \tag{21}$$

Note that historically the index 0 is called “auxiliary”, whereas the indices $1,2,\dots,N$ are called “quantum”, and they are usually suppressed for simplicity, i.e. we simply write $T_0(\lambda)$ and $\mathfrak{t}(\lambda)$.

The following Proposition 3.4 is quite general and holds for any $R(\lambda)=\lambda\mathscr{P}\check{r}+\mathscr{P}$, where $\check{r}$ is an involutive solution of the braid equation and $\mathscr{P}$ is the permutation operator. The key property that allows the derivation of local Hamiltonians is $R(\lambda=0)=\mathscr{P}$.

Proposition 3.4 *Consider the λ-series expansion of the monodromy matrix: $T(\lambda)=\lambda^N\sum_{k=0}^{N}\frac{T^{(k)}}{\lambda^k}$ for any $R(\lambda)=\lambda\mathscr{P}\check{r}+\mathscr{P}$, where $\check{r}$ is an involutive solution of the braid equation. Let also $H^{(k)}=\mathfrak{t}^{(k)}(\mathfrak{t}^{(N)})^{-1},\ \ k=0,\dots,N-1$ and $H^{(N)}=\mathfrak{t}^{(N)}$, where $\mathfrak{t}^{(k)}=tr_0(T_0^{(k)})$. Then the commuting quantities, $H^{(k)}$ for $k=1,\dots,N-1$, are expressed exclusively in terms of the elements $\check{r}_{nn+1}$, $n=1,\dots,N-1$, and $\check{r}_{N1}$.*

Proof. We refer the interested reader to [19] for the detailed proof.

The generic first neighbor Hamiltonian is given as $\mathscr{H}=\sum_{j=1}^{N}\check{r}_{jj+1}$. Higher commuting quantities can be found in [19] for periodic spin chains and in [18] for open spin chains. In the special case of set-theoretic solutions (10) the local Hamiltonian becomes

$$\mathscr{H}=\sum_{j=1}^{N}\sum_{a,b\in X}e^{(j)}_{a,\sigma_a(b)}e^{(j+1)}_{b,\tau_b(a)}. \tag{22}$$

A simple example within this class is the Lyubashenko solution (see Example 2.1), then the Hamiltonian (22) takes the simple form,

$$\mathscr{H}_c=\sum_{j=1}^{N}\sum_{a,b=1}^{n}e^{(j)}_{a,b+c}e^{(j+1)}_{b,a-c},$$

where recall $c\in\{1,2,\dots,n-1\}$ is fixed.

The ultimate goal in the context of quantum integrable systems, or any quantum system for that matter, is the identification of the eigenvalues and eigenvectors of the corresponding Hamiltonian. In the frame of quantum integrable systems there exists a set of mutually commuting “Hamiltonians”, guaranteed by the existence of a quantum R-matrix that satisfies the Yang-Baxter equation. An exhaustive analysis of the symmetries of periodic and open quantum spin chains constructed from Baxterized involutive set-theoretic solutions is presented in [18, 19]. The hierarchy of periodic and open mutually commuting Hamiltonians is also explicitly derived in [18, 19] exclusively in terms of the elements of the symmetric group.

4 Set-theoretic solutions as Drinfel’d twists: an example

We recall in this section the Drinfel’d twist for set-theoretic solutions. It was shown in [20] that all involutive, set-theoretic solutions can be obtained from the permutation operator via suitable twists (see also [60] for an analogous non-local map), whereas non-involutive, invertible solutions are obtained via the same twist from rack/quandle solutions [22, 24]. We consider here a simple example of set-theoretic solution of the braid equation obtained as a simple twist of the permutation to provide a key motivation for the general results presented in the subsequent section.

Simple non-trivial case: Lyubashenko's solution

We recall the Lyubashenko solution and show that is immediately obtained from the permutation operator as a simple twist. Although the construction is simple it has significant implications on the associated symmetries of the braid solutions. Inspired by the isotropic case a similar construction for the q-deformed analogue of Lyubashenko's solution is provided in [19, 18].

Before we derive the Lyubashenko solution as a suitable twist we first introduce a useful Lemma.

Lemma 1. *Let* $\check{r}' : V \otimes V \to V \otimes V$ *(*V *is an* n *dimensional vector space) satisfy the braid relation and* $(\check{r}')^2 = 1_V^{\otimes 2}$*. Let also* $u : V \to V$ *be an invertible map, such that* $(u \otimes u)\check{r}' = \check{r}'(u \otimes u)$*. We define* $\check{r} = (u \otimes 1_V)\check{r}'(u^{-1} \otimes 1_V) = (1_V \otimes u^{-1})\check{r}'(1_V \otimes u)$*, then:*

1. $\check{r}^2 = 1_V^{\otimes 2}$
2. $\check{r}$ *satisfies the braid relation.*

Proof. The proof is straightforward [18].

Proposition 4.1 *Let* $\tau,\ \sigma : X \to X,\ X = \{1, \dots, n\}$ *be isomorphisms, such that* $\sigma(\tau(x)) = \tau(\sigma(x)) = x$ *and let* $u = \sum_{x \in X} e_{x,\tau(x)}$ *and* $u^{-1} = \sum_{x \in X} e_{\tau(x),x}$*. Then any solution of the type (Lyubashenko's solution)*

$$\check{r} = \sum_{x,y \in X} e_{x,\sigma(y)} \otimes e_{y,\tau(x)}, \tag{1}$$

is obtained from the permutation operator $\mathscr{P} = \sum_{x,y \in X} e_{x,y} \otimes e_{y,x}$ *as*

$$\check{r} = (u \otimes 1_V)\mathscr{P}(u^{-1} \otimes 1_V) = (1_V \otimes u^{-1})\mathscr{P}(1_V \otimes u). \tag{2}$$

Proof. The proof relies on the definitions of $\mathscr{P},\ u,\ u^{-1}$, the fundamental property $e_{x,y}e_{z,w} = \delta_{y,z}e_{x,w}$ and by straightforward computation.

Note that $r = \mathscr{P}\check{r}$, and consequently $R = \mathscr{P}\check{R}$ take a simple form for this class of solutions:

$$r = u^{-1} \otimes u \ \Rightarrow\ R(\lambda) = \lambda u^{-1} \otimes u + \mathscr{P}. \tag{3}$$

Before we present our findings on the symmetry of Lyubashenko's $\check{r}$-matrix we first introduce a useful Lemma [19, 18].

Lemma 2. *Let $\mathfrak{l}_{x,y}$ be the generators of the $\mathfrak{gl}_n$ algebra:*

$$\Big[\mathfrak{l}_{x,y}, \mathfrak{l}_{z,w}\Big] = \delta_{y,z}\mathfrak{l}_{x,w} - \delta_{x,w}\mathfrak{l}_{z,y}. \tag{4}$$

The $\mathfrak{gl}_n$ algebra is equipped with a coproduct $\Delta : \mathfrak{gl}_n \to \mathfrak{gl}_n \otimes \mathfrak{gl}_n$, such that

$$\Delta(\mathfrak{l}_{x,y}) = \mathfrak{l}_{x,y} \otimes id + id \otimes \mathfrak{l}_{x,y}. \tag{5}$$

The N-coproduct is obtained by iteration $\Delta^{(N)} = (\Delta^{(N-1)} \otimes id)\Delta = (id \otimes \Delta^{(N-1)})\Delta$ and is given as $\Delta^{(N)}(\mathfrak{l}_{x,y}) = \sum_{n=1}^{N} id \otimes \ldots \otimes \underbrace{\mathfrak{l}_{x,y}}_{n^{th}\ position} \otimes \ldots \otimes id.$

Let also $\mathscr{F}^{(N)} \in \mathfrak{gl}_n^{\otimes N}$ be an invertible element ($\mathscr{F}^{(2)} =: \mathscr{F}$), and define $\Delta_T^{(N)}(\mathfrak{l}_{x,y}) := \mathscr{F}^{(N)}\Delta^{(N)}(\mathfrak{l}_{x,y})(\mathscr{F}^{(N)})^{-1}$, then $\Delta_T^{(N)}(\mathfrak{l}_{x,y})$ also satisfy the $\mathfrak{gl}_n$ algebraic relations.

Proof. The N-coproducts satisfy the $\mathfrak{gl}_n$ relations (4), i.e. $\Big[\Delta^{(N)}(\mathfrak{l}_{x,y}), \Delta^{(N)}(\mathfrak{l}_{z,w})\Big] = \delta_{y,z}\Delta^{(N)}(\mathfrak{l}_{x,w}) - \delta_{x,w}\Delta^{(N)}(\mathfrak{l}_{z,y})$. By acting from the left with $\mathscr{F}^{(N)}$ and with $(\mathscr{F}^{N})^{-1}$ from the right in the latter commutator we immediately obtain $\Big[\Delta_T^{(N)}(\mathfrak{l}_{x,y}), \Delta_T^{(N)}(\mathfrak{l}_{z,w})\Big] = \delta_{y,z}\Delta_T^{(N)}(\mathfrak{l}_{x,w}) - \delta_{x,w}\Delta_T^{(N)}(\mathfrak{l}_{z,y})$.

