

Quantum counting, and a relevant sign

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Abstract Two indispensable algorithms in an introductory course on Quantum Computing are Grover’s search algorithm and quantum phase estimation. Quantum counting is a simple yet beautiful blend of these two algorithms, and it is therefore an attractive topic for a student project in such a course. However, a sign that is irrelevant when implementing Grover’s algorithm becomes relevant. We briefly review these algorithms, highlighting the aforementioned sign.

1 Introduction

Two indispensable algorithms in an introductory undergraduate course on Quantum Computing are Grover’s search algorithm [4, 5] and quantum phase estimation (QPE) [6]. Quantum counting [2, 3] is a simple yet beautiful blend of these two algorithms, and it is therefore an attractive topic for a student project in such a course. However, a sign that is irrelevant when implementing Grover’s algorithm becomes relevant.

We start by briefly reviewing in Sec. 2 Grover’s algorithm for a single marked element, and its extension [2] to the case of multiple marked elements. QPE is briefly reviewed in Sec. 3. The key part of this paper is Sec. 4, where we review quantum counting and highlight the aforementioned sign. We finish in Sec. 5 with a brief conclusion.

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2 Grover's algorithm

Grover's search algorithm, one of the "crown jewels" of Quantum Computing, has been widely described in detail, see e.g. [7]. We content ourselves here with reminding the reader of the main steps, which also serves to set out our notations and conventions. We first treat in Sec. 2.1 the familiar case of a single marked element, and then consider in Sec. 2.2 the perhaps less-familiar case of multiple marked elements.

2.1 One marked element

Let x and a be n -bit integers, and $f(x)$ the search function

$$f(x) = \begin{cases} 0 & \text{if } x \neq a \\ 1 & \text{if } x = a \end{cases} \quad (1)$$

The "marked element" a is the object of our search. We define the corresponding unitary operator U_f , the so-called standard protocol acting on the n -qubit "input register" $|x\rangle$ and the 1-qubit "output register" $|y\rangle$, by

$$U_f(|y\rangle|x\rangle) = |y \oplus f(x)\rangle|x\rangle. \quad (2)$$

The initial state of the output register is taken to be

$$|y\rangle = H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \quad (3)$$

and therefore

$$U_f(H|1\rangle|x\rangle) = (-1)^{f(x)}H|1\rangle|x\rangle. \quad (4)$$

We define the operator V on the input register by

$$V|x\rangle = (-1)^{f(x)}|x\rangle = \begin{cases} |x\rangle & \text{if } x \neq a \\ -|x\rangle & \text{if } x = a \end{cases}, \quad (5)$$

which can be realized as

$$V = 1 - 2|a\rangle\langle a|, \quad (6)$$

so that (4) can re-expressed as

$$U_f(H|1\rangle|x\rangle) = H|1\rangle V|x\rangle. \quad (7)$$

The output register remains in the state $H|1\rangle$ throughout the algorithm, and so we henceforth focus only on the input register.

The initial state of the input register is taken to be

$$|\phi\rangle \equiv H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle. \quad (8)$$

The kets $|a\rangle$ and $|\phi\rangle$ determine a (real) plane; we define a normalized ket $|a_\perp\rangle$ in this plane that is perpendicular to $|a\rangle$, i.e. $\langle a|a_\perp\rangle = 0$. We define θ as the angle between $|a_\perp\rangle$ and $|\phi\rangle$, as in Fig. 1. We therefore have

$$|\phi\rangle = \cos\theta |a_\perp\rangle + \sin\theta |a\rangle. \quad (9)$$

We observe from (8) that $\langle a|\phi\rangle = \frac{1}{2^{n/2}}$; and from (9) that $\langle a|\phi\rangle = \sin\theta$. Hence,

$$\sin\theta = \frac{1}{2^{n/2}}. \quad (10)$$

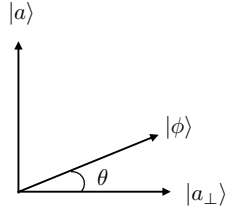


Fig. 1: The plane spanned by $|a\rangle$ and $|\phi\rangle$

We define the unitary operator W , the so-called diffuser, by

$$W = 2|\phi\rangle\langle\phi| - 1. \quad (11)$$

By explicit computation using (6), (9) and (11), we obtain

$$\begin{aligned} (WV)|a_\perp\rangle &= \cos(2\theta)|a_\perp\rangle + \sin(2\theta)|a\rangle, \\ (WV)|a\rangle &= -\sin(2\theta)|a_\perp\rangle + \cos(2\theta)|a\rangle. \end{aligned} \quad (12)$$