Corollary 4.2 *Let $\rho : \mathfrak{gl}_n \to End(\mathbb{C}^n)$ be the fundamental representation of $\mathfrak{gl}_n$, such that $\mathfrak{l}_{x,y} \mapsto e_{x,y}$, where recall $e_{x,y}$ are $n \times n$ matrices with elements $(e_{x,y})_{z,w} = \delta_{x,z}\delta_{y,w}$. The special solution $\check{r}$ (1) is $\mathfrak{gl}_n$ symmetric, i.e.*

$$\Big[\check{r},\ \Delta_i(e_{x,y})\Big] = 0, \quad x,\ y \in X, \tag{6}$$

where we define the "twisted" co-products ($i = 1,2$):

$$\begin{aligned} \Delta_1(e_{x,y}) &= e_{\sigma(x),\sigma(y)} \otimes 1_V + 1_V \otimes e_{x,y}, \\ \Delta_2(e_{x,y}) &= e_{x,y} \otimes 1_V + 1_V \otimes e_{\tau(x),\tau(y)}, \end{aligned} \tag{7}$$

($\Delta_1(e_{\tau(x),\tau(y)}) = \Delta_2(e_{x,y})$).

Proof. This can be shown using the form of the special class of solutions (1). The permutation operator is $\mathfrak{gl}_n$ symmetric, i.e.

$$\Big[\mathscr{P},\ \Delta(e_{x,y})\Big] = 0, \tag{8}$$

where the co-products $\Delta(e_{x,y})$ are defined in Lemma (2) ($\mathfrak{l}_{x,y} \mapsto e_{x,y}$).

Let $u = \sum_{x\in X} e_{x,\tau(x)}$, then (6) immediately follows from (8) and (2) after acting (8) from the left and right with $u \otimes 1_V,\ u^{-1} \otimes 1_V$ or $1_V \otimes u^{-1},\ 1_V \otimes u$ respectively. $\Delta_i(e_{x,y})$ are then defined as

$$\begin{aligned} \Delta_1(e_{x,y}) &= u e_{x,y} u^{-1} \otimes 1_V + 1_V \otimes e_{x,y}, \\ \Delta_2(e_{x,y}) &= e_{x,y} \otimes 1_V + 1_V \otimes u^{-1} e_{x,y} u \end{aligned} \tag{9}$$

and explicitly given by (7). Indeed, $u e_{x,y} u^{-1} = e_{\sigma(x),\sigma(y)}$ and $u^{-1} e_{x,y} u = e_{\tau(x),\tau(y)}$.

According to Lemma 2 $\Delta_i(e_{x,y})$ also satisfy the $\mathfrak{gl}_n$ algebra relations, thus $\check{r}$ (1) is $\mathfrak{gl}_n$ symmetric. In this particular case, as is clear from the computation above, two invertible linear maps are involved, $F_i \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n), i \in \{1,2\}$, such that $F_1 := u \otimes 1_V$ and $F_2 := 1_V \otimes u^{-1}$ and $F_i \Delta(e_{x,y}) F_i^{-1} = \Delta_i(e_{x,y})$.

By iteration one derives the N co-products: $\Delta_1^{(N)} = (\Delta_1^{(N-1)} \otimes \mathrm{id})\Delta_1$ and $\Delta_2^{(N)} = (\mathrm{id} \otimes \Delta_2^{(N-1)})\Delta_2$, which explicitly read as

$$\Delta_1^{(N)}(e_{x,y}) = \sum_{k=1}^{N} 1_V \otimes \ldots \otimes e_{\sigma^{N-k}(x),\sigma^{N-k}(y)} \otimes \ldots \otimes 1_V, \tag{10}$$

$$\Delta_2^{(N)}(e_{x,y}) = \sum_{k=1}^{N} 1_V \otimes \ldots \otimes e_{\tau^{k-1}(x),\tau^{k-1}(y)} \otimes \ldots \otimes 1_V. \tag{11}$$

The above expressions can be written in a compact form as: $\Delta_i^{(N)}(e_{x,y}) = F_i^{(N)} \Delta^{(N)}(e_{x,y}) (F_i^{(N)})^{-1}$, where $\Delta^{(N)}(e_{x,y}) = \sum_{k=1}^{N} \mathrm{id} \otimes \ldots \otimes \underbrace{e_{x,y}}_{k^{th}\text{ position}} \otimes \ldots \otimes \mathrm{id}$, and we define $F_1^{(N)} := u^{N-1} \otimes u^{N-2} \otimes \ldots \otimes u \otimes 1_V$ and $F_2^{(N)} := 1_V \otimes u^{-1} \otimes u^{-2} \otimes \ldots \otimes u^{-(N-1)}$ (see also relevant findings in [20, 21]), where $F_i^{(2)} =: F_i$, $i \in \{1,2\}$. Notice that co-associativity does not hold in this case. In fact, it turns out that the quantum algebra associated to generic set-theoretic solutions is a quasi-bialgebra (see e.g. [20, 21]).

We note that local open Hamiltonians for generic Baxterized solutions coming from involutive solutions were systematically derived in [19]. Specifically, the open Hamiltonian associated to Lyubashenko's solution with special boundary conditions is givens as

$$\mathscr{H}_c = \sum_{j=1}^{N-1} \sum_{a=1}^{n} e_{a,b+c}^{(j)} e_{b,a-c}^{(j+1)}, \tag{12}$$

$c \in \{1,2,\ldots,n-1\}$. This is a $\mathfrak{gl}_n$ symmetric Hamiltonian, i.e. $\left[\mathscr{H}_c,\ \Delta_i^{(N)}(y)\right] = 0$, $y \in \mathfrak{gl}_n$, $i \in \{1,2\}$, as opposed to the periodic one, which is not $\mathfrak{gl}_n$ symmetric. Recall $\Delta_i^{(N)}$ are defined in (10),(11).

Motivated by the case of Lyubashenko's solution and the suitable twist of $\mathfrak{gl}_n$ (see also [21]) we present in the subsequent section the full theory of quantum algebras associated to set-theoretic solutions. These are new types of quasi-triangular Hopf algebras introduced in [20, 22, 24]. The quantum algebras associated to the set-theoretic solutions of the parametric Yang-Baxter equation were also introduced in [25].

5 Set-theoretic quasi-triangular Hopf algebras & Drinfel'd twists

In this section we discuss the Yang-Baxter algebras associated to rack type and set-theoretic solutions as quasi-triangular Hopf algebras [20, 24]. The Hopf algebra theory associated to set-theoretic solutions has been developed in [20, 22, 24] (see also [30] and [3] in connection with pointed Hopf algebras and [49, 4] on Hopf algebras in connection to braces [55]–[57], [39]). The associated universal $\mathscr{R}$-matrices are also derived in this frame. A full analysis of set-theoretic twists in the Yangian $\mathfrak{gl}_n$ is presented in [20, 21, 26].

We start our analysis with the rack and quandle algebras and the construction of the associated universal $\mathscr{R}$-matrix. We then extend the algebra to the *decorated rack algebra* and via a suitable admissible Drinfel'd twist we construct the corresponding universal $\mathscr{R}-$matrix. For a more detailed exposition on the proofs presented in this section the interested reader is referred to [20, 24].

5.1 Rack & quandle algebras

We first define the *rack* and *quandle* algebras [24].

Definition 5.1 *(Rack algebra) Let* $(X,\triangleright)$ *be a finite magma, or such that* $a\triangleright$ *is surjective, for every* $a\in X$*. We say that the unital, associative algebra* $\mathscr{A}$*, over a field k generated by indeterminates* $1_{\mathscr{A}}$ *(the unit element),* $q_a,\ q_a^{-1},\ h_a\in\mathscr{A}$ $(h_a=h_b\Leftrightarrow a=b)$ *and relations,* $a,b\in X$ *:*

$$q_a^{-1}q_a=q_aq_a^{-1}=1_{\mathscr{A}},\quad q_aq_b=q_bq_{b\triangleright a},\quad h_ah_b=\delta_{a,b}h_a,\quad q_bh_{b\triangleright a}=h_aq_b,\quad (1)$$

is a rack algebra.

Definition 5.2 *(Quandle algebra) A rack algebra is called a quandle algebra if there is a magma* $(X,\bullet)$*, such that for all* $a,b\in X$*,* $a\bullet:X\to X$ *is a bijection and* $a\bullet b=b\bullet(b\triangleright a)$*.*

The following proposition fully justifies the names rack and quandle algebras.

Proposition 5.3 *Let* $\mathscr{A}$ *be a rack algebra, then for all* $a,b,c\in X$*,* $c\triangleright(b\triangleright a)=(c\triangleright b)\triangleright(c\triangleright a)$*, and* $a\triangleright$ *is bijective, i.e.* $(X,\triangleright)$ *is a rack. If* $\mathscr{A}$ *is a quandle algebra, then in addition, for all* $a\in X$*,* $a\triangleright a=a$*, i.e.* $(X,\triangleright)$ *is a quandle.*

Proof. We compute $h_aq_bq_c$ using the associativity of the rack algebra and due to invertibility of q_a for all $a\in X$ we conclude for all $a,b,c\in X$,

$$h_{c\triangleright(b\triangleright a)}=h_{(c\triangleright b)\triangleright(c\triangleright a)}\ \Rightarrow\ c\triangleright(b\triangleright a)=(c\triangleright b)\triangleright(c\triangleright a).$$

We assume $c\triangleright a=c\triangleright b$, then $q_ch_{c\triangleright a}=q_ch_{c\triangleright b}$, by the fourth relation in 1, we get $h_aq_c=h_bq_c$ and by the invertibility of q_c, $h_a=h_b$, hence $a=b$, i.e. $a\triangleright$ is bijective for all $a\in X$ and thus $(X,\triangleright)$ is a rack.

Moreover, we recall for the quandle algebra $a \bullet b = b \bullet (b \triangleright a)$, then for $a = b$ and by recalling bijectivity and hence left cancellation in $(X, \bullet)$ we conclude that $a \triangleright a = a$ for all $a \in X$, i.e. $(X, \triangleright)$ is a quandle.

Lemma 3. *Let $c = \sum_{a\in X} h_a$, then c is a central element of the rack algebra $\mathscr{A}$.*

Proof. The proof is direct by means of the definition of the algebra $\mathscr{A}$ and Proposition 5.3. Without loss of generality let $c = 1_{\mathscr{A}}$.

Having defined the rack algebra we are now in the position to identify the associated universal $\mathscr{R}$-matrix (solution of the Yang-Baxter equation).

Proposition 5.4 *Let $\mathscr{A}$ be a rack algebra and $\mathscr{R} \in \mathscr{A} \otimes \mathscr{A}$ be an invertible element, such that $\mathscr{R} = \sum_{a\in X} h_a \otimes q_a$. Then $\mathscr{R}$ satisfies the Yang-Baxter equation*

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12},$$

where $\mathscr{R}_{12} = \sum_{a\in X} h_a \otimes q_a \otimes 1_{\mathscr{A}}$, $\mathscr{R}_{13} = \sum_{a\in X} h_a \otimes 1_{\mathscr{A}} \otimes q_a$, and $\mathscr{R}_{23} = \sum_{a\in X} 1_{\mathscr{A}} \otimes h_a \otimes q_a$.

Proof. The proof is a direct computation of the two sides of the Yang-Baxter equation (and use of the fundamental relations (1)):

$$\begin{aligned}
\text{LHS}: \quad \sum_{a,b,c\in X} h_a h_b \otimes q_a h_c \otimes q_b q_c &= \sum_{a,b,c\in X} h_a \otimes q_a h_c \otimes q_a q_c \\
&= \sum_{a,b,c\in X} h_a \otimes q_a h_{a\triangleright c} \otimes q_a q_{a\triangleright c} \\
\text{RHS}: \quad \sum_{a,b,c\in X} h_b h_a \otimes h_c q_a \otimes q_c q_b &= \sum_{a,b,c\in X} h_a \otimes q_a h_{a\triangleright c} \otimes q_c q_a,
\end{aligned}$$

where we have used that $a\triangleright$ is bijective. Then due to the basic relation $q_a q_b = q_b q_{b\triangleright a}$, we show that LHS=RHS, and this concludes our proof.

Remark 5.5 *The universal $\mathscr{R}$-matrix is invertible, in fact, by Lemma 3, $\sum_{a\in X} h_a = 1_{\mathscr{A}}$, hence $\mathscr{R}^{-1} = \sum_{a\in X} h_a \otimes q_a^{-1}$.*

Remark 5.6 Fundamental representation: *Let $\mathscr{A}$ be a rack algebra and $\rho : \mathscr{A} \to End(V)$ be the map defined by*

$$q_a \mapsto \sum_{x\in X} e_{x,a\triangleright x}, \quad h_a \mapsto e_{a,a}, \tag{2}$$

where $(X, \triangleright)$ is a rack. Then $\mathscr{R} \mapsto r := \sum_{a,b\in X} e_{b,b} \otimes e_{a,b\triangleright a}$, is the linearized version of a rack solution of the Yang-Baxter equation. We note that r is invertible, because $a\triangleright : X \to X$ is a bijection for all $a \in X$, then $r^{-1} = \sum_{a,b\in X} e_{b,b} \otimes e_{b\triangleright a,a}$.

Let $\mathscr{P} = \sum_{a,b\in X} e_{a,b} \otimes e_{b,a}$ be the permutation (flip) operator, then the solution of the braid equation is the linearized version of rack solution, $\check{r} = \mathscr{P} r = \sum_{a,b} e_{a,b} \otimes e_{b,b\triangleright a}$.

Note that in the special case where $\check{r}$ is involutive, i.e. $\check{r}^2 = id$, then $a \triangleright b = b$, for all $a, b \in X$, which means that the rack algebra (Definition 5.1) becomes a commutative algebra. In this case $\check{r} = \sum_{a,b\in X} e_{a,b} \otimes e_{b,a}$, i.e. $\check{r}$ reduces to the permutation operator, whereas r reduces to the identity

Theorem 5.7 *Let $\mathscr{A}$ be a quandle algebra (Definition 5.2). Let also $\mathscr{R} = \sum_{a\in X} h_a \otimes q_a$ and for all $a, b \in X$, $q_a q_b = q_{a\bullet b}$. If $(X, \bullet, e)$ is a group, then $(\mathscr{A}, \Delta, \varepsilon, S, \mathscr{R})$ is a quasi-triangular Hopf algebra:*

- *Co-product.* $\Delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$, $\Delta(q_a^{\pm 1}) = q_a^{\pm 1} \otimes q_a^{\pm 1}$ *and* $\Delta(h_a) = \sum_{b,c\in X} h_b \otimes h_c\big|_{b\bullet c=a}$.
- *Co-unit.* $\varepsilon : \mathscr{A} \to k$, $\varepsilon(q_a^{\pm 1}) = 1$, $\varepsilon(h_a) = \delta_{a,e}$.
- *Antipode.* $S : \mathscr{A} \to \mathscr{A}$, $S(q_a^{\pm 1}) = q_a^{\mp 1}$, $S(h_a) = h_{a^*}$, *where a^* is the inverse in $(X, \bullet)$ for all $a \in X$.*

Proof. It is straightforward to check that Δ is an algebra homomorphism. Indeed, this can be explicitly checked via the distributivity condition $a \triangleright (b \bullet c) = (a \triangleright b) \bullet (a \triangleright c)$, which readily follows from, $a \triangleright b = a^* \bullet b \bullet a$.

We are now going to show that all the axioms of a quasi-triangular Hopf algebra hold.

Given the co-products of the generators we have to check co-associativity and also uniquely derive the counit $\varepsilon : \mathscr{A} \to k$ (homomorphism) and antipode $S : \mathscr{A} \to \mathscr{A}$ (anti-homomorphism).

(i) Co-associativity.:

$$(\mathrm{id} \otimes \Delta)\Delta(q_a) = (\Delta \otimes \mathrm{id})\Delta(q_a) = q_a \otimes q_a \otimes q_a,$$
$$(\mathrm{id} \otimes \Delta)\Delta(h_a) = (\Delta \otimes \mathrm{id})\Delta(h_a) = \sum_{b,c,d\in X} h_b \otimes h_c \otimes h_d\big|_{b\bullet c\bullet d=a}.$$

(ii) Counit: $(\varepsilon \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \varepsilon)\Delta(x) = x$, for all $x \in \{q_a,\ q_a^{-1}, h_a\}$.
The generators q_a are group-like elements, so $\varepsilon(q_a) = 1$, and

$$\sum_{a,b\in X} \varepsilon(h_a) h_b = \sum_{a,b} h_a \varepsilon(h_b)\big|_{a\bullet b=c} = h_c \Rightarrow \varepsilon(h_a) = \delta_{a,e}. \tag{3}$$

(iii) Antipode: $m\big((S\otimes \mathrm{id})\Delta(x)\big) = m\big((\mathrm{id}\otimes S)\Delta(x)\big) = \varepsilon(x) 1_{\mathscr{A}}$ for all $x \in \{q_a,\ q_a^{-1}, h_a\}$.
For q_a, we immediately have $S(q_a) = q_a^{-1}$ and (recall $h_a h_b = \delta_{a,b} h_a$ and $\sum_{a\in X} h_a = 1_{\mathscr{A}}$)

$$\sum_{a,b\in X} S(h_a) h_b\big|_{a\bullet b=c} = \sum_{a,b\in X} h_a S(h_b)\big|_{a\bullet b=c} = \delta_{c,e} 1_{\mathscr{A}} \ \Rightarrow\ S(h_a) = h_{a^*},$$

where a^* is the inverse in $(X, \bullet)$ for all $a \in X$. So $(\mathscr{A}, \Delta, \varepsilon, S)$ is a Hopf algebra.