We see that WV can be represented in the basis $|a_\perp\rangle, |a\rangle$ by the matrix

$$WV = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} = e^{-i2\theta Y}, \quad (13)$$

which rotates any ket in the plane by the angle 2θ counterclockwise (i.e. from $|a_\perp\rangle$ to $|a\rangle$). After k such Grover rotations, the input register is in the state

$$(WV)^k |\phi\rangle = \cos((2k+1)\theta)|a_\perp\rangle + \sin((2k+1)\theta)|a\rangle. \quad (14)$$

The number k is chosen such that the result (14) is the sought-after ket $|a\rangle$; measuring the input register will then give a with probability 1. To this end, we set $(2k+1)\theta = \pi/2$, and thus $k = \frac{\pi}{4\theta} - \frac{1}{2}$. We see from (10) that $\theta \approx \sin \theta = \frac{1}{2^{n/2}}$ for large n . Hence, the required number of Grover rotations is

$$k \approx \frac{\pi}{4} \sqrt{2^n}. \quad (15)$$

It is not difficult to show that the W operator (11) can be re-expressed as

$$W = -H^{\otimes n} X^{\otimes n} (c^{n-1} Z) X^{\otimes n} H^{\otimes n}, \quad (16)$$

where $c^{n-1}Z$ denotes the $(n-1)$ -fold controlled Z gate. This form for W is useful for implementing Grover's algorithm on, say, a quantum simulator. For the purpose of determining a , the minus sign in (16) is irrelevant, so it is typically dropped. (Note that multiplication by -1 is not a standard gate. Nevertheless, this sign could be implemented using a product of standard 1-qubit gates, for example $ZXZX = -\mathbb{I}$.) That is, when simulating Grover's algorithm, typically one uses

$$\tilde{W} \equiv -W = H^{\otimes n} X^{\otimes n} (c^{n-1} Z) X^{\otimes n} H^{\otimes n} \quad (17)$$

instead of (16). We will revisit this sign in Sec. 4.

2.2 Multiple marked elements

Suppose we want to search now for a set $S = \{a_1, a_2, \dots, a_m\}$ of m marked elements. The standard protocol U_f is still given by (2), except that the search function is now given by

$$f(x) = \begin{cases} 0 & \text{if } x \notin S \\ 1 & \text{if } x \in S \end{cases}. \quad (18)$$

The operator V is therefore now defined by

$$V|x\rangle = (-1)^{f(x)}|x\rangle = \begin{cases} |x\rangle & \text{if } x \notin S \\ -|x\rangle & \text{if } x \in S \end{cases}, \quad (19)$$

and is given by

$$V = 1 - 2 \sum_{j=1}^m |a_j\rangle\langle a_j|. \quad (20)$$

The key insight is to consider the plane spanned by $|s\rangle$ and $|\phi\rangle$, where $|s\rangle$ is an equal-weight superposition of all the $|a_j\rangle$'s

$$|s\rangle = \frac{1}{\sqrt{m}} \sum_{j=1}^m |a_j\rangle. \quad (21)$$

We define a vector $|s_\perp\rangle$ in this plane that is perpendicular to $|s\rangle$, i.e. $\langle s|s_\perp\rangle = 0$; moreover, we define θ as the angle between $|s_\perp\rangle$ and $|\phi\rangle$, see Fig. 2. We therefore have

$$|\phi\rangle = \cos \theta |s_\perp\rangle + \sin \theta |s\rangle. \quad (22)$$