(iv) Moreover,

$$\mathscr{R}_{13}\mathscr{R}_{12} = \sum_{a\in X} h_a\otimes q_a\otimes q_a = \sum_{a\in X} h_a\otimes \Delta(q_a) = (\mathrm{id}\otimes\Delta)\mathscr{R}, \tag{4}$$

$$\mathscr{R}_{13}\mathscr{R}_{23} = \sum_{a,b\in X} h_a\otimes h_b\otimes q_c\big|_{a\bullet b=c} = \sum_{c\in X}\Delta(h_c)\otimes q_c = (\Delta\otimes\mathrm{id})\mathscr{R}. \tag{5}$$

(v) It is also readily shown from the relations of the rack algebra that

$$\Delta^{(op)}(q_a)\mathscr{R} = \mathscr{R}\Delta(q_a) \quad \Delta^{(op)}(h_a)\mathscr{R} = \mathscr{R}\Delta(h_a), \tag{6}$$

where $\Delta^{(op)} = \pi\circ\Delta$, π is the flip map.
We conclude that $(\mathscr{A},\Delta,\varepsilon,S,\mathscr{R})$ is a quasi-triangular Hopf algebra.

5.2 Set-theoretic YB algebras

In this subsection we suitably extend the rack and quandle algebras in order to construct the universal $\mathscr{R}-$matrix associated to general set-theoretic solutions of the Yang-Baxter equation.

We define the *decorated rack* and the set-theoretic Yang-Baxter algebras.

Definition 5.8 *(Decorated rack algebra.) Let $\mathscr{A}$ be a rack algebra (Definition 1). Let also $\sigma_a,\ \tau_b : X\to X$, and σ_a be bijective for all $a\in X$. We say that the unital, associative algebra $\hat{\mathscr{A}}$ over k, generated by indeterminates $1_{\hat{\mathscr{A}}}, q_a, q_a^{-1}, h_a\in\mathscr{A}$ ($h_a = h_b \Leftrightarrow a=b$) and $w_a, w_a^{-1}\in\hat{\mathscr{A}}$, $a\in X$, and relations, $a,b\in X$:*

$$\begin{aligned}&q_a^{-1}q_a = q_aq_a^{-1} = 1_{\hat{\mathscr{A}}},\ \ q_aq_b = q_bq_{b\triangleright a},\ \ h_ah_b = \delta_{a,b}h_a,\ \ q_bh_{b\triangleright a} = h_aq_b,\\ &w_a^{-1}w_a = w_aw_a^{-1} = 1_{\hat{\mathscr{A}}},\ \ w_aw_b = w_{\sigma_a(b)}w_{\tau_b(a)}\ \ w_ah_b = h_{\sigma_a(b)}w_a,\ \ w_aq_b = q_{\sigma_a(b)}w_a\end{aligned} \tag{7}$$

is a decorated rack algebra.

Lemma 4. *Let $c=\sum_{a\in X}h_a$, then c is a central element of the decorated rack algebra $\hat{\mathscr{A}}$.*

Proof. The proof is straightforward by means of the definition of the algebra $\hat{\mathscr{A}}$. We consider henceforth, without loss of generality, $c = 1_{\hat{\mathscr{A}}}$ (see also Lemma 3).

Proposition 5.9 *Let $\hat{\mathscr{A}}$ be a decorated rack algebra, then for all $a,b,c\in X$,*

$$\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c)) \quad \& \quad \sigma_c(b)\triangleright\sigma_c(a) = \sigma_c(b\triangleright a).$$

Proof. We compute $w_aw_bh_c$ using the associativity of the algebra and the invertibility of w_a for all $a\in X$ and we deduce for all $a,b,c\in X$,

$$h_{\sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))} = h_{\sigma_a(\sigma_b(c))}\ \Rightarrow\ \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c)) = \sigma_a(\sigma_b(c)).$$

We also compute $h_a q_b w_c$, via associativity and the invertibility of $q_a,\ w_a$ for all $a \in X$, we obtain for all $a,b,c \in X$,

$$h_{\sigma_c^{-1}(b)\triangleright\sigma_c^{-1}(a)} = h_{\sigma_c^{-1}(b\triangleright a)} \ \Rightarrow\ \sigma_c^{-1}(b)\triangleright\sigma_c^{-1}(a) = \sigma_c^{-1}(b\triangleright a),$$

from the latter it immediately follows, $\sigma_c(b)\triangleright\sigma_c(a) = \sigma_c(b\triangleright a)$.

Definition 5.10 *(Set-theoretic Yang-Baxter algebra.) Let* $\mathscr{A}$ *be a quandle algebra. Let also* $\sigma_a,\ \tau_b : X \to X$, *and* σ_a *be bijective for all* $a \in X$. *We say that the unital, associative algebra* $\hat{\mathscr{A}}$ *over* k, *generated by indeterminates* $1_{\hat{\mathscr{A}}}, q_a, q_a^{-1}, h_a \in \mathscr{A}$ *(*$h_a = h_b \Leftrightarrow a = b$*) and* $w_a, w_a^{-1} \in \hat{\mathscr{A}}$, $a \in X$, *and relations, (7) is a set-theoretic Yang-Baxter algebra.*

Proposition 5.11 *(Hopf algebra) Let* $\hat{\mathscr{A}}$ *be a set-theoretic Yang-Baxter algebra,* $\mathscr{R} = \sum_{b\in X} h_b \otimes q_b$ *and* $(\mathscr{A},\Delta,\varepsilon,S,\mathscr{R})$ *be the quasi-triangular Hopf algebra of Theorem 5.7. If for all* $a,b,x \in X$,

$$\sigma_x(a)\bullet\sigma_x(b) = \sigma_x(a\bullet b), \tag{8}$$

then,

1. $(\hat{\mathscr{A}},\Delta,\varepsilon,S)$ *is a Hopf algebra with* $\Delta(w_a) = w_a \otimes w_a$, *for all* $a \in X$.
2. $\Delta(w_a)\mathscr{R} = \mathscr{R}\Delta(w_a)$, *for all* $a \in X$.

Proof. In our proof below we are using the Definition 5.10 and (8).

1. The coproduct Δ is an algebra homomorphism. It is sufficient to check below the consistency of all algebraic relations of Definition 5.10 for the corresponding coproducts and (8). Then we have for $Y_b \in \{h_b,\ q_b\}$ and for all $a,b \in X$,

   $$\Delta(w_a)\Delta(w_b) = \Delta(w_{\sigma_a(b)})\Delta(w_{\tau_b(a)}), \quad \Delta(w_a)\Delta(Y_b) = \Delta(Y_{\sigma_a(b)})\Delta(w_a).$$

   Also, w_a is a group-like element, thus the counit and antipode are given as: $\varepsilon(w_a) = 1$ and $S(w_a) = w_a^{-1}$. Recall that the coproducts, counits and antipodes of the generators $h_a,\ q_a$ are given in Theorem 5.7.
2. By a direct computation and using the algebraic relations of the Definition 5.10 we conclude for all $a \in X,\ \Delta(w_a)\mathscr{R} = \mathscr{R}\Delta(w_a)$.

5.3 Set-theoretic Drinfel'd twist

In this subsection we introduce the universal set-theoretic (or combinatorial) Drinfel'd twist ([20, 21, 24] (see also, relevant construction in [60, 50]). Using the twist, we will be able to obtain the universal $\mathscr{R}$-matrix associated with the set-theoretic Yang-Baxter (YB) algebra.

Before we introduce the set-theoretic twist, we recall a general statement [28].

Proposition 5.12 *(Admissible Drinfel'd twist [28]) Let* $\mathscr{A}$ *be a unital, associative algebra,* $\mathscr{F}, \mathscr{R} \in \mathscr{A} \otimes \mathscr{A}$ *be invertible elements and* $\mathscr{R}$ *satisfies the Yang-Baxter equation. Let also* $\mathscr{F}_{1,23}, \mathscr{F}_{12,3} \in \mathscr{A}^{\otimes 3}$*, such that*

1. $\mathscr{F}_{23}\mathscr{F}_{1,23} = \mathscr{F}_{12}\mathscr{F}_{12,3}$*, where recall* $\mathscr{F}_{12} = \mathscr{F} \otimes 1_{\mathscr{A}}$ *and* $\mathscr{F}_{23} = 1_{\mathscr{A}} \otimes \mathscr{F}$.
2. $\mathscr{F}_{1,32}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{F}_{1,23}$ *and* $\mathscr{F}_{21,3}\mathscr{R}_{12} = \mathscr{R}_{12}\mathscr{F}_{12,3}$.

That is, $\mathscr{F}$ *is an admissible Drinfel'd twist. Define also* $\mathscr{R}^F := \mathscr{F}^{(op)}\mathscr{R}\mathscr{F}$, $\mathscr{F}^{(op)} = \pi(\mathscr{F})$ *where* $\pi : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ *is the flip map. Then* $\mathscr{R}^F$ *also satisfies the Yang-Baxter equation.*

Proof. It is convenient to introduce some handy notation that can be used in the following. First, let $\mathscr{F}_{123} := \mathscr{F}_{12}\mathscr{F}_{1,23} = \mathscr{F}_{23}\mathscr{F}_{1,23}$. Let also $i, j, k \in \{1,2,3\}$, then $\mathscr{F}_{jik} = \pi_{ij}(\mathscr{F}_{ijk})$ and $\mathscr{F}_{ikj} = \pi_{jk}(\mathscr{F}_{ijk})$, where π is the flip map. This notation describes all possible permutations of the indices 1, 2, 3.

The proof is quite straightforward, [28], we just give a brief outline here. We first prove that $\mathscr{F}_{jik}\mathscr{R}_{ij}\mathscr{F}_{ijk}^{-1} = \mathscr{R}_{ij}^F$, indeed via condition (2) of the proposition the definition of $\mathscr{R}^F$ and the notation introduced above we have

$$\mathscr{F}_{jik}\mathscr{R}_{ij}\mathscr{F}_{ijk}^{-1} = \mathscr{F}_{ji}\mathscr{F}_{ji,k}\mathscr{R}_{ij}\mathscr{F}_{ijk} = \mathscr{F}_{ji}\mathscr{R}_{ij}\mathscr{F}_{ij,k}\mathscr{F}_{ijk}^{-1} = \mathscr{R}_{ij}^F\mathscr{F}_{ij}\mathscr{F}_{ij,k}\mathscr{F}_{ijk}^{-1} = \mathscr{R}_{ij}^F. \tag{9}$$

Similarly, it is shown that $\mathscr{F}_{ikj}\mathscr{R}_{jk}\mathscr{F}_{ijk}^{-1} = \mathscr{R}_{jk}^F$.

Then from the YBE we have,

$$\mathscr{F}_{321}\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{F}_{321}\mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12} \;\Rightarrow\; \mathscr{R}_{12}^F\mathscr{R}_{13}^F\mathscr{R}_{23}^F\mathscr{F}_{123} = \mathscr{R}_{23}^F\mathscr{R}_{13}^F\mathscr{R}_{12}^F\mathscr{F}_{123}.$$

But $\mathscr{F}_{123}$ is invertible, hence $\mathscr{R}^F$ indeed satisfies the Yang-Baxter equation.

Below we prove the main theorem on the set-theoretic Drinfel'd twist (see more details on the proof in [20, 24]).

Theorem 5.13 *(Set-theoretic Drinfel'd twist [20, 24]) Let* $\mathscr{R} = \sum_{a\in X} h_a \otimes q_a \in \mathscr{A} \otimes \mathscr{A}$ *be the universal rack* $\mathscr{R}-$*matrix. Let also* $\hat{\mathscr{A}}$ *be a decorated rack algebra,* $\mathscr{F} \in \hat{\mathscr{A}} \otimes \hat{\mathscr{A}}$*, such that* $\mathscr{F} = \sum_{b\in X} h_b \otimes w_b^{-1}$ *and* $\mathscr{R}_{ij}^F := \mathscr{F}_{ji}\mathscr{R}_{ij}\mathscr{F}_{ij}^{-1}$, $i, j \in \{1,2,3\}$. *We also define:*

$$\mathscr{F}_{1,23} := \sum_{a\in X} h_a \otimes w_a^{-1} \otimes w_a^{-1} =, \quad \mathscr{F}_{12,3}^{*} := \sum_{a,b\in X} h_a \otimes h_{\sigma_a(b)} \otimes w_b^{-1}w_a^{-1}. \tag{10}$$

Let also for every $a, b \in X$, $b \triangleright a = \sigma_b(\tau_{\sigma_a^{-1}(b)}(a))$. *Then, the following statements are true:*

1. $\mathscr{F}_{12}\mathscr{F}_{12,3}^{*} = \mathscr{F}_{23}\mathscr{F}_{1,23} =: \mathscr{F}_{123}$.
2. *For* $i, j, k \in \{1,2,3\}$*: (i)* $\mathscr{F}_{ikj}\mathscr{R}_{jk} = \mathscr{R}_{jk}^F\mathscr{F}_{ijk}$ *and (ii)* $\mathscr{F}_{jik}\mathscr{R}_{ij} = \mathscr{R}_{ij}^F\mathscr{F}_{ijk}$.

That is, $\mathscr{F}$ *is an admissible Drinfel'd twist.*

Proof. The proof is straightforward based on the underlying algebra $\hat{\mathscr{A}}$.

1. Indeed, this is proved by a direct computation and use of the decorated rack algebra. In fact, $\mathscr{F}_{123} = \sum_{a,b\in X} h_a \otimes h_b w_a^{-1} \otimes w_b^{-1} w_a^{-1}$.
2. Given the notation introduced in the proof of Proposition 5.12 it suffices to show that $\mathscr{F}_{132}\mathscr{R}_{23} = \mathscr{R}^F_{23}\mathscr{F}_{123}$ and $\mathscr{F}_{213}\mathscr{R}_{12} = \mathscr{R}^F_{12}\mathscr{F}_{123}$.
   (i) Due to the fact that for all $a \in X$, $\Delta(w_a)\mathscr{R} = \mathscr{R}\Delta(w_a)$ (see Proposition 5.11) we arrive at $\mathscr{F}_{1,32}\mathscr{R}_{23} = \mathscr{R}_{23}F_{1,23}$, then

   $$\mathscr{F}_{132}\mathscr{R}_{23} = \mathscr{F}_{32}\mathscr{F}_{1,32}\mathscr{R}_{23} = \mathscr{F}_{32}\mathscr{R}_{23}\mathscr{F}_{1,23} = \mathscr{R}^F_{23}\mathscr{F}_{123}.$$

   (ii) By means of the relations of the decorated rack algebra $\hat{\mathscr{A}}$ we compute:

   $$\mathscr{F}^*_{21,3}\mathscr{R}_{12} = \sum_{a,c\in X} h_a \otimes q_a h_{a\triangleright c} \otimes \left(w_c w_{\sigma_c^{-1}(a)}\right)^{-1},$$
   $$\mathscr{R}_{12}\mathscr{F}^*_{12,3} = \sum_{a,b\in X} h_a \otimes q_a h_{\sigma_a(b)} \otimes (w_a w_b)^{-1}.$$

   Due to the fact that $b \triangleright a = \sigma_b(\tau_{\sigma_a^{-1}(b)}(a))$ and $w_a w_b = w_{\sigma_a(b)} w_{\tau_b(a)}$ we conclude that $\mathscr{F}^*_{21,3}\mathscr{R}_{12} = \mathscr{R}_{12}\mathscr{F}^*_{12,3}$ and consequently (recall $\mathscr{F}_{213} = \mathscr{F}_{21}\mathscr{F}^*_{21,3}$)

   $$\mathscr{F}_{213}\mathscr{R}_{12} = \mathscr{F}_{21}\mathscr{F}^*_{21,3}\mathscr{R}_{12} = \mathscr{F}_{21}\mathscr{R}_{12}\mathscr{F}^*_{12,3} = \mathscr{R}^F_{12}\mathscr{F}_{123}.$$

Corollary 5.14 *Let $\hat{\mathscr{A}}$ be a decorated rack algebra, $\mathscr{F} \in \hat{\mathscr{A}} \otimes \hat{\mathscr{A}}$, such that $\mathscr{F} = \sum_{b\in X} h_b \otimes w_b^{-1}$ and $\mathscr{R}^F_{ij} := \mathscr{F}_{ji}\mathscr{F}^{-1}_{ij}$, $i, j \in \{1,2,3\}$. We also define:*

$$\mathscr{F}_{1,23} := \sum_{a\in X} h_a \otimes w_a^{-1} \otimes w_a^{-1} =, \quad \mathscr{F}^*_{12,3} := \sum_{a,b\in X} h_a \otimes h_{\sigma_a(b)} \otimes w_b^{-1} w_a^{-1}. \tag{11}$$

Let also for every $a,b \in X$, $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$. Then, the following statements are true:

1. *$\mathscr{F}_{12}\mathscr{F}^*_{12,3} = \mathscr{F}_{23}\mathscr{F}_{1,23} =: \mathscr{F}_{123}$.*
2. *For $i, j, k \in \{1,2,3\}$: (i) $\mathscr{F}_{ikj} = \mathscr{R}^F_{jk}\mathscr{F}_{ijk}$ and (ii) $\mathscr{F}_{jik} = \mathscr{R}^F_{ij}\mathscr{F}_{ijk}$.*
3. *$\mathscr{R}^F_{12}\mathscr{R}^F_{21} = 1_{\hat{\mathscr{A}}^{\otimes 2}}$, i.e. $\mathscr{R}^F$ is reversible.*

Proof. This is an immediate consequence of Theorem 5.13. $\mathscr{R}^F$ is reversible by construction.