We observe from (21) and (8) that $\langle s|\phi\rangle = \sqrt{\frac{m}{2^n}}$; and from (22) that $\langle s|\phi\rangle = \sin \theta$. Hence,

$$\sin \theta = \sqrt{\frac{m}{2^n}}. \quad (23)$$

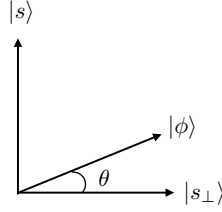


Fig. 2: The plane spanned by $|s\rangle$ and $|\phi\rangle$

It is easy to check that $V|s\rangle = -|s\rangle$; furthermore, $\langle a_j|s_\perp\rangle = 0$, and hence $V|s_\perp\rangle = |s_\perp\rangle$. We see that, in the plane of $|s_\perp\rangle$ and $|s\rangle$, V acts as

$$V = 1 - 2|s\rangle\langle s|, \quad (24)$$

cf. (6). The computation of WV on $|s_\perp\rangle$ and $|s\rangle$ is therefore the same as the previous computation of WV on $|a_\perp\rangle$ and $|a\rangle$, respectively (12); and in the basis $|s_\perp\rangle, |s\rangle$, WV is given by the same matrix (13). After k Grover rotations, the input register is in the state

$$(WV)^k |\phi\rangle = \cos((2k+1)\theta) |s_\perp\rangle + \sin((2k+1)\theta) |s\rangle. \quad (25)$$

Choosing again $(2k+1)\theta = \pi/2$, so that

$$k \approx \frac{\pi}{4} \sqrt{\frac{2^n}{m}}, \quad (26)$$

the result (25) is the ket $|s\rangle$ (21); measuring the input register will then give any one of the $a_j \in S$ with equal probability. Of course, (26) reduces to (15) for $m = 1$.

In order to use this algorithm, we must know the number of marked elements (m), which enters into the formula (26) for the number of Grover rotations. What do we do if we do *not* know m ahead of time? One way to proceed, the so-called quantum counting considered in Sec. 4, involves using QPE.

3 Quantum phase estimation

QPE is another important quantum algorithm with many uses, including an elegant formulation of Shor's celebrated period-finding algorithm [11]. Given a unitary operator \mathcal{U} and an eigenvector $|\psi\rangle$, whose corresponding eigenvalue necessarily has the form $e^{i\alpha}$ with α real,

$$\mathcal{U}|\psi\rangle = e^{i\alpha}|\psi\rangle, \quad (27)$$

QPE provides an estimate for α . A key ingredient of QPE is the quantum Fourier transform, which we recall (see e.g. [7]) is defined by

$$U_{FT}|x\rangle = \frac{1}{2^{\frac{n}{2}}} \sum_{y=0}^{2^n-1} e^{\frac{2\pi ixy}{2^n}}|y\rangle, \quad 0 \leq x < 2^n, \quad (28)$$

and which can be shown to be written more explicitly as

$$U_{FT}|x\rangle = \frac{1}{2^{\frac{n}{2}}} \left(|0\rangle + e^{\frac{2\pi ix}{2}}|1\rangle \right) \otimes \dots \otimes \left(|0\rangle + e^{\frac{2\pi ix}{2^{n-1}}}|1\rangle \right) \otimes \left(|0\rangle + e^{\frac{2\pi ix}{2^n}}|1\rangle \right). \quad (29)$$

The QPE circuit diagram, with t auxiliary qubits, is shown in Fig. 3.

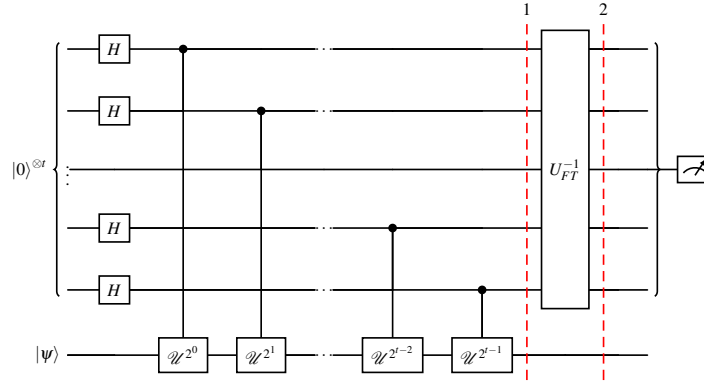


Fig. 3: Circuit diagram for QPE

At slice 1, one can check with the help of (29) that the circuit is in the state

$$|\psi\rangle U_{FT} \left| \frac{2^t \alpha}{2\pi} \right\rangle. \quad (30)$$

At slice 2, after applying the inverse quantum Fourier transform, the state becomes simply

$$|\psi\rangle \left| \frac{2^t \alpha}{2\pi} \right\rangle. \quad (31)$$

Thus, the measurement of the ancillary qubits gives an integer j that is within $1/2$ of $\frac{2^t \alpha}{2\pi}$. This estimate can be improved by increasing the value of t .