We now examine the twisted $\mathscr{R}-$matrix as well as the twisted co-products of the set-theoretic Yabg-Baxter algebra (see also [20, 22]).

Remark 5.15 *(Twisted universal $\mathscr{R}$-matrix) We extract now the explicit expressions of the twisted universal $\mathscr{R}-$matrix and the twisted coproducts of the algebra. We recall the admissible twist $\mathscr{F} = \sum_{b\in X} h_b \otimes w_b^{-1}$.*

- *The twisted $\mathscr{R}-$matrix:*

$$\mathscr{R}^F = \mathscr{F}^{(op)}\mathscr{R}\mathscr{F}^{-1} = \sum_{a,b\in X} h_b w_a^{-1} \otimes h_a q_{\sigma_a(b)} w_{\sigma_a(b)}.$$

- *The twisted coproducts:* $\Delta_F(y) = \mathscr{F}\Delta(y)\mathscr{F}^{-1}$, $y \in \hat{\mathscr{A}}$ *(in this Remark* $\hat{\mathscr{A}}$ *denotes the set-theoretic Yang-Baxter algebra) and we recall, for* $a \in X$,

$$\Delta(w_a) = w_a \otimes w_a, \quad \Delta(h_a) = \sum_{b,c\in X} h_b \otimes h_c\big|_{b\bullet c=a}, \quad \Delta(q_a) = q_a \otimes q_a.$$

Then, the twisted coproducts read as: $\Delta_F(w_a) = \sum_{b\in X} h_{\sigma_a(b)} w_a \otimes w_{\tau_b(a)}$, $\Delta_F(h_a) = \sum_{b\in X} h_b \otimes w_b^{-1} h_c w_b\big|_{b\bullet c=a}$, $\Delta_F(q_a) = \sum_{b\in X} q_a h_{a\triangleright b} \otimes w_b^{-1} q_a w_{a\triangleright b}$,

and it immediately follows that $\mathscr{R}^F \Delta_F(Y) = \Delta_F^{(op)}(Y)\mathscr{R}^F$, $Y \in \hat{\mathscr{A}}$.

Remark 5.16 Fundamental representation & the set-theoretic solution:
Let $\rho : \hat{\mathscr{A}} \to End(V)$, *such that*

$$q_a \mapsto \sum_{x\in X} e_{x,a\triangleright x}, \quad h_a \mapsto e_{a,a}, \quad w_a \mapsto \sum_{b\in X} e_{\sigma_a(b),b}, \tag{12}$$

then $\mathscr{F} \mapsto F := \sum_{a,b\in X} e_{a,a} \otimes e_{\sigma_a(b),b}$ *and* $\mathscr{R}^F \mapsto r^F := \sum_{a,b\in X} e_{b,\sigma_a(b)} \otimes e_{a,\tau_b(a)}$, *where we recall that for all* $a,b,c \in X$, $\sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c)) = \sigma_a(\sigma_b(c)$ *and* $\tau_b(a) := \sigma^{-1}_{\sigma_a(b)}(\sigma_a(b) \triangleright a)$ *and* $(X,\triangleright)$ *is a rack (see also [60, 50, 22]). This is the linearized version of the general set-theoretic solution of the Yang-Baxter equation.*

Recall, $\mathscr{P} = \sum_{a,b\in X} e_{a,b} \otimes e_{b,a}$ *is the permutation (flip) operator, then the solution of the braid equation is the linearized set-theoretic solution,* $\check{r}^F = \mathscr{P} r^F = \sum_{a,b\in X} e_{a,\sigma_a(b)} \otimes e_{b,\tau_b(a)}$.

In the special case, where the set-theoretic solution of the braid equation is invertible, i.e. $(\check{r}^F)^2 = id$, *then* $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$, *which leads to* $a \triangleright b = b$ *for all* $a,b \in X$ *(see also Remark 5.6). That is to say that all involutive set-theoretic* $\check{r}$-*matrices are coming from the permutation operator via the set-theoretic twist, i.e.* $\check{r}^F = F\mathscr{P}F^{-1}$ *(and* $r^F = F^{(op)}F^{-1}$*) where recall* $\mathscr{P}$ *is the permutation operator and* $F = \sum_{a,b\in X} e_{a,a} \otimes e_{\sigma_{a(b),b}}$, *such that for all* $a,b,c \in X$, $\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))$ *and* $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$ *(see also Corollary 5.14 and [24, 26]).*

6 Solutions from Drinfel'd twists

The main aim of this section is the derivation of general invertible solutions of the set-theoretic Yang-Baxter equation via an admissible Drinfel'd twist.

From the analysis of Section 5 we conclude that any admissible set-theoretic twist satisfies two fundamental conditions, for all $a,b,c \in X$,
$\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))$, $\sigma_{\sigma_a(b)}(\tau_b(a)) = \sigma_a(b) \triangleright a$ and $(X,\triangleright)$ is a rack (see also Theorem 5.13). It is thus convenient to introduce an alternative definition for the admissible set-theoretic twist at the fundamental representation (see also Remark 5.16 and a similar definition first introduced in [24]).

Definition 6.1 *Let* $(X, \triangleright)$ *be a rack and define* $F := \sum_{x,y \in X} e_{x,x} \otimes e_{\sigma_x(y),y}$, *such that* $\sigma_x : X \to X$ *is bijective. F is called an admissible set-theoretic Drinfel'd twist if for all* $a, b, c \in X$ *(see also Theorem 5.13)*

(a) $\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))$.
(b) $\sigma_{\sigma_a(b)}(\tau_b(a)) = \sigma_a(b) \triangleright a$.

Notice that in the involutive case the second condition above becomes $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$ for all $a, b \in X$.

6.1 Involutive case

We first focus on the systematic derivation of involutive, set-theoretic solutions of the braid equation by exploiting the existence of an admissible Drinfel'd twist.

The following useful proposition can be now formulated (see also [56, 57, 23]).

Proposition 6.2 *Let* $(X, \circ, 1)$ *be a group and let* $\sigma_a, \tau_b : X \to X$, *such that for all* $a, b \in X$, σ_a *is a bijection,* $a \circ b = \sigma_a(b) \circ \tau_b(a)$ *and* $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$. *Moreover, define* $+ : X \times X \to X$, *such that* $a + b := a \circ \sigma_a^{-1}(b)$ *and assume that* $+$ *is associative and for all* $a, b, c \in X$, $a \circ (b + c) = a \circ b - a + a \circ c$. *Then for all* $a, b, c \in X$,

1. $(X, +, \circ)$ *is a brace and* $\sigma_a(b) = -a + a \circ b$.
2. $\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a}(c))$.

Proof.

1. For the proof of $(X, +)$ being a group we refer the interested reader to [23] for a step by step constructive approach of the algebraic structure. Then due to distributivity condition $(X, +, \circ)$ is a skew brace (see the original works on (skew) braces [55, 56, 57, 39]). To prove that $(X, +, \circ)$ is a brace we need to show that $(X, +)$ is an abelian group.
Indeed, from condition $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$ and the structure group condition $a \circ b = \sigma_a(b) \circ \tau_b(a)$ we obtain for all $a, b \in X$,
$$\sigma_a(b)^{-1} \circ a \circ b = \sigma^{-1}_{\sigma_a(b)}(a) \;\Rightarrow\; a \circ \sigma_a^{-1}(b) = b \circ \sigma_b^{-1}(a) \;\Rightarrow\; a + b = b + a,$$
i.e. $(X, +)$ is abelian. We used in the last part of the proof above the definition $a + b := a \circ \sigma_a^{-1}(b)$.
Moreover, for all $a, b \in X$
$$a \circ \sigma_a^{-1}(b) = a + b \;\Rightarrow\; \sigma_a^{-1}(b) = a^{-1} \circ (a + b)$$
and
$$\sigma_a^{-1}(\sigma_a(b)) = b \;\Rightarrow\; a^{-1} \circ (a + \sigma_a(b)) = b \;\Rightarrow\; \sigma_a(b) = -a + a \circ b.$$

2. Condition (2) is condition (8) and its proof is given in the proof of Proposition 2.14.

Notice that in Proposition 6.2 we assume a non-standard distributivity condition as the usual one does not apply (see for more details [23]). Indeed, let us assume that the usual distributivity condition holds, then

$$a\circ(1+0)=a \ \Rightarrow\ a\circ 1+a\circ 0=a \ \Rightarrow\ a\circ 0=0 \ \Rightarrow\ a=a\circ 0^{-1}=1, \quad \forall a\in X.$$

The latter statement is false given that we consider sets with not just the unit element 1. In general, when a non-empty set X is equipped with two groups operation $\circ$ and $+$, then the non-standard distributivity condition of Proposition 6.2 (or slight variations of it [23]) applies.

We conclude from Proposition 6.2 that the maps $\sigma_a(b)$ and $\tau_b(a)$ provide an involutive solution $\check{r}(a,b)=(\sigma_a(b),\tau_b(a))$ of the set-theoretic braid equation (Rump's solution, Proposition 2.14).

6.2 Non-involutive case

We recall from Section 4 (see also [24]) that invertible, non-involutive, set-theoretic solutions, which are the main focus of this subsection, are constructed from the rack-quandle solutions via an admissible Drinfel'd twist.

The following proposition is useful in describing general set-theoretic solutions.

Proposition 6.3 *Let $\sigma_a,\tau_b : X\to X$, such that for all $a,b\in X$, σ_a is a bijection and $\tau_b(a)=\sigma^{-1}_{\sigma_a(b)}(\sigma_a(b)\triangleright a)$. Moreover, let $(X,\circ,1)$ be a group, such that for all $a,b\in X$, $a\circ b=\sigma_a(b)\circ\tau_b(a)$ and define $\bullet : X\times X\to X$, such that $a\bullet b := a\circ\sigma_a^{-1}(b)\circ\xi$, $\xi\in X$ is a fixed element. Then, for all $a,b\in X$,*

1. *(a) $a\bullet b=b\bullet(b\triangleright a)$*
   (b) $a\bullet\sigma_a(b)=a\circ b\circ\xi$.
2. *Assume that $(X,\bullet)$ is a group and set $a\bullet b=: a+b$ and $\xi=1$.*
   a. *Then, $b\triangleright a=-b+a+b$, where $-b$ is the inverse of b in $(X,+)$ (conjugate quandle), and $\sigma_a(b)=-a+a\circ b$.*
   b. *If $(X,+,\circ)$ is a skew brace, then $\sigma_a(\sigma_b(c))=\sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))$ and $\sigma_a(b\triangleright c)=\sigma_a(b)\triangleright\sigma_a(c)$.*

Proof.

1. (a) Using the definitions $a\bullet b=a\circ\sigma_a^{-1}(b)\circ\xi$ and $b\triangleright a=\sigma_b(\tau_{\sigma_a^{-1}(b)}(a))$ for all $a,b\in X$, we compute

$$b\bullet(b\triangleright a)=b\bullet\sigma_b(\tau_{\sigma_a^{-1}(b)}(a))=b\circ\sigma_b^{-1}(\sigma_b(\tau_{\sigma_a^{-1}(b)}(a)))\circ\xi=b\circ\tau_{\sigma_a^{-1}(b)}(a))\circ\xi.$$

But due to the condition $a\circ b=\sigma_a(b)\circ\tau_b(a)$ we conclude that $a\bullet b=b\bullet(b\triangleright a)$.

(b) From the definition of $a \bullet b$, and the fact that σ_a is bijection:

$$a \bullet \sigma_a(b) = a \circ \sigma_a^{-1}(\sigma_a(b)) \circ \xi = a \circ b \circ \xi.$$

2. (a) The first part follows immediately from (1) (a). From (1) (b) we immediately conclude that $\sigma_a(b) = -a + a \circ b$.
(b) This is condition (8) and it is shown in the proof of Proposal 2.14.

Notice that the binary operation $\bullet$ such that it satisfies condition (1) (a) in Proposition 6.3 is not uniquely defined (see [24] for a more detailed discussion). However, based on the definition of the operation $\bullet$ given in Proposition 6.3 we provide below a classification of non-involutive set-theoretic solutions given a specific rack/quandle.

In what follows we assume the existence of the map $\sigma_a : X \to X$ being a bijection and $(X, \circ)$ is a group.

1. **The conjugate quandle.** This case corresponds to Part (2) of Proposition 6.3. Recall that σ_a satisfies condition (a) of Definition 6.1 and provides a solution to the Yang-Baxter equation. We also confirm that condition (b) of Definition 6.1 is equivalent to $a \circ b = \sigma_a(b) \circ \tau_b(a)$. This corresponds to the Guarnieri-Vendramin solution [39].
2. **The affine quandle.** We generalize the definition of an affine quandle as follows. Let $f : X \to X$ be a a bijection and

$$f(-a+b+c) = -f(a) + f(b) + f(c) \tag{1}$$

for all $a, b, c \in X$. Define also $\triangleright : X \times X \to X$, such that $b \triangleright a = -f(b) + f(a) + b$ then $(X, \triangleright)$ is called an affine quandle. Note that, due to condition (1) it is shown that $(X, \triangleright)$ is a quandle. We define for all $a, b \in X$ $a \bullet b := f(a) + b$, and conclude that $f(a) + b = f(b) + (b \triangleright a)$. Then we obtain from the definition of σ_a^{-1} of Proposition 6.3, $\sigma_a^{-1}(b) = a^{-1} \circ (f(a) + b) \circ \xi^{-1}$ and consequently $\sigma_a(b) = -f(a) + a \circ b \circ \xi$. Note that σ_a satisfies condition (a) of Definition 6.1 if and only if

$$a \circ f(b) - a + f(a) \circ \xi^{-1} = \sigma_a(b) \circ f(\tau_b(a)) - \sigma_a(b) + f(\sigma_a(b)) \circ \xi^{-1}. \tag{2}$$

For instance, let $(X, +, \circ)$ be a skew brace and consider $f(a) := a \circ z - z$, $z \in X$ is a fixed element and $\xi = 1$, then conditions (1) and (2) are satisfied. We also confirm that condition (b) of Definition 6.1 is equivalent to $a \circ b = \sigma_a(b) \circ \tau_b(a)$, (see also [22, 24]).
3. **The core quandle.** Recall the core quandle. Recall that $(X, +)$ is a group and we define for all $a, b \in X$, $b \triangleright a = b - a + b$. We also define for all $a, b \in X$, $a \bullet b := -a + b$, then $-a + b = -b + (b \triangleright a)$. Then according to Proposition 6.3 for $\xi = 1$, $\sigma_a(b) = a + a \circ b$. We confirm that σ_a satisfies condition (a) and (b) of Definition 6.1. Notice, in particular that $\sigma_a(\sigma_b(c)) = \sigma_{a \circ b}(c)$, for all $a, b, c \in X$, if and only if $(X, +, \circ)$ is a brace, i.e. $(X, +)$ is abelian.

And with this we conclude our discussion on the derivation of generic solutions of the set-theoretic Yang-Baxter equation from admissible twists. A more exhaustive analysis of admissible twists and general set-theoretic solutions in accordance to Proposition 6.3 will be presented in a forthcoming publication.

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References

1. M. J. Ablowitz, B. Prinari and A. D. Trubatch, *Soliton interactions in the vector NLS equation*, Inverse Problems, vol. 20, no. 4 (2004) 1217.
2. V.E. Adler, A.I. Bobenko and Yu.B. Suris, *Classification of integrable equations on quad-graphs. The consistency approach*, Comm. Math. Phys. 233 (2003) 513.
3. N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math., 178 (2003) 2, 177–243.
4. I. Angiono, C. Galindo and L. Vendramin, *Hopf braces and Yang-Baxter operators*, Proc. Amer. Math. Soc. 1 45 (2017) no. 5, 1981–1995.
5. R. Baxter, *Exactly solved models in statistical mechanics*, Academic Press (1982).
6. D. Bachiller, F. Cedó, E. Jespers and J. Okniński, *Iterated matched products of finite braces and simplicity; new solutions of the Yang-Baxter equation*, Trans. Amer. Math. Soc. 370 (2018) 4881–4907.
7. A. Berenstein and D. Kazhdan, Geometric and unipotent crystals, in Visions in 13 Mathematics: GAFA special volume (2000) 1p. 88.
8. T. Brzeziński, *Lie trusses and heaps of Lie affebras*, Proceedings of Corfu Summer Institute 2021 ”School and Workshops on Elementary Particle Physics and Gravity” — PoS(CORFU2021), 406.
9. T. Brzeziński and B. Rybołowicz, *Congruence classes and extensions of rings with an application to braces*, Comm. Contemp. Math. **23** (2021) 2050010.
10. T. Brzeziński, S. Mereta and B. Rybołowicz, *From pre-trusses to skew bracess,* Publicacions Matemàtiques 66, Issue 2 (2022).
11. C. Burstinm and W. Mayer, *Distributive Gruppen von endlicher Ordnung*, J. Reine Angew. Math. **160** (1929) 111–130. (English translation arXiv:1403.6326)
12. V. Caudrelier and Q.C. Zhang, *Yang-Baxter and reflection maps from vector solitons with a boundary*, Nonlinearity 27 (2014) 1081.
13. M. Castelli, F. Catino and P. Stefanelli, *Left non-degenerate set-theoretic solutions of the Yang-Baxter equation and dynamical extensions of q-cycle sets*, J. Algebra Appl., 21 (2022) 8, Paper No. 2250154, 22.
14. F. Cedó, E. Jespers and J. Okniński, *Braces and the Yang-Baxter equation*, Comm. Math. Phys. 327 (2014) 1, 101–116.
15. F. Chouraqui, *On some Garsideness properties of structure groups of set-theoretic solutions of the Yang-Baxter equation*, Comm. in Alg., Volume 51 (2023) Issue 8.
16. I. Colazzo, E. Jespers, L. Kubat, A. Van Antwerpen and C. Verwimp, *Finite Idempotent Set-Theoretic Solutions of the Yang-Baxter Equation* , International Mathematics Research Notices (2023); rnad183.
17. I. Colazzo, E. Jespers, A. Van Antwerpen and C. Verwimp, *Left non-degenerate set-theoretic solutions of the Yang-Baxter equation and semitrusses*, J. Algebra, 610 (2022) 409–462.