4 Quantum counting

We are now in position to address the question raised at the end of Sec. 2.2: what can we do if we are searching for multiple marked elements, but we do not know their number m ?

An answer, known as quantum counting, is to apply QPE to the Grover rotation operator; that is, recalling (13), set

$$\mathcal{U} = WV = e^{-i2\theta Y}, \quad (32)$$

and implement the QPE circuit in Fig. 3 using $|\psi\rangle = |\phi\rangle$ (8). Evidently, \mathcal{U} has eigenvalues $e^{\pm 2i\theta}$. Although $|\phi\rangle$ is not an eigenket of \mathcal{U} , $|\phi\rangle$ lies in the subspace spanned by $|s_\perp\rangle$ and $|s\rangle$ (22); hence, QPE provides estimates for both $\pm 2\theta$. More precisely, QPE gives integers j_+ and j_- that are closest to $\frac{2^t(2\theta)}{2\pi}$ and $\frac{2^t(2\pi-2\theta)}{2\pi}$, respectively, see (31). Solving for θ , we see that

$$\theta \approx \frac{\pi j_+}{2^t}, \quad \pi - \frac{\pi j_-}{2^t}. \quad (33)$$

Using (23), we conclude that the number of marked elements is given by

$$m = 2^n \sin^2 \theta \approx 2^n \sin^2 \left(\frac{\pi j_\pm}{2^t} \right) \quad (\text{implementing } W). \quad (34)$$

Using \tilde{W} (17) instead of W (that is, dropping the sign in (16)) leads to a result different from (34). Indeed, a simple way to account for the different sign is to observe from (32) that changing $W \mapsto -W = \tilde{W}$ implies $\mathcal{U} \mapsto -\mathcal{U}$; and the corresponding eigenvalues transform accordingly $e^{\pm 2i\theta} \mapsto -e^{\pm 2i\theta} = e^{\pm 2i(\theta \pm \frac{\pi}{2})}$. Performing the shift $\theta \mapsto \theta - \frac{\pi}{2}$ in (34), we obtain

$$m = 2^n \sin^2 \left(\theta - \frac{\pi}{2} \right) \approx 2^n \sin^2 \left[\pi \left(\frac{j_\pm}{2^t} - \frac{1}{2} \right) \right] \quad (\text{implementing } \tilde{W} = -W). \quad (35)$$

In our simulations, we implemented \tilde{W} , and the number of marked elements indeed matched with (35). For example, we searched for the set of **three** 3-bit integers $S = \{2, 4, 6\}$ using 5 ancillas, so that $n = m = 3$ and $t = 5$. In 1024 shots, the two most frequent integers obtained from measuring the ancillas were $j = 9$ and $j = 23$. Eq. (35) then correctly gave $m \approx 3.22 \approx 3$, while (34) would imply the incorrect result $m \approx 4.78 \approx 5$.

5 Conclusion

Although quantum counting is not considered part of the standard undergraduate introductory Quantum Computing curriculum (see e.g. [1, 8]), it entails only modest – yet very interesting – extensions of two topics that *are* part of the standard repertoire, namely Grover’s algorithm and QPE. Hence, coding and simulating quantum counting, using e.g. qiskit [9] or cirq [10], is an attractive topic for a student project in such a course, especially for students who have already coded and simulated Grover and QPE.¹ However, the sign of the diffuser W (16) should not be neglected.

Acknowledgements RN was supported in part by the National Science Foundation under Grant No. PHY 2310594 and by a Cooper fellowship.

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¹ Sample qiskit code is included as Supplementary Material; it can be found at <https://arxiv.org/abs/2310.07428>.