18. A. Doikou and A. Smoktunowicz, *Set-theoretic Yang-Baxter* & *reflection equations and quantum group symmetries*, Lett. Math. Phys. 111 (2021) 105.
19. A. Doikou and A. Smoktunowicz, *From Braces to Hecke algebras* & *Quantum Groups*, J. of Algebra and its Applications (2022) 2350179.
20. A. Doikou, *Set-theoretic Yang-Baxter equation, braces and Drinfel'd twists*, J. Phys. A 54 (2021) 41.
21. A. Doikou, A. Ghionis and B. Vlaar, *Quasi-bialgebras from set-theoretic type solutions of the Yang-Baxter equation*, Lett. Math. Phys. 112 (2022) 4, Paper No. 78, 29.
22. A. Doikou and B. Rybołowicz, *Novel non-involutive solutions of the Yang-Baxter equation from (skew) braces*, Journal of the London Math. Soc. 110, 4 (2024) e12999.
23. A. Doikou and B. Rybołowicz, *Near braces and p-deformed braided groups*, Bull. London Math. Soc. 56 (2024) 124-139.
24. A. Doikou, B. Rybołowicz and P. Stefanelli, *Quandles as pre-Lie skew braces, set-theoretic Hopf algebras* & *universal R-matrices*, J. Phys. A: Math. Theor. 57 (2024) 405203.
25. A. Doikou, *Parametric set-theoretic Yang-Baxter equation: p-racks, solutions* & *quantum algebras*,
arXiv:2405.04088 [math-ph].
26. A. Doikou, *Combinatorial twists in* $\mathfrak{gl}_n$ *Yangian*, arXiv:2504.21690 [math-QA].
27. V.G. Drinfel'd, *On some unsolved problems in quantum group theory*, in: Quantum groups (Leningrad, 1990), vol. 1510 of Lecture Notes in Math., Springer, Berlin, (1992) pp. 1–8.
28. V.G. Drinfel'd, *Hopf algebras and the quantum Yang–Baxter equation*, Soviet. Math. Dokl. 32 (1985) 254.
29. M. Elhamdadi, *A Survey of Racks and Quandles: Some Recent Developments*, Algebra Colloquium, Vol. 27, No. 03, pp. 509-522 (2020).
30. P. Etingof, T. Schedler and A. Soloviev, *Set-theoretical solutions to the Quantum Yang-Baxter equation*, Duke Math. J. 100(2) (1999) 169–209.
31. P. Etingof, *Geometric crystals and set-theoretical solutions to the quantum Yang-Baxter equation*, Comm. Algebra 31 (2003) 1961.
32. L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. 1 (1990) 193.
33. R. Fenn, M. Jordan-Santana, and L. Kauffman, Biquandles and virtual links, Topology Appl., 145 (2004) pp. 157–175.
34. T. Fritz, *Self-distributive structures in physics*, arXiv:2403.14458 [quant-ph].
35. T. Gateva-Ivanova and S. Majid, *Matched pairs approach to set theoretic solutions of the Yang–Baxter equation*, J. Algebra 319 (2008) 1462–1529.
36. T. Gateva-Ivanova, *Set-theoretic solutions of the Yang–Baxter equation, braces and symmetric groups*, Adv. Math. 388(7):649–701 (2018).
37. T. Gateva-Ivanova, *Quadratic algebras, Yang-Baxter equation, and Artin- Schelter regularity*, Adv. in Math. 230 (2012) 2152–2175.
38. T. Gateva-Ivanova and S. Majid, *Quadratic algebras and idempotent braided sets*, arXiv:2409.02939.
39. L. Guarnieri, L. Vendramin, *Skew braces and the Yang-Baxter equation*, Math. Comput. 86(307) (2017) 2519–2534.
40. G. Hatayama, A. Kuniba and T. Takagi, *Soliton cellular automata associated with crystal bases*, Nucl. Phys. B577 (2000) 619.
41. P. Jedlicka, A. Pilitowska and A. Zamojska-Dzienio, *The retraction relation for biracks*, J. Pure Appl. Algebra 223 (2019) 3594–3610.
42. E. Jespers, E. Kubat and A. Van Antwerpen, *The structure monoid and algebra of a non-degenerate set-theoretic solution of the Yang-Baxter equation*, Transactions of the American Mathematical Society, 372, 10 (2019) 7191-7223.
43. E. Jespers, E. Kubat, A. Van Antwerpen and L. Vendramin, *Factorizations of skew braces*, Math. Ann. 375 (2019) no. 3-4, 1649–1663.
44. M. Jimbo, A q-difference analogue of U(g) and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985) 63.

45. M. Jimbo, Quantum R-matrix for the generalized Toda system, Comm. Math. Phys. 102 (1986) 537–547
46. D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, 23 (1982) 1, 37–65.
47. L.H. Kauffman, Virtual knot theory, European Journal of Combinatorics 20 (1999) 663–691.
48. V. Lebed, *Applications of self-distributivity to Yang–Baxter operators and their cohomology*, Journal of Knot Theory and Its RamificationsVol. 27, No. 11, 1843012 (2018).
49. V. Lebed and L. Vendramin, *On structure groups of set-theoretical solutions to the Yang-Baxter equation*, Proc. Edinburgh Math. Soc., Volume 62, Issue 3 (2019) 683 -717.
50. V. Lebed and L. Vendramin, *Reflection equation as a tool for studying solutions to the Yang-Baxter equation*, J. Algebra, 607 (2022) 360–380.
51. J.-H. Lu, M. Yan, and Y.-C. Zhu, *On the set-theoretical Yang-Baxter equation*, Duke Math. J., 104 (2000) 1, 1–18.
52. S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) 119(161) (1982) no. 1, 78–88, 160 (Russian).
53. V.G. Papageorgiou, A.G. Tongas and A.P. Veselov, *Yang-Baxter maps and symmetries of integrable equations on quad-graphs*, J. Math. Phys. 47, 083502 (2006).
54. V.G. Papageorgiou, Yu.B. Suris, A.G. Tongas and A.P. Veselov, *On quadrirational Yang-Baxter Maps*, SIGMA 6 (2010) 033.
55. W. Rump, *A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equation*, Adv. Math. 193 (2005) 40–55.
56. W. Rump, *Braces, radical rings, and the quantum Yang-Baxter equation*, J. Algebra 307 (1) (2007) 153–170.
57. W. Rump, *A covering theory for non-involutive set-theoretic solutions to the Yang–Baxter equation*, J. Algebra 520 (2019) 136–170.
58. J. Scott Carter, *A Survey of Quandle Ideas*, Introductory Lectures on Knot Theory, pp. 22-53 (2011).
59. A. Smoktunowicz, L. Vendramin, *On Skew Braces* (with an appendix by N. Byott and L. Vendramin), Journal of Combinatorial Algebra Volume 2, Issue 1, (2018) 47-86.
60. A. Soloviev, Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation, Math. Res. Lett. 7(5-6) (2000) 577-596.
61. D. Takahashi and J. Satsuma, *A soliton cellular automaton*, J. Phys. Soc. Japan 59 (1990) 3514.
62. T. Tsuchida, *N-Soliton Collision in the Manakov Model*, Progress of Theoretical Physics, vol. 111 (2004) 151–182.
63. A.P. Veselov, *Yang-Baxter maps and integrable dynamics*, Phys. Lett. A314 (2003) 214.
64. C.N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Rev. Lett. 19 (1967) 1312.